

Boundedness of Maximal and Singular Operators in Morrey Spaces with Variable Exponent

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Abstract. In this paper the boundedness of Hardy-Littlewood maximal and singular operators in variable exponent Morrey spaces $M_{q(\cdot)}^{p(\cdot)}(X)$ defined on spaces of homogeneous type is established provided that p and q satisfy Dini-Lipschitz (log-Hölder continuity) condition.

Key words: Maximal functions, singular integrals, spaces of homogeneous type, Morrey spaces, function spaces with variable exponent, boundedness of operators.
Mathematics Subject Classification 2000: 42B20, 42B25, 47B38.

INTRODUCTION

Let (X, ρ, μ) be a space of homogeneous type with $L := \text{diam}(X) < \infty$. The paper is devoted to the boundedness of the operators

$$(Mf)(x) = \sup_{\substack{x \in X \\ 0 < r < L}} (\mu B(x, r))^{-1} \int_{B(x, r)} |f(y)| d\mu(y), \quad x \in X,$$

$$(Kf)(x) = \lim_{\varepsilon \rightarrow 0} \int_{X \setminus B(x, \varepsilon)} k(x, y) f(y) d\mu(y), \quad x \in X$$

in Morrey spaces with variable exponent defined on X , where k is the Calderón-Zygmund kernel on X .

In the paper [2] some properties of variable Morrey spaces over a bounded open set $\Omega \subset \mathbf{R}^n$ and the boundedness of maximal and Riesz potential operators in these spaces were studied.

For mapping properties of maximal functions and singular integrals on Lebesgue spaces with variable exponent we refer to [9], [10], [7], [6], [19]-[20], [26], [13] (see also [29], [17] and references therein).

We notice that the main results of this paper were announced in [18].

The paper is organized as follows. In Section 1 we formulate and prove some properties of variable exponent spaces. Section 2 is devoted to the boundedness of M in variable exponent Morrey spaces, while in Section 3 we study the same problem for the operator K . Section 4 deals with applications of the derived results to singular integrals on fractal sets.

Constants (often different constants in the same series of inequalities) will be denoted by c or C .

1. PRELIMINARIES

A space of homogeneous type (*SHT*) (X, d, μ) is a topological space X with a complete measure μ such that the space of compactly supported continuous functions is dense in $L^1_\mu(X)$ and there is a non-negative function (quasimetric) $\rho : X \times X \rightarrow \mathbf{R}_+$ which satisfies the following conditions:

- (i) $\rho(x, x) = 0$ for all $x \in X$.
- (ii) $\rho(x, y) > 0$ for all $x \neq y, x, y \in X$.
- (iii) There exists a positive constant a_0 such that $\rho(x, y) \leq a_0 \rho(y, x)$ for every $x, y \in X$.
- (iv) There exists a constant a_1 such that $\rho(x, y) \leq a_1(\rho(x, z) + \rho(z, y))$ for every $x, y, z \in X$.
- (v) For every neighbourhood V of the point $x \in X$ there exists $r > 0$ such that the ball $B(x, r) = \{y \in X : \rho(x, y) < r\}$ is contained in V .
- (vi) Balls $B(x, r)$ are measurable for every $x \in X$ and for arbitrary $r > 0$.
- (vii) There exists a constant $b > 0$ such that

$$\mu B(x, 2r) \leq b\mu(B(x, r)) < \infty \quad (1.1)$$

for every $x \in X$ and $r, 0 < r < \infty$.

It is known (see [24], [14], p.2) that there exists another quasimetric ρ' , equivalent to ρ , for which every ball is open.

Throughout the paper we assume that $L := \text{diam}(X) < \infty$; $\mu\{x\} = 0$ for all $x \in X$, and

$$\mu(B(x_0, R) \setminus B(x_0, r)) > 0 \quad (1.2)$$

for all $x_0 \in X$ and r, R with $0 < r < R < L/2$. For the definition and some properties of *SHT* see, e.g., [24], [5], [14], Ch.1, [12], Ch. 1.

