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MULTILINEAR MAXIMAL FUNCTIONS AND SINGULAR INTEGRALS IN WEIGHTED GRAND LEBESGUE SPACES

Introduction

In this talk we give one–weight estimates for multi(sub)linear Hardy–Littlewood maximal and multilinear Calderón–Zygmund operators in weighted grand Lebesgue spaces under the vector Muckenhoupt condition on weights. We note that operators are defined on metric measure spaces.

Nowadays the theory of grand Lebesgue spaces introduced by T. Iwaniec and C. Sbordone [15] is one of the intensively developing directions of the modern analysis. It was realized the necessity for the study of these spaces because of their rather essential role and applications in various fields. These spaces naturally arise, for example, in the integrability problems of the Jacobian under minimal hypothesis (see [15] for the details).

Structural properties of grand Lebesgue spaces were studied in the papers [5], [1]. In [6] the authors proved that for the boundedness of the Hardy–Littlewood maximal operator defined on $[0,1]$ in weighted grand Lebesgue spaces $L^{p)}_w([0,1])$ it is necessary and sufficient that the weight w belongs to the Muckenhoupt's class $A_p([0,1])$. The same phenomenon was noticed by the first and third authors of this paper for the Hilbert transform in [18]. We refer to the papers [17], [16], [21] for one–weight results regarding maximal and singular integrals of various type in these spaces.

Multisublinear maximal operators appeared naturally in connection with multilinear Calderón-Zygmund theory. A multisublinear maximal operator that acts on the product of m-Lebesgue spaces and is smaller than the m-fold product of the Hardy–Littlewood maximal function was studied in [19]. It was used to obtain a precise control on multilinear singular integral operators of Calderón-Zygmund type and to build a theory of weights adapted to the multilinear setting.

Multilinear analysis on spaces of homogeneous type (SHT) was developed in [9]. In particular, the results of that work treats with Hardy–

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Littlewood maximal functions and Calderón-Zygmund operators defined on metric spaces with doubling measure.

It should be stressed that the results of this paper are new even for Euclidean spaces.

1. Preliminaries

Let X be a set and let $d : X \times X \mapsto \mathbb{R}_+$ be a quasi-metric, i.e., d satisfies the following conditions:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) \leq \kappa(d(x, z) + d(z, y))$ for every $x,\ y,\ z \in X$;
(iii) $d(x, y) = d(y, x)$ for every $x,\ y \in X$.

The couple (X, d) is called a quasi-metric space. If $\kappa = 1$, then d is called a metric and (X, d) is a metric space. For all $x \in X$ and $r > 0$, as usual $B(x, r) := \{y \in X : d(x, y) < r\}$ is called a ball.

Suppose that μ is a measure defined on a σ-algebra of subsets that contains the balls of X. Then the triple $X := (X, d, \mu)$ is called a quasi-metric measure space. Throughout the paper we assume that $0 < \mu(B(x, r)) < \infty$ and $\mu\{x\} = 0$ for all $x \in X$ and $r > 0$.

If μ satisfies the doubling condition

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)),$$

with the positive constant C independent of $x \in X$ and $r > 0$, then (X, d, μ) is called a space of homogeneous type (SHT). For the definition, examples and some properties of an SHT see, e.g., monographs [22], [3], [4], [9].

Let $\ell := diam\ (X) = \sup_{x,y\in X} d(x, y)$. Notice that the condition $\ell < \infty$ implies that $\mu(X) < \infty$.

Definition A. The triple (X, d, μ) is called an *RD*-space if it is an SHT and μ satisfies the reverse doubling condition: there exist constants $a, b > 1$ such that for all $x \in X$ and $0 < r < \ell/a$,

$$b\mu(B(x, r)) \leq \mu B(x, ar).$$

It is known that (X, d, μ) is an *RD*- space if and only if it is an SHT and there is a constant a_0 such that for all $x \in X$ and $0 < r < \ell/a_0$,

$$B(x, a_0 r) \setminus B(x, r) \neq 0.$$

Let $1 \leq r < \infty$. We denote by $L^r(X, \mu)$ the Lebesgue space on X with an exponent r.

If w is a weight (locally integrable, μ- a.e. positive function on X), then then we denote the Lebesgue spaces with weight w by $L^r_w(X)$.

