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ON ONE-SIDED OPERATORS IN VARIABLE EXPONENT LEBESGUE SPACES

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This note is devoted to the boundedness of one-sided maximal functions, singular integrals and potentials in Lebesgue spaces with variable exponent.

Let $I := (a, b) \subseteq \mathbf{R}$. We denote

$$p_-(E) = \operatorname*{ess\,inf}_E p, \quad p_+(E) = \operatorname*{ess\,sup}_E p$$

for measurable functions $p : I \to \mathbf{R}$ and measurable sets $E \subseteq I$.

Let $\mathcal{P}_-(I)$ be the class of all measurable functions $p : I \to \mathbf{R}$ such that

(i) $\quad 1 < p_-(I) \le p(t) \le p_+(I) < \infty, \quad t \in I;$ (1)

(ii) there exists a positive constant c such that for almost all $x \in I$ and all r, $0 < r \le \min\big\{1/2, x - a\big\}$, the inequality

$$r^{p_-\big((x-r,x]\big)-p(x)} \le c \tag{2}$$

holds.

Analogously, we define the class $\mathcal{P}_+(I)$ to be the set of all measurable $p : I \to \mathbf{R}$ satisfying (1) and

$$r^{p_-\big([x,x+r)\big)-p(x)} \le c \tag{3}$$

for almost all $x \in I$ and all r, $0 < r \le \min\big\{1/2, b - x\big\}$.

It is easy to see that if p is a non-increasing function on I, then condition (2) is satisfied, while for non-decreasing p condition (3) holds.

Further, let $1 \le p(x) \le p_+(I) < \infty$. For measurable function $f : I \to \mathbf{R}$ we say that $f \in L^{p(x)}(I)$ (or $f \in L^{p(\cdot)}(I)$) if

$$S_{p(\cdot)}(f) = \int_I \big|f(x)\big|^{p(x)} dx < \infty.$$

It is known that $L^{p(x)}(I)$ is a Banach space with the norm

$$\|f\|_{L^{p(x)}(I)} = \inf\Big\{\lambda > 0 : S_{p(\cdot)}\Big(\frac{f}{\lambda}\Big) \le 1\Big\}.$$

For the basic properties of $L^{p(x)}$ spaces see e.g. [9], [13], [6].

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Let $-\infty \le a < b \le +\infty$ and let us introduce the following maximal operators:

$$\big(\mathcal{M}_R f\big)(x) = \sup_{0<h<b-x} \frac{1}{h} \int\limits_{x-h}^{x+h} |f|; \ \ \big(\mathcal{M}_L f\big)(x) = \sup_{0<h<x-a} \frac{1}{h} \int\limits_{x-h}^{x+h} |f|,$$

$$\big(\mathcal{M} f\big)(x) = \sup_{0<h<\min\{x-a,b-x\}} \frac{1}{2h} \int\limits_{x-h}^{x+h} |f|,$$

where $x \in (a,b)$.

Definition 1. Let $I = \mathbf{R}_+$ or $I = \mathbf{R}$. Suppose that p is a constant, $1 < p < \infty$. We say that $w \in A_p^+(I)$ if there exists $c > 0$ such that

$$\bigg(\frac{1}{h}\int\limits_{x-h}^{x} w\bigg)^{1/p}\bigg(\frac{1}{h}\int\limits_{x}^{x+h} w^{1-p'}\bigg)^{1/p'} \le c, \quad h, x > 0, \ \ h < x,$$

for $I = \mathbf{R}_+$ and

$$\bigg(\frac{1}{h}\int\limits_{x-h}^{x} w\bigg)^{1/p}\bigg(\frac{1}{h}\int\limits_{x}^{x+h} w^{1-p'}\bigg)^{1/p'} \le c; \quad x \in \mathbf{R}, \ h > 0,$$

for $I = \mathbf{R}$, where $p' = \frac{p}{p-1}$.

The weight $w \in A_1^+(I)$ if there exists $c > 0$ such that $\mathcal{M}_L w \le c w(x)$ for a.a. $x \in \mathbf{R}$ when $I = \mathbf{R}$ and for a.a $x \in \mathbf{R}_+$ whenever $I = \mathbf{R}_+$.