Notice that conditions (1.1) and $L < \infty$ imply $\mu X < \infty$.

If b is the smallest constant for which the measure μ satisfies (1.1), then the number $Q = \log_2 b$ is called the doubling order of μ . Iterating (1.1) we find that

$$\frac{\mu(B(x, R))}{\mu(B(y, r))} \leq C_\mu \left(\frac{R}{r}\right)^Q \quad (1.1')$$

for all balls $B(x, R)$ and $B(y, r)$ with $B(y, r) \subset B(x, R)$. For example, in the case of \mathbf{R}^n with the Lebesgue measure $Q = n$.

Let p and q be measurable functions on X such that $1 < p_- \leq p(x) \leq p_+ < \infty$, $1 < q_- \leq q(x) \leq q_+ < \infty$, where

$$p_- = \inf_X p, \quad p_+ := \sup_X p, \quad q_- = \inf_X q, \quad q_+ := \sup_X q.$$

We shall use the following notation:

$$p_-(E) = \inf_E p, \quad p_+(E) := \sup_E p,$$

where E is a μ -measurable subset of X .

The Lebesgue space with variable exponent $L^{p(\cdot)}(X)$ (or $L^{p(x)}(X)$) is the class of all measurable μ - functions f on X for which

$$S_p(f) := \int_X |f(x)|^{p(x)} dx < \infty.$$

The norm in $L^{p(\cdot)}(X)$ is defined as follows

$$\|f\|_{L^{p(\cdot)}(X)} = \inf \left\{ \lambda > 0 : S_p(f/\lambda) \leq 1 \right\}.$$

It is known that $L^{p(\cdot)}(X)$ is a Banach space (see [23], [16]). For other properties of $L^{p(\cdot)}$ spaces we refer to [31], [23], [28], [16].

Definition 1.1. *We say that p satisfies the Dini-Lipschitz condition on X ($p \in DL(X)$) if there exists a positive constant A such that*

$$|p(x) - p(y)| \leq \frac{A}{-\log(\rho(x, y))}$$

for all $x, y \in X$ with $\rho(x, y) \leq 1/2$.

Remark 1.1. It is easy to check that if $1 < p_-(X) \leq p_+(X) < \infty$ and $p \in DL(X)$, then $1/p(\cdot)$ and $p'(\cdot)$ belong to $DL(X)$, where $p'(\cdot) := p(\cdot)/(p(\cdot) - 1)$.

The next lemma was proved in [16] for metric measure spaces.

Lemma 1.1. *Let $p \in DL(X)$. Then there exists a positive constant c such that for all balls $B \subset X$ the inequality*

$$\mu(B)^{p_-(B) - p_+(B)} \leq c$$

holds.

Proof. First notice that (1.1') implies that μ is lower Ahlfors Q -regular, i.e. there is a positive constant c independent of y and r such that

$$\mu B(y, r) \geq cr^Q$$

for all $y \in X$ and $0 < r < L$. Further, suppose that $c_0 = \min \left\{ \frac{1}{2}, a_1(a_0 + 1) \right\}$, where a_0 and a_1 are constants from the definition of the quasimetric ρ . If $0 < r < c_0$, then for $B := B(y, r)$, we have

$$(\mu B)^{p_-(B) - p_+(B)} \leq cr^{p_-(B) - p_+(B)} \leq cr^{\frac{QA}{\log(c_0 r)}} \leq c.$$

□

Remark 1.2. It is easy to check that if $p \in DL(X)$, then there is a positive constant c such that

$$(\mu B)^{p(x)} \leq c(\mu B)^{p(y)}$$

for all balls B and all $x, y \in B$.

In [2] variable Morrey spaces over a bounded open set $\Omega \subset \mathbf{R}^n$ were introduced.