Let $\mu(X) < \infty$, $1 < p < \infty$ and let φ be a continuous positive function on $(0, p-1)$ such that it is non-decreasing on $(0, \sigma)$ for some small positive σ

and satisfies the condition $\lim_{x\to 0+}\varphi(x)=0$. The generalized grand Lebesgue space $L^{p),\varphi}(X,\mu)$ is the class of those $f: X\to\mathbb{R}$ for which the norm

$$\|f\|_{L^{p),\varphi}(X,\mu)} = \sup_{0<\varepsilon<p-1}\left(\varphi(\varepsilon)\int_X |f(x)|^{p-\varepsilon}d\mu(x)\right)^{1/(p-\varepsilon)}$$

is finite. If w is a weight on X, then the weighted grand Lebesgue space with weight w is denoted by $L_w^{p),\varphi}(X)$ and coincides with the class $L^{p),\varphi}(X,wd\mu)$. In this case we assume that $\|f\|_{L_w^{p),\varphi}(X)} = \|f\|_{L^{p),\varphi}(X,wd\mu)}$.

If $\varphi(x)=x^\theta$, where θ is a positive number, then we denote $L^{p),\varphi}(X,\mu)$ (resp. $L_w^{p),\varphi}(X)$) by $L^{p),\theta}(X,\mu)$ (resp. by $L_w^{p),\theta}(X)$).

The space $L^{p),\theta}(X,\mu)$ is a Banach space.

It is easy to check that the following continuous embeddings hold:

$$L^p(X,\mu)\hookrightarrow L^{p),\theta_1}(X,\mu)\hookrightarrow L^{p),\theta_2}(X,\mu)\hookrightarrow L^{p-\varepsilon}(X,\mu),$$

where $0<\varepsilon\le p-1$ and $\theta_1<\theta_2$.

It turns out that in the theory of PDEs the generalized grand Lebesgue spaces are appropriate to the solutions of existence and uniqueness, and, also, the regularity problems for various nonlinear differential equations. The space $L^{p),\theta}$ (defined on bounded domains in $\mathbb{R}^n$) for arbitrary positive θ was introduced in the paper [12], where the authors studied the nonhomogeneous n-harmonic equation div $A(x,\nabla u)=\mu$. If $\theta=1$, then $L^{p),\theta}(X,\mu)$ coincides with the Iwaniec–Sbordone space, which we denote by $L^{p)}(X,\mu)$.

In the sequel the following notation will be used:

$$\overrightarrow{p} := (p_1,\dots,p_m),$$

where $p_i\in(0,\infty)$ for each $1\le i\le m$.

$$\overrightarrow{f} = (f_1,\dots,f_m),$$

where f_i are $\mu-$ measurable functions defined on X.

$$d\mu(\overrightarrow{y}) := d\mu(y_1)\cdots d\mu(y_m);\quad d\overrightarrow{y} = dy_1\cdots dy_m.$$

$$\nu_{\overrightarrow{w}} := \prod_{j=1}^m w_j^{p/p_j};\quad B_{xy} := \mu(B(x,d(x,y))).$$

Let $s\in[1,\infty]$. As usual we put $s' := \frac{s}{s-1}$ if $s\in(1,\infty)$ and $s':=\infty$ for $s=1$ and $s':=1$ for $s=\infty$.

Let $\mu(X)<\infty$ and let $1<p_j<\infty$ for each $1\le j\le m$, $\frac{1}{p}=\sum_{j=1}^m\frac{1}{p_j}$.

We define the class $\prod_{j=1}^m\mathcal{L}^{p_j),\varphi}(X,\mu_j)$ of vector functions $\overrightarrow{f}$ as follows: $\overrightarrow{f}\in\prod_{j=1}^m\mathcal{L}^{p_j),\varphi}(X,\mu_j)$ if

$$
\begin{aligned}
&\|\overrightarrow{f}\|_{\prod_{j=1}^m \mathcal{L}^{p_j),\varphi}(X,\mu_j)} = \\
= &\sup_{1<r<p} \left\{ \varphi\Big(\frac{p}{r'}\Big)^{\frac{r}{p}} \prod_{j=1}^m \|f_j\|_{L^{p_j/r}(X,\mu_j)} \right\} = \\
= &\sup_{1<r<p} \left\{ \prod_{j=1}^m \varphi\big(\frac{p}{r'}\big)^{\frac{r}{p_j}} \|f_j\|_{L^{p_j/r}(X,\mu_j)} \right\} < \infty.
\end{aligned}
$$

