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ON TWO-WEIGHT ESTIMATES FOR STRONG FRACTIONAL MAXIMAL FUNCTIONS AND POTENTIALS WITH MULTIPLE KERNELS

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In this note necessary and sufficient conditions governing two-weight inequalities for strong fractional maximal functions and potentials with multiple kernels are presented, provided that the weight on the right-hand side is a product of one-dimensional weights. This enables us, for example, to obtain criteria guaranteeing the trace inequalities for the operators mentioned above.

In our opinion one of the challenging problems in the weight theory currently is to solve two-weight problem for integral operators with product kernels. The one-weight problem for the Riesz potentials with multiple kernels has been derived in [13]. Necessary and sufficient conditions guaranteeing the trace inequalities for one-sided potentials with multiple kernels have been established in [16-17] (see also [18] for some two-weight estimates for the Riesz and other potentials with multiple kernels).

Historically the one-weight inequality for the classical Riesz potentials

$$I_\alpha f(x)=\int\limits_{R^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy,\quad 0<\alpha<n,$$

has been derived in [20], while the pioneering result concerning the two-weight problem for I_α has been obtained in [25-26]. In the case $1<p<q<\infty$ two-weight criteria in more transparent form were given in [7], [9] (see also [10], [27] for two-weight criteria for integral transforms with positive kernels). Namely, the following statement holds:

Theorem A. *Let $1<p<q<\infty$. Then I_α is bounded from $L^p_w(R^n)$ into $L^q_v(R^n)$ if and only if*

$$\sup_{\substack{x\in R^n\\ r>0}}\bigg(\int\limits_{B(x,2r)} v\bigg)^{1/q}\bigg(\int\limits_{|x-y|>r}|x-y|^{(\alpha-n)p'}w^{1-p'}(y)dy\bigg)^{1/p'}<\infty$$

and

$$\sup_{\substack{x\in R^n\\ r>0}}\bigg(\int\limits_{B(x,2r)} w^{1-p'}\bigg)^{1/p'}\bigg(\int\limits_{|x-y|>r}|x-y|^{(\alpha-n)q}v(y)dy\bigg)^{1/q}<\infty,$$

where $p'=p/(p-1)$ and $B(x,r)$ is a ball centered at x and of radius r.

The proof of Theorem A is based on the two-weight weak-type criterion for the Riesz potentials given in [24] and on more transparent one established in [6-7] (see also [15]).

In the case $w\equiv 1$, Theorem A (trace inequality) has been obtained in [1].

For $p=q$ a two-weight criterion guaranteeing the trace inequality for I_α is due to [19] (see also [29] for more general case).

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For the solution of the two-weight problem for fractional maximal operators

$$M_\alpha f(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/n}} \int_B |f|, \ \ 0 < \alpha < n,$$

where the supremum is taken over all balls B containing x, we refer to [21-22], [30], [11] (see also [10]).

A two-weight criterion for the strong Hardy–Littlewood maximal functions has been obtained in [22], provided that the weight on the right-hand side satisfies some additional conditions, for instance, belongs to the Muckenhoupt's A_p class in each variable separately, or is product of one-dimensional weights.

A criterion which guarantees the trace inequality for the truncated Riesz potential

$$J_\alpha f(x) = \int_{|y|<2|x|} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \ x \in R^n,$$

has been given in [23] for $p = q$ (for the simple proof in the case $1 < p \leq q < \infty$ see [4], Section 5.1).

Let us introduce the following two-dimensional operators:

$$(M_{\alpha,\beta} f)(x,y) = \sup_{I \times J \ni (x,y)} \frac{1}{|I|^{1-\alpha}|J|^{1-\beta}} \int_I \int_J |f(t,\tau)| dt d\tau;$$

$$(M_\alpha I_\beta f)(x,y) = \sup_{I \ni x} \frac{1}{|I|^{1-\alpha}} \int_I \left| \int_R |y-\tau|^{\beta-1} f(t,\tau) d\tau \right| dt;$$

$$(I_\alpha J_\beta f)(x,y) = \int_R \int_{|\tau|<2|y|} f(t,\tau)|x-t|^{\alpha-1}|y-\tau|^{\beta-1} dt d\tau;$$

$$(I_{\alpha,\beta} f)(x,y) = \int_R \int_R |x-t|^{\alpha-1}|y-\tau|^{\beta-1} f(t,\tau) dt d\tau,$$

where I and J are arbitrary intervals in R.