Definition 1.2. Let $1 < q_- \leq q(\cdot) \leq p(\cdot) \leq p_+ < \infty$. We say that a measurable locally integrable function f on X belongs to the class $M_{q(\cdot)}^{p(\cdot)}(X)$ if

$$\|f\|_{M_{q(\cdot)}^{p(\cdot)}(X)} = \sup_{\substack{x \in X \\ 0 < r < L}} (\mu B(x, r))^{1/p(x)-1/q(x)} \|f\|_{L^{q(\cdot)}(B(x, r))}.$$

It is easy to see that if $p = q$, then $M_{q(\cdot)}^{p(\cdot)}(X) = L^{p(\cdot)}(X)$. When $p(x) \equiv p$ and $q(x) \equiv q$ are constants, the space $M_{q(\cdot)}^{p(\cdot)}$ coincides with the classical Morrey space M_q^p (see [25], [27], [15], [1] for the definition and some properties of M_q^p). For the boundedness of maximal and singular integrals in the spaces M_q^p we refer to [4], [11], [3], [30].

Taking into account Remark 1.2 we have the next statement.

Proposition 1.1. If $p, q \in DL(X)$, then there are positive constants c_1 and c_2 such that

$$c_1 \|f\|_{\tilde{M}_{q(\cdot)}^{p(\cdot)}(X)} \leq \|f\|_{M_{q(\cdot)}^{p(\cdot)}(X)} \leq c_2 \|f\|_{\tilde{M}_{q(\cdot)}^{p(\cdot)}(X)}$$

for all f , where

$$\|f\|_{\tilde{M}_{q(\cdot)}^{p(\cdot)}(X)} := \sup_{\substack{x \in X \\ 0 < r < L}} \|(\mu(B(x, r)))^{1/p(\cdot)-1/q(\cdot)} f(\cdot)\|_{L^{q(\cdot)}(B(x, r))}.$$

This follows easily from Lemma 1.1 and Remark 1.2.

For the next statement we refer to [23], [28].

Proposition 1.2. Let f be a measurable function on X and let E be a measurable subset of X . Then the following inequalities hold:

$$\begin{aligned} \|f\|_{L^{p(\cdot)}(E)}^{p_+(E)} &\leq S_p(f\chi_E) \leq \|f\|_{L^{p(\cdot)}(E)}^{p_-(E)}, & \|f\|_{L^{p(\cdot)}(E)} &\leq 1; \\ \|f\|_{L^{p(\cdot)}(E)}^{p_-(E)} &\leq S_p(f\chi_E) \leq \|f\|_{L^{p(\cdot)}(E)}^{p_+(E)}, & \|f\|_{L^{p(\cdot)}(E)} &\geq 1. \end{aligned}$$

Hölder's inequality in variable exponent Lebesgue spaces has the following form (see e.g. [23], [28]):

$$\int_E fgd\mu \leq \left(1/p_-(E) + 1/(p')_-(E)\right) \|f\|_{L^{p(\cdot)}(E)} \|g\|_{L^{p'(\cdot)}(E)}. \quad (1.3)$$

The following statement holds (for Euclidean spaces see [2], Lemma 7).

Proposition 1.3. Let $1 < (q_1)_- \leq q_1(\cdot) \leq q_2(\cdot) \leq (q_2)_+ < \infty$ and let $q_1, q_2 \in DL(X)$. Then

$$M_{q_2(\cdot)}^{p(\cdot)}(X) \hookrightarrow M_{q_1(\cdot)}^{p(\cdot)}(X).$$

Proof. By Lemma 1.1, Remark 1.2 and Proposition 1.2 it is enough to see that there is a positive constant c independent of f , x and r such that

$$\|(\mu B(x, r))^{1/q_2(\cdot)-1/q_1(\cdot)} f(\cdot)\|_{L^{q_1(\cdot)}(B(x, r))} \leq c \|f\|_{L^{q_2(\cdot)}(B(x, r))}.$$

