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ON THE MAXIMAL AND FOURIER OPERATORS IN WEIGHTED LEBESGUE SPACES WITH VARIABLE EXPONENT

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Let J be a subinterval of $\mathbf{R}$. Suppose that p is measurable function on J with the condition

$$1 < p_-(J) \le p(x) \le p_+(J) < \infty,$$

where

$$p_-(J) := \inf_J p; \quad p_+(J) := \sup_J p.$$

Suppose also that ρ is an almost everywhere positive locally integrable function on J, i.e. ρ is a weight. We say that a measurable function $f : J \to \mathbf{R}$, belongs to $L^{p(\cdot)}_\rho(J)$ (or $L^{p(x)}_\rho(J)$) if

$$S_{p,\rho}(f) = \int_J \big|f(x)\rho(x)\big|^{p(x)} dx < \infty.$$

It is known that $L^{p(x)}_\rho(J)$ is a Banach space with the norm

$$\|f\|_{L^{p(x)}_\rho(J)} = \inf\big\{\lambda > 0 : S_{p,\rho}\big(f/\lambda\big) \le 1\big\}.$$

If $p = \text{const}$, then $L^{p(\cdot)}_\rho(J)$ coincides with the classical Lebesgue space with the weight ρ. Further, if $\rho \equiv 1$, then we use the symbol $L^{p(\cdot)}(J)$ for $L^{p(\cdot)}_\rho(J)$.

For some basic properties of $L^{p(\cdot)}$ spaces we refer, e.g., to [4-6].

We say that $p : J \to \mathbf{R}$ satisfies the Dini-Lipschitz (log-Hölder continuity) condition on J ($p \in DL(J)$) if there exists a positive constant A such that

$$|p(x) - p(y)| \le \frac{A}{-\ln|x-y|}; \quad x, y \in J; \quad |x - y| \le 1/2.$$

A weight function ρ satisfies the doubling condition on J ($\rho \in DC(J)$) if there exists a positive constant b such that

$$\int_{I(x,2r)} \rho \le b \int_{I(x,r)} \rho$$

for all $x \in J$ and $r > 0$, where $I(x, r) := (x - r, x + r)$.

Let $T := [-\pi, \pi]$ and let

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k coskx + b_k sinkx)$$

be the Fourier series of the function $f \in L^1(T)$.

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The Cesàro mean of order $\alpha > 0$, σ_n^α, is defined as

$$\sigma_n^\alpha(f,x) = \frac{1}{\binom{n+\alpha}{n}} \sum_{k=0}^{n} \binom{n-k+\alpha}{n-k} A_k(x), \qquad \alpha > 0$$

where

$$A_0 = \frac{a_0}{2} \quad \text{and} \quad A_k(x) = a_k coskx + b_k sinkx.$$

Let also

$$u_r(f,x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} A_k(x) r^k$$

be the Abel-Poisson means of function $f(x)$.

The following statements are true:

Theorem 1. *Let $1 < p_-(T) \le p(x) \le p_+(T) < \infty$ and let $p \in DL(T)$. If $(w(\cdot))^{-p'(\cdot)}$ satisfies the doubling condition on T, then the following conditions are equivalent:*

i) $\| \sup_n \sigma_n^\alpha(f,\cdot)\|_{L_v^{p(\cdot)}(T)} \le c\|f\|_{L_w^{p(\cdot)}(T)}$

ii) $\| \sup_{0<r<1} u_r(f,\cdot)\|_{L_v^{p(\cdot)}(T)} \le c\|f\|_{L_w^{p(\cdot)}(T)}$

iii) there exists a constant $c > 0$ such that

$$\int_I (v(x))^{p(x)} \Big(M\big(w^{-p'(\cdot)}(\cdot)\chi_I(\cdot)\big)(x)\Big)^{p(x)} dx \le c \int_I w^{-p'(x)} dx \tag{1}$$

for an arbitrary interval $I \subset T$.

Theorem 2. *Let $1 < p_-(T) \le p(x) \le p_+(T) < \infty$ and let $p \in DL(T)$. Suppose that $(w(\cdot))^{-p'(\cdot)} \in DC(T)$ and the condition (1) holds with $v = w$. Then for arbitrary $f \in L_w^p(T)$ we have*

$$\lim_{n\to\infty} \|\sigma_n^\alpha(f,\cdot) - f(\cdot)\|_{L_w^{p(\cdot)}(T)} = 0$$

and

$$\lim_{r\to 1} \|u_r(f,\cdot) - f(\cdot)\|_{L_w^p(T)} = 0.$$

Two-weight estimates for the Cesàro means enable us to obtain the extended Bernstein inequality for the derivative of trigonometric polynomial and its conjugate in two-weighted setting.