Let $\mathcal{D}$ be the set of all dyadic intervals in R. By dyadic interval we mean an interval of the form $[2^k n, 2^k(n+1))$, where k and n are integers. The main property of the dyadic intervals is that if $|I'| \leq |I|$, then $I' \subset I$ or $I' \cap I = \emptyset$. Let us denote $\Lambda_k = 2^{-k}Z$ for $k \in Z$. Suppose that $\mathcal{D}^{(k)}$ is the collection of the intervals determined by Λ_k. It is clear that $\mathcal{D} = \bigcup_{k \in Z} \mathcal{D}^{(k)}$. Each $I \in \mathcal{D}^{(k)}$ is the union of 2 nonoverlapping intervals belonging to $\mathcal{D}_{k+1}$ (for details and some properties of the dyadic intervals see, for instance, [8], p. 136).

To formulate the main results of this note we need some definitions of weight classes.

Definition 1. *We say that the weight function ρ satisfies the dyadic reverse doubling condition ($\rho \in RD^{(d)}(R)$) if there exists a constant $d > 1$ such that*

$$d\rho(I') \leq \rho(I),$$

for all $I', I \in \mathcal{D}$, where $I' \subset I$ and $|I| = 2|I'|$.

It is obvious that the constant d in Definition 1 is equal to 2 when $\rho \equiv 1$. It is also easy to see that if a measure μ satisfies the doubling condition $\mu([x-2r, x+2r]) \leq b\mu([x-r, x+r])$ (i.e., $\mu \in DC(R)$), where the constant b is independent of $x \in R$ and $r > 0$, then $\mu \in DC^{(d)}(R)$, i.e., $\mu(I) \leq b_1 \mu(I')$, where $I, I' \in \mathcal{D}$, $I' \subset I$ and $|I'| = |I|/2$. Consequently (see, e.g., [28], p. 21) if $\mu \in DC(R)$, then $\mu \in RD^{(d)}(R)$.

Definition 2. We say that the weight ρ on R satisfies $A_\infty(R)$ condition ($\rho \in A_\infty(R)$) is there exist constants $c, \delta > 0$ such that for all intervals I and measurable sets $E \subset I$

the inequality

$$\frac{\rho(E)}{\rho(I)} \le c\left(\frac{|E|}{|I|}\right)^{\delta}$$

holds, where $\rho(E) = \int\limits_E \rho$. Further, we say that a two-dimensional weight u belongs to the class $A_\infty(R)$ with respect to the first variable uniformly to the second one ($u \in A_\infty^{(x)}(R)$) if the inequality

$$\frac{u_y(E)}{u_y(I)} \le c\left(\frac{|E|}{|I|}\right)^{\delta}$$

holds for all $y \in R$, all intervals $I \subset R$ and measurable sets $E \subset I$, where $u_y(E) = \int\limits_E u(x,y)dx$.

It is known (see [12], [2], [8], Ch. IV) that $\rho \in A_\infty(R)$ if and only if ρ belongs to the Muckenhoupt's class $A_p(R)$ for some $p \ge 1$.

It should be mentioned that some essential properties of Muckenhoupt's A_p classes defined on rectangles has been studied in [14], [5] (see also [3], [8]: Ch. 4).