Indeed, suppose that $\|f\|_{L^{q_2(\cdot)}(B(x,r))} \leq 1$; using Hölder's inequality and Remark 1.1 we have

$$S_{q_1}\left((\mu B(x,r))^{1/q_2(\cdot)-1/q_1(\cdot)} f(\cdot)\right) = \int_{B(x,r)} \left((\mu B(x,r))^{1/q_2(y)-1/q_1(y)} |f(y)| \right)^{q_1(y)} dy \leq c(\mu B(x,r))^{q_1(x)/q_2(x)-1} \| |f|^{q_1(\cdot)} \|_{L^{q_2(\cdot)/q_1(\cdot)}(B(x,r))} \| \chi_{B(x,r)}(\cdot) \|_{L^{(q_2(\cdot)/q_1(\cdot))'}(X)} \leq c.$$

□

Lemma 1.2. *Let β be a measurable function on X satisfying $\beta(x) < -1$ for all X . Suppose that r is a small positive number. Then there exists a positive constant c independent of r and x such that*

$$A(x,r) := \int_{X \setminus B(x,r)} (\mu(B_{xy}))^{\beta(x)} d\mu(y) \leq c \frac{\beta(x)+1}{\beta(x)} (\mu(B(x,r)))^{\beta(x)+1},$$

where

$$B_{xy} := \mu(B(x, \rho(x,y))).$$

Proof. We have

$$A(x,r) = \int_0^\infty \mu((X \setminus B(x,r)) \cap \{y \in X : (\mu B_{xy})^{\beta(x)} > \lambda\}) d\lambda = \int_0^{(\mu B(x,r))^{\beta(x)}} + \int_{(\mu B(x,r))^{\beta(x)}}^\infty := A_1(x,r) + A_2(x,r).$$

First observe that $A_2(x,r) = 0$ for all $x \in X$ and small r . Indeed, let $x \in X$ and $\lambda > (\mu B(x,r))^{\beta(x)}$. We denote

$$E_\lambda(x) := \{y \in X : (\mu B_{xy})^{\beta(x)} > \lambda\}.$$

Suppose that $y \in (X \setminus B(x,r)) \cap E_\lambda(x)$. Then we have

$$\mu B_{xy} < \lambda^{1/\beta(x)}.$$

On the other hand, if $\lambda > (\mu B_{xy})^{\beta(x)}$, then $\mu B_{xy} < \lambda^{1/\beta(x)} < \mu B(x,r)$.

When $y \in X \setminus B(x,r)$ we have $\rho(x,y) \geq r$ and therefore, $\mu B_{xy} \geq \mu B(x,r)$. Consequently, $(X \setminus B(x,r)) \cap E_\lambda(x) = \emptyset$ if $\lambda > (\mu B(x,r))^{\beta(x)}$, which implies that $A_2(x,r) = 0$.

Now we estimate $A_1(x,r)$. First we show that

$$\mu E_\lambda \leq b^2 \lambda^{1/\beta(x)}, \tag{1.4}$$

where b is the constant from the doubling condition for μ . If $\mu E_\lambda = 0$, then (1.4) is obvious. If $\mu E_\lambda \neq 0$, then $0 < t_0 < \infty$, where

$$t_0 = \sup\{s \in (0, L) : \mu B(x, s) < \lambda^{1/\beta(x)}\}.$$

Indeed, since $L < \infty$, we have $t_0 < \infty$. Assume now that $t_0 = 0$. Then $E_\lambda = \{x\}$; otherwise there exists $y, y \in E_\lambda(x)$, such that $\rho(x,y) > 0$ and $\mu B_{xy} < \lambda^{1/\beta(x)}$ which contradicts the assumption $t_0 = 0$. Hence we conclude that $0 < t_0 < \infty$.

Further, let $z \in E_\lambda(x)$. Then $\mu B_{xz} < \lambda^{1/\beta(x)}$. Consequently, $\rho(x, z) \leq t_0$. From this we have $z \in B(x, 2t_0)$, which due to the doubling condition yields

$$\mu E_\lambda(x) \leq \mu B(x, 2t_0) \leq b^2 \mu B(x, t_0/2) \leq b^2 \lambda^{1/\beta(x)}.$$

This implies (1.4). Since $\beta(x) < -1$, we have

$$A_1(x, r) \leq b^2 \int_0^{(\mu B(x, r))^{\beta(x)}} \mu E_\lambda(x) d\lambda = \frac{b^2 \beta(x)}{1 + \beta(x)} (\mu B(x, r))^{\beta(x)+1}.$$

□

Lemma 1.3 ([5], [33]). *Conditions (1.1) and (1.2) imply the reverse doubling condition (RDC) i.e. there exist constants $\alpha, \beta, 0 < \alpha, \beta < 1$ such that*

$$\mu B(x, \alpha r) \leq \beta \mu B(x, r) \quad (1.5)$$

for all $x \in X$ and all sufficiently small positive r .