The expression $\|\overrightarrow{f}\|_{\prod_{j=1}^m \mathcal{L}^{p_j),\varphi}(X,\mu_j)}$ can be rewritten as follows:

$$
\begin{aligned}
&\|\overrightarrow{f}\|_{\prod_{j=1}^m \mathcal{L}^{p_j),\varphi}(X,\mu_j)} = \\
= &\sup_{0<\eta<p-1} \left\{ \prod_{j=1}^m \varphi\Big(\eta\Big)^{\frac{1}{p_j-\eta_j}} \|f_j\|_{L^{p_j-\eta_j}(X,\mu_j)} : \frac{p}{p-\eta} = \right. \\
= &\left. \frac{p_j}{p_j-\eta_j},\ j=1,\dots,m \right\}.
\end{aligned}
$$

It is easy to check that

$$
\prod_{j=1}^m L^{p_j)\varphi}(X,\mu_j) \hookrightarrow \prod_{j=1}^m \mathcal{L}^{p_j),\varphi}(X,\mu_j),
$$

in particular,

$$
\|\overrightarrow{f}\|_{\prod_{j=1}^m \mathcal{L}^{p_j),\varphi}(X,\mu_j)} \le \|\overrightarrow{f}\|_{\prod_{j=1}^m L^{p_j),\varphi}(X,\mu_j)}.
$$

Let $1<r<\infty$. We say that a weight function w belongs to the Muckenhoupt class $A_r(X)$ if

$$
\|w\|_{A_r} := \sup_B \left(\frac{1}{\mu(B)} \int_B w d\mu \right) \left(\frac{1}{\mu(B)} w^{1-r'} d\mu \right)^{r-1} < \infty.
$$

Let us recall the definition of the vector Muckenhoupt condition (see [19] for Euclidean spaces and [9] for metric measure spaces).

Definition B. Let $1 \le p_i < \infty$, $0<p<\infty$. We say that $\overrightarrow{w}$ satisfies the $A_{\overrightarrow{p}}$ condition ($\overrightarrow{w} \in A_{\overrightarrow{p}}$) if

$$
\|\overrightarrow{w}\|_{A_{\overrightarrow{p}}} :=
$$

$$
= \sup_{B\subset X} \left(\frac{1}{\mu(B)} \int_B \nu_{\overrightarrow{w}}(x) d\mu(x) \right) \prod_{j=1}^m \left(\frac{1}{\mu(B)} \int_B w_j^{1-p'}(x) d\mu(x) \right)^{p/p'_j} < \infty,
$$

where the supremum is taken over all balls B in X. For $p_j = 1$, the expression $\left(\frac{1}{\mu(B)} \int\limits_B w_j^{1-p'}(x) d\mu(x)\right)^{1/p'_j}$ is understood as $(\inf_B w_j)^{-1}$.

The expression $\|\overrightarrow{w}\|_{A_{\overrightarrow{p}}}$ is called $A_{\overrightarrow{p}}$ characteristic of $\overrightarrow{w}$.

The following fact gives so called openness property of the vector Muckenhoupt condition holds (see [19] for Euclidean spaces but the proof allows us to formulate it for an SHT).

Let $1 < p_i < \infty$, $i = 1, \cdots, m$. If that $\overrightarrow{w} \in A_{\overrightarrow{p}}(X)$, then there is a constant $r > 1$ such that $\overrightarrow{w} \in A_{\overrightarrow{p}/r}(X)$.

In the same paper (see Remark 7.3) the authors showed that the classes $A_{\overrightarrow{p}}$ are not increasing under the natural partial order; however it is noticed in [2] that the classes $A_{s\overrightarrow{p}}$ are strictly increasing as s increases, in particular, the following estimate holds:

$$\|\overrightarrow{w}\|_{A_{s_1\overrightarrow{p}}} \leq \|\overrightarrow{w}\|_{A_{s_2\overrightarrow{p}}}, \quad s_2 \leq s_1. \tag{1}$$

In the linear case ($m = 1$) the class $A_{\overrightarrow{p}}$ coincides with the well- known Muckenhoupt class A_p.