We begin with the operator $M_\alpha I_\beta$:

Theorem 1. *Let $1 < p < q < \infty$ and let $0 < \alpha, \beta < 1$. Suppose that $w(x,y) = w_1(x)w_2(y)$ with $w_1^{1-p'} \in RD^{(d)}(R)$. Then $M_\alpha I_\beta$ is bounded from $L^p_w(R^2)$ to $L^q_v(R^2)$ if and only if*

$$A_1 := \sup_{\substack{a\in R, r>0\\ I\subset R}} |I|^{\alpha-1}\left(\int\limits_I \int\limits_{|y-a|<r} w^{1-p'}(x,y)dxdy\right)^{1/p'} \times$$
$$\times\left(\int\limits_I \int\limits_{|y-a|>r} \frac{v(x,y)}{|y-a|^{(1-\beta)q}}dxdy\right)^{1/q} < \infty;$$
$$A_2 := \sup_{\substack{a\in R, r>0\\ I\subset R}} |I|^{\alpha-1}\left(\int\limits_I \int\limits_{|y-a|>r} w^{1-p'}(x,y)|y-a|^{(\beta-1)p'}dxdy\right)^{1/p'} \times$$
$$\times\left(\int\limits_I \int\limits_{|x-a|<r} v(x,y)dxdy\right)^{1/q} < \infty, .$$

where I is an arbitrary interval in R.

For the strong fractional maximal functions we have

Theorem 2. *Let $1 < p < q < \infty$ and let $0 < \alpha, \beta < 1$. Suppose that $w(x,y) = w_1(x)w_2(y)$ with $w_1^{1-p'}, w_2^{1-p'} \in RD^{(d)}(R)$. Then $M_{\alpha,\beta}$ is bounded from $L^p_w(R^2)$ to $L^q_v(R^2)$ if and only if*

$$\sup_{I,J\subset R} |I|^{\alpha-1}|J|^{\beta-1}\left(\int\limits_I \int\limits_J v(x,y)dxdy\right)^{1/q} \times$$
$$\times\left(\int\limits_I \int\limits_J w^{1-p'}(x,y)dxdy\right)^{1/p'} < \infty,$$

where the supremum is taken over all intervals I and J in R.

The next statement concerns the Riesz potentials with multiple kernels $I_{\alpha,\beta}$:

Theorem 3. *Let $1 < p < q < \infty$ and let $0 < \alpha, \beta < 1$. Suppose that $w(x,y) = w_1(x)w_2(y)$ with $w_1^{1-p'} \in RD^{(d)}(R)$ and $v \in A_\infty^{(x)}(R)$ uniformly to the second variable. Then $I_{\alpha,\beta}$ is bounded from $L^p_w(R^2)$ to $L^q_v(R^2)$ if and only if* $\max\{A_1, A_2\} < \infty$.

The following statement is also true for the operator $I_\alpha J_\beta$:

Theorem 4. *Let $1 < p < q < \infty$. Suppose that $0 < \alpha < 1$ and $\beta > 1/p$. Then the two-weight inequality*

$$\Big(\int_R\int_R |(I_\alpha J_\beta f)(x,y)|^q v(x,y)dxdy\Big)^{1/q} \le c\Big(\int_R\int_R |f(x,y)|^p u(x)dxdy\Big)^{1/p}$$

holds if and only if

(i)
$$\sup_{\substack{a\in R\\ r>0\\ k\in Z}}\Big(\int_{|x-a|>r}\int_{2^k<|y|<2^{k+1}} \frac{v(x,y)}{|x-a|^{(1-\alpha)q}}dxdy\Big)^{1/q}\times$$
$$\times\Big(\int_{|x-a|<r} u^{1-p'}(x)dx\Big)^{1/p'} 2^{k(\beta-1/p)} < \infty;$$

(ii)
$$\sup_{\substack{a\in R\\ r>0\\ k\in Z}}\Big(\int_{|x-a|<r}\int_{2^k<|y|<2^{k+1}} v(x,y)dxdy\Big)^{1/q}\times$$
$$\times\Big(\int_{|x-a|>r} \frac{u^{1-p'}(x)}{|x-a|^{(\alpha-1)p'}}dx\Big)^{1/p'} 2^{k(\beta-1/p)} < \infty.$$

In the diagonal ($p=q$) case we have

Theorem 5. *Let $1 < p < \infty$, $0 < \alpha < 1/p$, $\beta > 1/p$. Then the operator $J_{\alpha,\beta}$ is bounded from $L^p(R^2)$ to $L^p_v(R^2)$ if and only if there exists a positive constant c such that for a.a. $x \in R$ and all $k \in \mathbb{Z}$ the inequality*

$$I_\alpha[I_\alpha \mathcal{V}_j]^{p'}(x) \le I_\alpha[\mathcal{V}_j](x)$$

holds, where I_α is the one-dimensional potential and

$$\mathcal{V}_j(x) \equiv \int_{2^j<|y|<2^{j+1}} v(x,y)|y|^{\beta p-1}dy.$$

The remaining part of this note is devoted to the trace inequalities.