2. MAXIMAL FUNCTIONS

In this section we establish the boundedness of the operator M in $M_{q(\cdot)}^{p(\cdot)}(X)$.

The following statement is well-known (see [9] for Euclidean spaces and [16] for metric measure spaces).

Theorem A. *Let $1 < q_- \leq q(x) \leq q_+ < \infty$. Suppose that $q \in DL(X)$. Then M is bounded in $L^{q(x)}(X)$.*

Our aim in this section is to prove the next statement.

Theorem 2.1. *Let $1 < q_- \leq q(\cdot) \leq p(\cdot) \leq p_+ < \infty$ and let $p, q \in DL(X)$. Then M is bounded in $M_{q(\cdot)}^{p(\cdot)}(X)$.*

Proof. Let r be a small positive number. Represent f as follows: $f = f_1 + f_2$, where $f_1 = f \chi_{B(x, \bar{a}r)}$, $f_2 = f - f_1$, where $\bar{a} = a_1(a_1(a_0 + 1) + 1)$. We have

$$\begin{aligned} & (\mu B(x, r))^{1/p(x)-1/q(x)} \|Mf\|_{L^{q(\cdot)}(B(x, r))} \leq \\ & (\mu B(x, r))^{1/p(x)-1/q(x)} \|Mf_1\|_{L^{q(\cdot)}(B(x, r))} + \\ & (\mu B(x, r))^{1/p(x)-1/q(x)} \|Mf_2\|_{L^{q(\cdot)}(B(x, r))} := I_1 + I_2. \end{aligned}$$

Taking into account the condition $q \in DL(X)$, Theorem A and the doubling condition we have

$$I_1 \leq c (\mu B(x, r))^{1/p(x)-1/q(x)} \|\chi_{B(x, \bar{a}r)} f\|_{L^{q(\cdot)}(X)} \leq c \|f\|_{M_{q(\cdot)}^{p(\cdot)}(X)}.$$

Now observe that for $y \in B(x, r)$,

$$Mf_2(y) \leq \sup_{B \supset B(x, r)} \frac{1}{\mu B} \int_B |f| d\mu,$$

because $B(x, r) \subset B(y, a_1(a_0 + 1)r) \subset B(x, \bar{a}r)$.

Hence by Hölder's inequality, Lemma 1.1 and Proposition 1.2 we have

$$I_2 \leq c (\mu B(x, r))^{1/p(x)-1/q(x)} \left[\sup_{B \supset B(x, r)} \frac{1}{\mu B} \int_B |f| d\mu \right] \|\chi_{B(x, r)}(\cdot)\|_{L^{q(\cdot)}(X)} \leq$$

$$\begin{aligned}
& c(\mu B(x, r))^{1/p(x)} \sup_{\substack{B \\ B \supset \bar{B}(x, r)}} (\mu B)^{-1} \|f\|_{L^{q(\cdot)}(B)} \|\chi_B\|_{L^{q'(\cdot)}(X)} \leq \\
& c(\mu B(x, r))^{1/p(x)} \sup_{\substack{B \\ B \supset \bar{B}(x, r)}} (\mu B)^{-1} (\mu B)^{1/(q_-(B))'} \|f\|_{L^{q(\cdot)}(B)} \leq \\
& c \sup_{\substack{B \\ B \supset \bar{B}(x, r)}} (\mu B)^{1/p(x)-1/q(x)} \|f\|_{L^{q(\cdot)}(B)} \leq \\
& c \sup_B (\mu B)^{1/p(x_0)-1/q(x_0)} \|f\|_{L^{q(\cdot)}(B)} \leq c \|f\|_{M_{q(\cdot)}^{p(\cdot)}(X)},
\end{aligned}$$

where x_0 is the center of B .