Theorem 3. *Let $1 < p_-(T) \le p(x) \le p_+(T) < \infty$ and let $p \in DL(T)$. Suppose that $(w(\cdot))^{-p'(\cdot)} \in DC(T)$ and condition (1) is satisfied. Then for an arbitrary trigonometric polynomial $t_n(x)$ and its conjugate $\widetilde{t_n}(x)$ we have*

$$\|t_n' v\|_{L^{p(\cdot)}(T)} \le cn\|t_n w\|_{L^{p(\cdot)}(T)}$$

and

$$\|\widetilde{t_n}' v\|_{L^{p(\cdot)}(T)} \le cn\|t_n w\|_{L^{p(\cdot)}(T)}.$$

For special pairs (v,w) the above mentioned results were obtained in [1]. For the constant p we refer to [2].

Now we discuss the Hardy-Littlewood maximal operators.

Let $M_{\mathbf{R}_+}$ and $M_{\mathbf{R}}$ be maximal operators given by

$$\big(M_{\mathbf{R}_+}f\big)(x) = \sup_{r>0} \frac{1}{2r} \int\limits_{(x-r,x+r)\cap R_+} |f(t)|dt, \quad x \in \mathbf{R}_+,$$

$$\big(M_{\mathbf{R}}f\big)(x) = \sup_{r>0} \frac{1}{2r} \int\limits_{x-r}^{x+r} |f(t)|dt, \quad x \in \mathbf{R}$$

respectively.

We have the following statements:

Theorem 4. *Let $1 < p_-(\mathbf{R}_+) \le p(x) \le p_+(\mathbf{R}_+) < \infty$ and let $p \in DL(\mathbf{R}_+)$. Suppose that there is a bounded interval $[0,a]$ such that $(w(\cdot))^{-p'(\cdot)} \in DC([0,a])$ and p is constant outside $[0,a]$. Then $M_{\mathbf{R}_+}$ is bounded from $L_w^{p(\cdot)}(\mathbf{R}_+)$ to $L_v^{p(\cdot)}(\mathbf{R}_+)$ if and only if there is a positive constant c such that for all bounded subintervals I of $\mathbf{R}_+$,*

$$\|M_{\mathbf{R}_+}(w^{-p'(\cdot)}\chi_I)\|_{L_v^{p(\cdot)}(I)} \le c\|w^{1-p'(\cdot)}(\cdot)\|_{L^{p(\cdot)}(I)} < \infty.$$

Theorem 5. *Let $1 < p_-(\mathbf{R}) \le p(x) \le p_+(\mathbf{R}) < \infty$ and let $p \in DL(\mathbf{R})$. Suppose that there is positive number a such that $(w(\cdot))^{-p'(\cdot)} \in DC([-a,a])$ and $p := p_c =$ const outside $[-a,a]$. Then for the boundedness of $M_{\mathbf{R}}$ from $L_w^{p(\cdot)}(\mathbf{R})$ to $L_v^{p(\cdot)}(\mathbf{R})$ it is necessary and sufficent that there exists a positive constant c such that for all bounded subintervals I of $\mathbf{R}$,*

$$\|M_{\mathbf{R}}(w^{-p'(\cdot)}\chi_I)\|_{L_v^{p(\cdot)}(\mathbf{R})} \le c\|w^{1-p'(\cdot)}\|_{L^{p(\cdot)}(I)} < \infty.$$

Finally we notice that two-weight Sawyer-type criteria for maximal functions in Lebesgue spaces defined on finite intervals were announced in [3].

In the sequel by V we denote the class of all measurable functions $f : R^1 \longrightarrow R^1$ for which

$$\int\limits_{-\infty}^{\infty} \frac{f(x)}{(1+|x|)^2}dx < \infty.$$

Theorem 6. *Let the conditions of Theorem 5 hold with $v = w$. Then for arbitrary $f \in L_w^p(R^1) \bigcap V$ we have*

$$\lim_{t\longrightarrow 0} \|f - U_t(\cdot, f)\|_{L_w^{p(\cdot)}} = 0$$

where

$$U_t(x,f) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{f(y)t}{t^2 + (x-y)^2}dy.$$

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