Analogously is defined the classes $A_p^-(I)$.

The following statement is a one-sided version of Rubio de Francia's extrapolation theorem for variable exponent Lebesgue spaces. For the related statement in the two-sided case see [2].

Theorem 1. *Let $I = \mathbf{R}_+$ or $I = \mathbf{R}$. Let $\mathcal{F}$ be a family of pairs of functions such that for some p_0 and q_0 with $0 < p_0 \le q_0 < \infty$ and for every weight $w \in A_1^+(I)$ (resp. $A_1^-(I)$) the inequality*

$$\bigg(\int\limits_I f(x)^{q_0} w(x)dx\bigg)^{\frac{1}{q_0}} \le c_0 \bigg(\int\limits_I g(x)^{p_0} w(x)^{p_0/q_0} dx\bigg)^{\frac{1}{p_0}}$$

holds for all $(f,g) \in \mathcal{F}$. Given p satisfying (1) and also the condition $p_0 < p_- \le p_+ < \frac{p_0 q_0}{q_0 - p_0}$ define a function q by

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}, \quad x \in I.$$

Let $\tilde{q}(x) = (\frac{q(x)}{q_0})'$. If $\mathcal{M}_L$ $\big($resp. $\mathcal{M}_R\big)$ is bounded in $L^{\tilde{q}(\cdot)}(I)$, then for all $(f,g) \in \mathcal{F}$ such that $f \in L^{q(\cdot)}(I)$ the inequality

$$\|f\|_{L^{q(\cdot)}(I)} \le c\|g\|_{L^{p(\cdot)}(I)}$$

holds.

Now we formulate the statements regarding one-sided maximal functions.

Theorem 2. *Let $I = (0,b)$ be a bounded interval. Then*

(a) *there exists a discontinuous function p on I such that $\mathcal{M}_L$ is bounded in $L^{p(\cdot)}(I)$ but $\mathcal{M}$ is not bounded in $L^{p(\cdot)}(I)$.*

(b) *there exists a discontinuous function* p *on* I *such that* $\mathcal{M}_R$ *is bounded in* $L^{p(\cdot)}(I)$ *but* $\mathcal{M}$ *is not bounded in* $L^{p(\cdot)}(I)$.

Theorem 3. *Let* I *be a bounded interval and let* $p \in \mathcal{P}_-(I)$. *Then* $\mathcal{M}_L$ *is bounded in* $L^{p(\cdot)}(I)$.

Theorem 4. *Let* I *be a bounded interval and let* $p \in \mathcal{P}_+(I)$. *Then* $\mathcal{M}_R$ *is bounded in* $L^{p(\cdot)}(I)$.

Theorem 5. *Let* $I = \mathbf{R}_+$ *and suppose that* $p \in \mathcal{P}_+(I)$. *Assume also that there exists a number* $b > 0$ *such that* $p(x) = p_c \equiv \text{const}$ *when* $x > b$. *Then* $\mathcal{M}_R$ *is bounded in* $L^{p(\cdot)}(\mathbf{R}_+)$.

Theorem 6. *Let* $I = \mathbf{R}_+$. *Suppose that* $p \in P_-(I)$ *and* $p(x) = p_c \equiv \text{const}$ *when* $x > b$ *for some positive* b. *Then* $\mathcal{M}_L$ *is bounded in* $L^{p(\cdot)}(I)$

Theorem 7. *Let* $I = \mathbf{R}$ *and let* $p \in \mathcal{P}_+(I)$. *Suppose that there is a bounded interval* (a, b) *such that* $p(x) = p_c \equiv \text{const}$ *when* $x \notin (a, b)$. *Then* $\mathcal{M}_R$ *is bounded in* $L^{p(x)}(I)$.

Theorem 8. *Let* $I = \mathbf{R}$ *and let* $p \in \mathcal{P}_-(I)$. *Suppose that* $p_c = p(x) \equiv \text{const}$ *when* $x \notin (a, b)$. *Then* $\mathcal{M}_L$ *is bounded in* $L^{p(x)}(I)$.