Finally we conclude that M is bounded in $M_{q(\cdot)}^{p(\cdot)}(X)$. \square

3. SINGULAR INTEGRALS

Let $k : X \times X \setminus \{(x, x) : x \in X\} \rightarrow \mathbf{R}$ be a measurable function satisfying the conditions:

$$|k(x, y)| \leq \frac{c}{\mu B(x, \rho(x, y))}, \quad x, y \in X, \quad x \neq y;$$

$$|k(x_1, y) - k(x_2, y)| + |k(y, x_1) - k(y, x_2)| \leq c\omega\left(\frac{\rho(x_2, x_1)}{\rho(x_2, y)}\right) \frac{1}{\mu B(x_2, \rho(x_2, y))}$$

for all x_1, x_2 and y with $\rho(x_2, y) > \rho(x, x_2)$, where ω is a positive, non-decreasing function on $(0, \infty)$ satisfying Δ_2 condition ($\omega(2t) \leq c\omega(t)$, $t > 0$) and the Dini condition $\int_0^1 \omega(t)/tdt < \infty$.

We also assume that for some p_0 , $1 < p_0 < \infty$, and all $f \in L^{p_0}(X)$ the limit

$$(Kf)(x) = p.v. \int_X k(x, y)f(y)d\mu(y)$$

exists almost everywhere on X and that K is bounded in $L^{p_0}(X)$.

The following statement is known (see [21], [22]).

Theorem B. *Let $p \in DL(X)$. Then K is bounded in $L^{p(\cdot)}(X)$.*

Theorem 3.1. *Let $1 \leq q_- \leq q(\cdot) \leq p(\cdot) \leq p_+ < \infty$. Suppose that $q, p \in DL(X)$. Then K is bounded in $M_{q(\cdot)}^{p(\cdot)}(X)$.*

Proof. We take small $r > 0$ and represent f as follows: $f_1 + f_2$, where $f_1 = f\chi_{B(x, 2a_1r)}$, $f_2 = f - f_1$. First observe that if $y \in B(x, r)$ and $z \in X \setminus B(x, 2a_1r)$, then $\mu B(x, \rho(x, z)) \leq c\mu B(y, \rho(y, z))$. Indeed,

$$\rho(x, z) \leq a_1\rho(x, y) + a_1\rho(y, z) \leq a_1r + a_1\rho(y, z) \leq \rho(x, z)/2 + a_1\rho(y, z).$$

Hence $\rho(x, z) \leq 2a_1\rho(y, z)$. This implies

$$\mu B(x, \rho(x, z)) \leq c\mu B(x, \rho(y, z)).$$

Further, if $t \in B(x, \rho(y, z))$, then

$$\rho(y, t) \leq a_1\rho(y, z) + a_1\rho(z, t) \leq a_1(\rho(y, z) + a_1\rho(z, x) + a_1\rho(x, t)) \leq$$

$$a_1(\rho(y, z) + 2a_1^2 a_0 \rho(y, z) + a_1 \rho(y, z)) = 2a_1^2(1 + a_0 a_1) \rho(y, z).$$

Thus, $\mu B(x, \rho(y, z)) \leq c \mu B(y, \rho(y, z))$. Finally,

$$\mu B(x, \rho(x, z)) \leq c_1 \mu B(x, \rho(y, z)) \leq c_2 \mu B(y, \rho(y, z)).$$