2. Main Results

Regarding the Hardy–Littlewood maximal operator

$$(\mathcal{M}\overrightarrow{f})(x) = \sup_{B \ni x} \prod_{i=1}^{m} \frac{1}{\mu(B)} \int\limits_B |f_i(y_i)| d\mu(y_i), \quad x \in X$$

we have the following result:

Theorem 2.1. *Let $\mu(X) < \infty$, $1 < p_i < \infty$ for each $1 \leq i \leq m$, and let $\theta > 0$. Suppose that $\frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j}$ and that $\overrightarrow{w} \in A_{\overrightarrow{p}}$. Then $\mathcal{M}$ is bounded from $\prod_{j=1}^{m} \mathcal{L}_{w_j}^{p_j),\theta}(X)$ to $L_{\nu_{\overrightarrow{w}}}^{p),\theta}(X)$ (hence, from $\prod_{j=1}^{m} L_{w_j}^{p_j),\theta}(X)$ to $L_{\nu_{\overrightarrow{w}}}^{p),\theta}(X)$).*

Let $1 < p_i < \infty$ for each $1 \leq i \leq m$, and let w_j, $j = 1, \cdots, m$ be weight functions on X. Let us denote by $M_{\overrightarrow{p}}$ the class of multilinear bounded operators from $\prod_{j=1}^{m} L_{w_j}^{p_j}(X)$ to $L_{\nu_{\overrightarrow{w}}}^{p}(X)$.

To get the one-weight estimate for the Calderón-Zygmund singular integrals we proved the general-type theorem for multilinear operators.

Theorem 2.2 (General Statement)**.** *Let $\mu(X) < \infty$ and let $1 < p_i < \infty$ for each $1 \leq i \leq m$. Suppose that $1 < p < \infty$, where $\frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j}$. Let $\theta > 0$. Suppose that a multilinear operator K belongs to $M_{\overrightarrow{p}} \cap M_{\frac{\overrightarrow{p}}{r}}$ for some r satisfying the condition $1 < r < \min_j\{p_j\}$. Then the operator K is bounded from $\prod_{j=1}^{m} \mathcal{L}_{w_j}^{p_j),\theta}(X)$ to $L_{\nu_{\overrightarrow{w}}}^{p),\theta}$ (hence, from $\prod_{j=1}^{m} L_{w_j}^{p_j),\theta}(X)$ to $L_{\nu_{\overrightarrow{w}}}^{p),\theta}(X)$).*

Let (X, d, μ) be an RD space $(X., d, \mu)$ with metric d. We give the definition of the Calderón-Zygmund singular integrals on (X, d, μ). First let us introduce some classes of functions.

We denote by $C_b(X)$ the class of continuous functions on X with bounded support (it is clear that if diameter of X is finite, then $C_b(X)$ is the class of all continuous functions on X and is denoted by $C(X)$). For any $\eta \in (0, 1]$, let $C^\eta(X)$ be the set of all functions $g : X \to \mathcal{C}$ such that

$$\|g\|_{\dot{C}^\eta(X)} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)^\eta} < \infty.$$

Further,

$$C_b^\eta := \{g \in C^\eta(X) : g \in BS(X)\},$$

where $BS(X)$ is the class of functions on X with bounded support.

L_b^∞ denotes the class of all bounded functions on X with bounded support;

Given $m \in \mathbb{N}$, set

$$\Omega_m := X^{m+1} \setminus \{(y_0, y_1, \dots, y_m) : y_0 = y_j,\, 1 \leq j \leq m\}.$$

Suppose that $k : \Omega_m \mapsto \mathbb{C}$ is locally integrable. The function k is called a Calderón-Zygmund kernel if there exist constants $C_k > 0$ and $\delta \in (0, 1]$ such that for every $(y_0, \dots, y_m) \in \Omega_m$,

(i)

$$|k(y_0, y_1, \dots, y_k, \dots, y_m)| \leq \frac{C_k}{\left(\sum_{k=1}^m \mu B_{y_0 y_k}\right)^m}; \tag{2}$$

(ii) for each $1 \leq k \leq m$,

$$|k(y_0, y_1, \dots, y_k, \dots, y_m) - k(y_0, y_1, \dots, y_k', \dots, y_m)| \leq$$
$$\leq \quad C_k \left[\frac{d(y_k, y_k')}{\max_{0 \leq k \leq m} d(y_0, y_k)}\right]^\delta \frac{1}{[\sum_{k=1}^m \mu B_{y_0 y_k}]^m},$$

whenever $d(y_k, y_k') \leq \max_{0 \leq k \leq m} d(y_0, y_k)/2$.