Now we assume that $I = (0, b)$, where $0 < b \le \infty$ and let

$$\big(\mathcal{R}_{\alpha(\cdot)} f\big)(x) = \int_0^x f(t)(x-t)^{\alpha(x)-1} dt; \quad \big(\mathcal{W}_{\alpha(\cdot)} f\big)(x) = \int_x^b f(t)(t-x)^{\alpha(x)-1} dt;$$

$$\big(\mathcal{I}_{\alpha(\cdot)} f\big)(x) = \int_0^b f(t)|x-t|^{\alpha(x)-1} dt,$$

where $x \in (0, b)$ and $0 < \alpha(x) < 1$.

If $\alpha(x) := \alpha = \text{const}$, then we denote $\mathcal{I}_{\alpha(\cdot)}$, $\mathcal{R}_{\alpha(\cdot)}$, $\mathcal{W}_{\alpha(\cdot)}$ by $\mathcal{I}_\alpha$, $\mathcal{R}_\alpha$ and $\mathcal{W}_\alpha$ respectively.

For one-sided potentials we have:

Theorem 9. *Let* $I = (0, b)$ *be a bounded interval and let* $\alpha \in (0, 1)$ *be a constant. Then*

(a) *there exists a discontinuous function* p *on* I *such that* $\mathcal{R}_\alpha$ *is bounded from* $L^{p(\cdot)}(I)$ *to* $L^{q(\cdot)}(I)$ *and* $\mathcal{I}_\alpha$ *is not bounded from* $L^{p(\cdot)}(I))$ *to* $L^{q(\cdot)}(I)$, *where* $q(x) = \frac{p(x)}{1-\alpha p(x)}$ *and* $0 < \alpha < 1/p_+(I)$.

(b) *there exists a discontinuous function* p *on* I *such that* $\mathcal{W}_\alpha$ *is bounded from* $L^{p(\cdot)}(I)$ *to* $L^{q(\cdot)}(I)$ *and* $\mathcal{I}_\alpha$ *is not bounded from* $L^{p(\cdot)}(I)$ *to* $L^{q(\cdot)}(I)$, *where* $q(x) = \frac{p(x)}{1-\alpha p(x)}$ *and* $0 < \alpha < \frac{1}{p_+(I)}$.

Theorem 10. *Let* $I = \mathbf{R}_+$ *and let* $1 < p_-(I) \le p(x) \le p_+(I) < \infty$. *Suppose that* α *is a constant on* I, $0 < \alpha < \frac{1}{p_+(I)}$, $q(x) = \frac{p(x)}{1-\alpha p(x)}$. *Suppose also that the condition*

$$r^{q(x)-q_+((x-r,x])} \le c, \quad 0 < r \le \min\big\{1/2, x\big\},$$

holds. Assume that p *is a constant outside the interval* $[0, b)$ *for some positive* b. *Then* $\mathcal{W}_\alpha$ *is bounded from* $L^{p(x)}(I)$ *to* $L^{q(x)}(I)$.

Theorem 11. *Let* $I = \mathbf{R}_+$ *and let* $1 < p_-(I) \le p(x) \le p_+(I) < \infty$. *Let* α *be a constant on* I, $0 < \alpha < \frac{1}{p_+(I)}$ *and let* $q(x) = \frac{p(x)}{1-\alpha p(x)}$. *Suppose that* $p(\cdot)$ *is constant outside an interval* $(0, b]$ *and that*

$$r^{q(x)-q_+\left([x,x+r)\right)} \le c, \quad 0 < r < \frac{1}{2}.$$

Then $\mathcal{R}_\alpha$ is bounded from $L^{p(x)}(I)$ to $L^{q(x)}(I)$.