Corollary 1. *Let $1 < p < q < \infty$. Suppose that $0 < \alpha, \beta < 1/p$. Then the following statements are equivalent:*

(i) $M_\alpha I_\beta$ is bounded from $L^p(R^2)$ to $L^q_v(R^2)$;

(ii) $M_{\alpha,\beta}$ is bounded from $L^p(R^2)$ to $L^q_v(R^2)$;

(iii)
$$B \equiv \sup_{I,J}\Big(\int_I\int_J v(x,y)dxdy\Big)|I|^{q(\alpha-1/p)}|J|^{q(\beta-1/p)} < \infty,$$

where I and J are arbitrary intervals in R.

Corollary 2. *Let $1 < p < q < \infty$ and let $0 < \alpha, \beta < 1/p$. Suppose that the two-dimensional weight $v(x,y)$ belongs to $A^{(x)}_\infty(R)$ uniformly to y, or $v \in A^{(y)}_\infty(R)$ uniformly with respect to x. Then $I_{\alpha,\beta}$ is bounded from $L^p(R^2)$ to $L^q_v(R^2)$ if and only if $B < \infty$.*

Corollary 3. *Let $1 < p < q < \infty$. Suppose that $0 < \alpha < 1$ and $\beta > 1/p$. Then the operator $I_\alpha J_\beta$ is bounded from $L^p(R^2)$ to $L^q_v(R^2)$ if and only if*

$$\sup_{\substack{\in R\\ r>0\\ k\in Z}}\Big(\int_{a-r}^{a+r}\int_{2^k<|y|<2^{k+1}} v(x,y)dxdy\Big)^{1/q} r^{\alpha-1/p}2^{k(\beta-1/p)} < \infty.$$