Let us take an integer m so that $\alpha^m \text{diam}(X)$ is sufficiently small, where α is the constant from (1.5). Now by the reverse doubling condition and the latter inequality, for $y \in B(x, r)$, we have

$$\begin{aligned} |Kf_2(y)| &\leq c \int_{X \setminus B(x, 2a_1 r)} |f(z)| \mu B(x, \rho(x, z))^{-1} d\mu(z) \leq \\ &\int_{X \setminus B(x, 2a_1 r)} |f(z)| \left(\int_{B(x, \alpha^m \rho(x, z)) \setminus B(x, \alpha^{m-1} \rho(x, z))} (\mu B(x, \rho(x, t)))^{-2} d\mu(t) \right) d\mu(z) \leq \\ &\int_{X \setminus B(x, \alpha^{m-1} 2a_1 r)} (\mu B(x, \rho(x, t)))^{-2} \left(\int_{B(x, \alpha^{1-m} \rho(x, t))} |f(z)| d\mu(z) \right) d\mu(t) \leq \\ &\int_{X \setminus B(x, \alpha^{m-1} 2a_1 r)} (\mu B(x, \rho(x, t)))^{-1} \bar{f}(x, t) d\mu(t), \end{aligned}$$

where

$$\bar{f}(x, t) := \left[\mu B(x, \alpha^{1-m} \rho(x, t)) \right]^{-1} \int_{B(x, \alpha^{1-m} \rho(x, t))} |f(z)| d\mu(z).$$

Taking into account the condition $q \in DL(X)$ and (1.1) we find that

$$\begin{aligned} \bar{f}(x, t) &\leq (\mu(x, \alpha^{1-m} \rho(x, t)))^{-1} \|f\|_{L^{q(\cdot)}(B(x, \alpha^{1-m} \rho(x, t)))} \|\chi_{B(x, \alpha^{1-m} \rho(x, t))}\|_{L^{q'(\cdot)}(X)} \leq \\ &\|f\|_{M_{p(\cdot)}^{q(\cdot)}(B(x, \alpha^{1-m} \rho(x, t)))} (\mu(x, \alpha^{1-m} \rho(x, t)))^{-1/q'(x)-1/p(x)+1/q'(x)} \leq \\ &c \|f\|_{M_{q(\cdot)}^{p(\cdot)}(X)} (\mu(x, \rho(x, t)))^{-1/p(x)}. \end{aligned}$$

Hence, Lemma 1.2 yields

$$\begin{aligned} |Kf_2(y)| &\leq c \|f\|_{M_{q(\cdot)}^{p(\cdot)}(X)} \int_{X \setminus B(x, \alpha^{m-1} 2a_1 r)} (\mu(x, \rho(x, t)))^{-1/p(x)-1} d\mu(t) \leq \\ &c_p \|f\|_{M_{q(\cdot)}^{p(\cdot)}(X)} (\mu B(x, r))^{-1/p(x)} \leq c \|f\|_{M_{q(\cdot)}^{p(\cdot)}(X)} (\mu B(x, r))^{-1/p(x)}. \end{aligned}$$

Further, by the last inequality, Theorem B, Lemma 1.1 and Remark 1.2 we have

$$\begin{aligned} &(\mu B(x, r))^{1/p(x)-1/q(x)} \|Kf\|_{L^{q(\cdot)}(B(x, r))} \leq \\ &(\mu B(x, r))^{1/p(x)-1/q(x)} \|Kf_1\|_{L^{q(\cdot)}(B(x, r))} + \\ &(\mu B(x, r))^{1/p(x)-1/q(x)} \|Kf_2\|_{L^{q(\cdot)}(B(x, r))} \leq \\ &(\mu B(x, r))^{1/p(x)-1/q(x)} \|f\|_{L^{q(\cdot)}(B(x, 2a_1 r))} + \\ &(\mu B(x, r))^{1/p(x)-1/q(x)} \|K_2 f\|_{L^{q(\cdot)}(B(x, r))} \leq c \|f\|_{M_{q(\cdot)}^{p(\cdot)}(X)} + \end{aligned}$$

$$c(\mu B(x, r))^{-1/q(x)} \|\chi_{B(x, r)}\|_{L^{q(\cdot)}(X)} \|f\|_{M_{q(\cdot)}^{p(\cdot)}(X)} \leq c \|f\|_{M_{q(\cdot)}^{p(\cdot)}(X)}.$$