Theorem 12. *Let $I := (0,b)$ be a bounded interval, $p \in \mathcal{P}_+(I)$, $0 < \alpha(x) < \frac{1}{p(x)}$, $q(x) = \frac{p(x)}{1-\alpha(x)p(x)}$. Then $\mathcal{W}_{\alpha(\cdot)}$ is bounded from $L^{p(x)}(I)$ to $L^{q(x)}(I)$.*

Theorem 13. *Let $I = (0,b)$ be a bounded interval and let $p \in \mathcal{P}_-(I)$. Suppose that $0 < \alpha(x) < \frac{1}{p(x)}$, $q(x) = \frac{p(x)}{1-\alpha(x)p(x)}$. Then $\mathcal{R}_{\alpha(\cdot)}$ is bounded from $L^{p(x)}(I)$ to $L^{q(x)}(I)$.*

Theorem 14. *Let $I = \mathbf{R}_+$, α and p be functions defined on $\mathbf{R}_+$ which are constants α_c, p_c respectively outside some interval $(0,a)$ and satisfy the conditions: $p \in \mathcal{P}_-(I)$, $0 < \alpha(x) < \frac{1}{p(x)}$, $q(x) = \frac{p(x)}{1-\alpha(x)p(x)}$, $\alpha_c < \min\left\{\frac{1}{p_c}, \frac{1}{(q_c)'}\right\}$. Then $\mathcal{R}_{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.*

Definition 2. We say that a function k in $L^1_{loc}\big(\mathbf{R} \setminus \{0\}\big)$ is a Calderón-Zygmund kernel if the following properties are satisfied:

(a) there exists a finite constant B_1 such that

$$\Bigg| \int\limits_{\varepsilon<|x|<N} k(x)dx \Bigg| \le B_1$$

for all ε and all N, with $0 < \varepsilon < N$, and furthermore

$$\lim_{\varepsilon \to 0} \int\limits_{\varepsilon<|x|<N} k(x)dx$$

exists;

(b) there exists a positive constant B_2 such that

$$\big|k(x)\big| \le \frac{B_2}{|x|}, \quad x \ne 0;$$

(c) there exists a positive constant B_3 such that for all x and y with $|x| > 2|y| > 0$ the inequality

$$|k(x-y) - k(x)| \le B_3 \frac{|y|}{|x|^2}$$

holds.

It is known (see [10], [1]) that conditions (a)-(c) are sufficient for the boundedness of the operators:

$$T^* f(x) = \sup_{\varepsilon>0} \big|T_\varepsilon f(x)\big|;$$
$$Tf(x) = \lim_{\varepsilon \to 0} T_\varepsilon f(x),$$

where

$$T_\varepsilon f(x) = \int\limits_{|x-y|>\varepsilon} k(x-y) f(y) dy,$$

in $L^p(\mathbf{R})$.

The following example shows that there exists a non-trivial Calderón-Zygmund kernel with support contained in $(0, +\infty)$

Example. The function

$$k(x) = \frac{1}{x} \, \frac{\sin(\ln x)}{\ln x} \chi_{(0,+\infty)}(x)$$

is a Calderón-Zygmund kernel (see e.g. [10], [1] for details).

There exists also a non-trivial Calderón-Zygmund kernel supported in $(-\infty, 0)$.

Theorem 15. *Let $I = \mathbf{R}$ and let p satisfy (1). Assume that $r^{p(x)-p_+((x-r,x])} \leq c$ for $x \in I$ and $0 < r < 1/2$. Suppose that p is a constant outside some bounded interval (a, b). Then T^*, with kernel k supported in $(-\infty, 0)$, is bounded in $L^{p(\cdot)}(I)$.*

An analogous result can be formulated for T^* with kernel supported in $(0, +\infty)$. Namely we have

Theorem 16. *Let $I = \mathbf{R}$ and let p satisfy (1). Assume that $r^{p(x)-p_+((x,x+r))} \leq c$ for $x \in I$ and $0 < r < 1/2$. Suppose that p is a constant outside some bounded interval (a, b). Then T^*, with kernel k supported in $(0, +\infty)$ is bounded in $L^{p(\cdot)}(I)$.*

Finally we mention that the boundedness of classical operators of various type in $L^{p(x)}$ spaces was established in [11], [3]–[5], [2]. For weighted $L^{p(x)}$ spaces with power-type weights we refer to [7]-[8], [12].

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