References

1. D. R. Adams, A trace inequality for generalized potentials. *Studia Math.* **48**(1973), 99–105.
2. R. R. Coifman and C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals. *Studia Math.* **51**(1974), 241–249.
3. E. M. Dynkin and B. P. Osilenker, Weighted estimates for singular integrals and their applications. (Russian) *Mathematical analysis, Vol.* 21, 42–129, *Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow*, 1983.
4. D. E. Edmunds, V. Kokilashvili, and A. Meskhi, Bounded and compact integral operators. *Mathematics and its Applications,* 543. *Kluwer Academic Publishers, Dordrecht*, 2002.
5. R. Fefferman, Multiparameter Fourier analysis. *Beijing lectures in harmonic analysis* (*Beijing*, 1984), 47–130, *Ann. of Math. Stud.,* 112*, Princeton Univ. Press, Princeton, NJ*, 1986.
6. M. Gabidzashvili, Weighted inequalities for anisotropic potentials. (Russian) *Trudy Tbilis. Mat. Inst. Razmadze* **82**(1986), 25–36.
7. M. Gabidzashvili and V. Kokilashvili, Two weight weak type inequalities for fractional type integrals. *Preprint,* 45*, Mathematical Institute Czech Acad. Sci., Prague,* 1989.
8. J. García–Cuerva and J. L. Rubio de Francia, Weighted norm inequalities and related topics. *North-Holland Mathematics Studies,* 116*. Notas de Matematica* [*Mathematical Notes*]*,* 104*. North-Holland Publishing Co., Amsterdam*, 1985.
9. I. Genebashvili, A. Gogatishvili, and V. Kokilashvili, Solution of two-weight problems for integral transforms with positive kernels. *Georgian Math. J.* **3**(1996), No. 4, 319–342.
10. I. Genebashvili, A. Gogatishvili, V. Kokilashvili, and M. Krbec, Weight theory for integral transforms on spaces of homogeneous type. *Pitman Monographs and Surveys in Pure and Applied Mathematics,* 92*, Longman, Harlow*, 1998.
11. A. Gogatishvili and V. Kokilashvili, Criteria of strong type two-weighted inequalities for fractional maximal functions. *Georgian Math. J.* **3**(1996), No. 5, 423–446.
12. R. Hunt, B. Muckenhoupt, and R. Wheeden, Weighted norm inequalities for the conjugate function and Hilbert transform. *Trans. Amer. Math. Soc.* **176**(1973), 227–251.
13. V. M. Kokilashvili, Weighted Lizorkin–Triebel spaces. Singular integrals, multipliers, imbedding theorems. (Russian) *Studies in the theory of differentiable functions of several variables and its applications*, IX, *Trudy Mat. Inst. Steklov* **161**(1983), 125–149; English transl.: *Proc. Steklov Inst. Math.* **3**(1984), 135–162.
14. V. M. Kokilashvili, Bisingular integral operators in weighted spaces. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **101**(1981), No. 2, 289–292.
15. V. Kokilashvili and M. Krbec, Weighted inequalities in Lorentz and Orlicz spaces. *World Scientific Publishing Co., Inc., River Edge, NJ,* 1991.
16. V. Kokilashvili and A. Meskhi, On one-sided potentials with multiple kernels. *Integral Transforms Special Functions* (*accepted for publication*).
17. V. Kokilashvili and A. Meskhi, On a trace inequality for one-sided potentials with multiple kernels. *Frac. Calc. Appl. Anal.* **6**(2003), No. 4, 461–472.
18. V. Kokilashvili and A. Meskhi, Two-weighted criteria for integral transforms with multiple kernels. *In Proceedings of Conference on the occasion of the* 70 *anniversary of Professor Zbigniew Ciesielski Bedlewo, September,* 20–24, 2004, *Banach Center Publications, Institute of Mathematics, Polish Academy of Sciences* (to appear).
19. V. G. Maz'ya and I. E. Verbitsky, Capacitary inequalities for fractional integrals, with applications to partial differential equations and Sobolev multipliers. *Ark. Mat.* **33**(1995), 81–115.

20. B. Muckenhoupt and R. Wheeden, Weighted norm inequalities for fractional integrals. *Trans. Amer. Math. Soc.* **192**(1974), 261–276.
21. E. T. Sawyer, Weighted norm inequalities for fractional maximal operators. *C.M.S. Conference Proceedings C.M.S.-American Math. Soc.* **1**(1981), 283-309.
22. E. T. Sawyer, Two weight norm inequalities for certain maximal and integral operators. *Harmonic analysis* (*Minneapolis, Minn.*, 1981), pp. 102–127, *Lecture Notes in Math.,* 908*, Springer, Berlin–New York*, 1982.
23. E. T. Sawyer, Multipliers of Besov and power-weighted L^2 spaces. *Indiana Univ. Math. J.* **33**(1984), No. 3, 353–366.
24. E. Sawyer, A two–weight weak type inequality for fractional integrals. *Trans. Amer. Math. Soc.* **281**(1984), No. 1, 339–345.
25. E. T. Sawyer, A characterization of two weight norm inequalities for fractional and Poisson integrals. *Trans. Amer. Math. Soc.* **308**(1988), No. 2, 533–545.
26. E. Sawyer and R. L. Wheeden, Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces. *Amer. J. Math.* **114**(1992), No. 4, 813–874.
27. E. T. Sawyer, R. L. Wheeden, and S. Zhao, Weighted norm inequalities for operators of potential type and fractional maximal functions. *Potential Anal.*, **5**(1996), No. 6, 523–580.
28. J.-O. Strömberg and A. Torchinsky, Weighted Hardy spaces. *Lecture Notes in Mathematics* 1381*, Springer-Verlag, Berlin,* 1989.
29. I. E. Verbitsky and R. L. Wheeden, Weighted norm inequalities for integral operators. *Trans. Amer. Math. Soc.* **350**(1998), No. 8, 3371–3391.
30. R. L. Wheeden, A caracterization of some weighted norm inequalities for the fractional maximal functions. *Studia Math.* **107**(1993), No. 3, 257–272.

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