□

4. APPLICATIONS TO SINGULAR INTEGRALS ON FRACTAL SETS

Let $\Gamma \subset \mathbb{C}$ be a connected rectifiable curve and let ν be arc-length measure on Γ . By definition, Γ is regular if

$$\nu(D(z, r) \cap \Gamma) \leq r$$

for every $z \in \Gamma$ and all $r > 0$, where $D(z, r)$ is a disc in \mathbb{C} with center z and radius r . The reverse inequality

$$\nu(D(z, r) \cap \Gamma) \geq cr$$

holds for all $z \in \Gamma$ and $r < L/2$, where L is a diameter of Γ . If we equip Γ with the measure ν and the Euclidean metric, the regular curve becomes an SHT.

The associate kernel in which we are interested is

$$k(z, w) = \frac{1}{z - w}.$$

The Cauchy integral

$$S_\Gamma f(t) = \int_\Gamma \frac{f(\tau)}{t - \tau} d\nu(\tau)$$

is the corresponding singular operator.

The above-mentioned kernel in the case of regular curves is a Calderón-Zygmund kernel. As was proved by G. David [8], a necessary and sufficient condition for continuity of the operator S_Γ in $L^r(\Gamma)$, where r is a constant ($1 < r < \infty$), is that Γ is regular.

Definition 4.1. *Let $1 < q_- \leq q(\cdot) \leq p(\cdot) \leq p_+ < \infty$. We say that a measurable locally integrable function f on Γ belongs to the class $M_{q(\cdot)}^{p(\cdot)}(\Gamma)$ if*

$$\|f\|_{M_{q(\cdot)}^{p(\cdot)}(\Gamma)} = \sup_{\substack{z \in \Gamma \\ 0 < r < L}} (\nu(D(z, r) \cap \Gamma))^{1/p(z) - 1/q(z)} \|f\|_{L^{q(\cdot)}(D(z, r) \cap \Gamma)}.$$

Theorem 3.1 implies the following statement.

Proposition 4.1. *Let Γ be a regular curve. Suppose that $1 < q_- \leq q(z) \leq p(z) \leq p_+ < \infty$ for all $z \in \Gamma$. Assume that $L < \infty$ and $p, q \in DL(\Gamma)$. Then the Cauchy integral S_Γ is a bounded operator in $M_{q(\cdot)}^{p(\cdot)}(\Gamma)$.*

Let now Γ be a subset of \mathbf{R}^n which is an s -set ($0 \leq s \leq n$) in the sense that there is a Borel measure μ in \mathbf{R}^n such that

- (i) $\text{supp } \mu = \Gamma$;
- (ii) there are positive constants c_1 and c_2 such that for all $z \in \Gamma$ and all $r \in (0, 1)$,

$$c_1 r^s \leq \mu(B(x, r) \cap \Gamma) \leq c_2 r^s.$$

It is known (see [32], Theorem 3.4) that μ is equivalent to the restriction of Hausdorff s -measure H_s to Γ . We shall thus identify μ with $H_s|_\Gamma$.

Let $\Gamma(x, r) = B(x, r) \cap \Gamma$, where $x \in \Gamma$. Suppose that K_Γ is a Calderón-Zygmund singular integral defined on an s -set Γ .

The next statement is a consequence of Theorem 3.1.

Proposition 4.2. *Let $1 < q_- \leq q(x) \leq p(x) \leq p_+ < \infty$ for all $x \in \Gamma$. Suppose that Γ is bounded. Assume that $p, q \in DL(\Gamma)$. Then the operator K_Γ is a bounded operator in $M_{q(\cdot)}^{p(\cdot)}(\Gamma)$.*

Acknowledgements. The authors express their gratitude to the referee for valuable remarks and suggestions.

The work was partially supported by the INTAS grant "Variable Exponent Analysis", Nr. 06-1000017-8792 and the Georgian National Foundation Grant "Some Topics of Harmonic and Non-linear Analysis in Non-standard Setting with Applications in Differential Equations" No. GNSF/STO7/3-169.

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