In this case, write $k \in Ker(m, C_k, \delta)$.

Definition C ([9]). Let $\eta \in (0, 1]$. An m-linear Calderón-Zygmund operator is a continuous operator

$$T : C_b^\eta(X) \times \cdots \times C_b^\eta(X) \mapsto (C_b^\eta(X))'$$

such that for every $f_1, \dots, f_m \in C_b^\eta(X)$ and $x \notin \bigcap_{i=1}^m \; supp\, f_j$,

$$T(f_1, \dots, f_m)(x) = \int\limits_{X^m} k(x, y_1, \dots, y_m) \prod_{i=1}^m f_i(y_i) d\mu(\overrightarrow{y}), \tag{3}$$

where the kernel $k \in Ker(m, C_k, \delta)$ for some C_k, $\delta \in (0,1]$. As an m-linear operator, T has m formal transposes. The j-the transpose T^{*j} of T is defined by

$$\langle T^{*j}(f_1, \dots, f_m), g\rangle = \langle T(f_1, \dots, f_{j-1}, g, f_{j+1}, \dots, f_m), f_j\rangle$$

for all $f_1, \dots, f_m, g \in C_b^\eta(X)$. The kernel k^{*j} is related to one of T by

$$k^{*j}(x, y_1, \dots, y_{j-1}, y_j, y_{j+1}, \dots, y_m) = k(y_j, y_1, \dots, y_{j-1}, x, y_{j+1}, \dots, y_m).$$

The next statement gives the one-weight inequality for the Calderón-Zygmund operator T (see [9]).

Theorem A. *Let $1 < p_j < \infty$ for each $1 \leq i \leq m$, and let $\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j}$. If $\overrightarrow{w} \in A_{\overrightarrow{p}}(X)$, then T can be extended to a bounded operator from $\prod_{j=1}^m L_{w_j}^{p_j}(X)$ to $L_{\nu_{\overrightarrow{w}}}^p(X)$.*

Our result regarding the operator T reeds as follows:

Theorem 2.3. *Let (X, d, μ) be an RD space with metric d, $\mu(X) < \infty$, $1 < p_i < \infty$ for each $1 \leq i \leq m$, $\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j}$ and that $\theta > 0$. Suppose that T is a Calderón-Zygmund operator and that $\overrightarrow{w} \in A_{\overrightarrow{p}}$. Then there is a positive constant C such that for all $f_j \in L_b^\infty$, the following inequality holds:*

$$\|T\overrightarrow{f}\|_{L^{p),\theta}_{\nu_{\overrightarrow{w}}}(X)} \leq C\|\overrightarrow{f}\|_{\prod_{j=1}^m \mathcal{L}^{p_j),\theta}_{w_j}(X)}.$$

Let

$$H\overrightarrow{f}(x) = p.v. \int\limits_{(a,b)^m} \frac{\sum_{j=1}^m (x_j - y_j)}{\left(\sum_{j=1}^m |x_j - y_j|^2\right)^{\frac{m+1}{2}}} f_1(y_1) \dots f_m(y_m) d\overrightarrow{y}$$

be the multilinear Hilbert transform.

As a corollary of Theorem 2.3 we have

Corollary 2.4. *Let $1 < p_i < \infty$ for for each $1 \leq i \leq m$, and let $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$. Suppose that $\theta > 0$. Let w_j be weights on a bounded interval (a, b). Suppose that $\overrightarrow{w} \in A_{\overrightarrow{p}}((a, b))$. Then the following condition*

$$\sup_I \left(\frac{1}{|I|} \int\limits_I \nu_{\overrightarrow{w}}(x) dx\right)^{1/p} \prod_{i=1}^m \left(\frac{1}{|I|} \int\limits_I w_i^{1-p_i'}(t_i) dt_i\right)^{1/p_i'} < \infty,$$

where I is a subinterval of (a, b) and $|I|$ denotes the Lebesgue measure of I, guarantees the one-weight inequality:

$$\|H\overrightarrow{f}\|_{L^{p),\theta}_{\nu_{\overrightarrow{w}}}((a,b))} \leq C\|\overrightarrow{f}\|_{\prod_{j=1}^m \mathcal{L}^{p_j),\theta}_{w_j}((a,b))}, \; f_j \in C_0^\infty((a, b)).$$

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