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On a Measure of Non-Compactness for the Riesz Transforms

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Let

$$R_j f(x) = \lim_{r\to 0} \gamma_n \int\limits_{\mathbb{R}^n\setminus B(x,r)} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\,dy$$

be the Riesz transform of the measurable function $f : \mathbb{R}^n \to \mathbb{R}$, where $j = 1,\dots,n$, $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, $\gamma_n = \frac{\Gamma[(n+1)/2]}{\pi^{(n+1)/2}}$. If $n = 1$, then $R_1 f(x)$ is the Hilbert transform defined by

$$Hf(x) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int\limits_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\,dy.$$

Let $L^p_w(\mathbb{R}^n)$ $(1 < p < \infty)$ be the weighted Lebesgue space with the weight w. If $w \equiv 1$, then we shall use the notation $L^p_w(\mathbb{R}^n) \equiv L^p(\mathbb{R}^n)$.

We denote by $\|A\|_{\mathcal{K}}$ the measure of non-compactness of the bounded linear operator $A : L^p_w(\mathbb{R}^n) \to L^p_v(\mathbb{R}^n)$, i.e.,

$$\|A\|_{\mathcal{K}} := \inf\Big\{\|A-P\| : P \in \mathcal{K}(L^p_w(\mathbb{R}^n),\, L^p_v(\mathbb{R}^n))\Big\},$$

where $\mathcal{K}(L^p_w(\mathbb{R}^n), L^p_v(\mathbb{R}^n))$, is the class of compact linear operators acting from $L^p_w(\mathbb{R}^n)$ to $L^p_v(\mathbb{R}^n)$. The measure of non-compactness of the operator A is also called the essential norm of the operator A.

The essential norm $\|S\|_{\mathcal{K}} = \operatorname{dist}\{A, \mathcal{K}(L^p(T))\}$ $\big($where $\mathcal{K}(L^p(T)) := \mathcal{K}(L^p(T), L^p(T))\big)$ for the operator

$$Sf(x) = \text{p.v. } \frac{1}{\pi i} \int\limits_T \frac{f(\tau)}{\tau - t}\,d\tau, \quad t \in T,$$

where T is the unit circle, were calculated by I. Gohberg and N. Krupnic (see [1]–[2]) for $p = 2^n$ and $p = \frac{2^n}{2^n-1}$. Lower estimate for $\|S\|_{\mathcal{K}}$ were also derived in these spaces for all $p \in (1,\infty)$. Upper estimates for all $p \in (1,\infty)$ were obtained by S. K. Pichorides [3]. The essential norm of Cauchy singular integtal over Lyapunov curves in the case of weighted Lebesgue spaces with power (Khvedelidze) weights was calculated by N. Krupnik and I. Verbitsky [4]. The case of general Muckenhoupt weight w over the unit circle T was considered by I. Feldman, N. Krupnik and I. Spitkovsky [5]. In that paper it was proved that $\|S\|_{\mathcal{K}} = \operatorname{dist}\{A, \mathcal{K}(L^2_w(T))\} = 1$ if and only if w has vanishing mean oscillation.

Now we formulate the main results.

Theorem 1. *Let $1 < p < \infty$. Then there exist no pair of weights (v,w) and integer j, $1 \le j \le n$, such that the operator R_j is compact from $L^p_w(\mathbb{R}^n)$ to $L^p_v(\mathbb{R}^n)$. Moreover, if R_j is bounded from $L^p_w(\mathbb{R}^n)$ to $L^p_v(\mathbb{R}^n)$ for some j, then the following inequality holds:*

$$\|R_j\|_{\mathcal{K}} \ge A_n \operatorname*{ess\,sup}_{a\in\mathbb{R}^n} \bigg(\frac{v(a)}{w(a)}\bigg)^{1/p},$$

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where the positive constant A_n depends only on n.

The next statement is true for the operator

$$Rf(x) = \sum_{j=1}^{n} R_j f(x).$$

Theorem 2. *Let $1 < p < \infty$. Then there exists no pair of weights (v, w) such that the operator R is compact from $L^p_w(\mathbb{R}^n)$ to $L^p_v(\mathbb{R}^n)$. Moreover, if R is bounded from $L^p_w(\mathbb{R}^n)$ to $L^p_v(\mathbb{R}^n)$ then the inequality*

$$\|R\|_{\mathcal{K}} \geq B_n \operatorname*{ess\,sup}_{a \in \mathbb{R}^n} \left(\frac{v(a)}{w(a)} \right)^{1/p}$$

holds, where the positive constant B_n depends only on n.

Analogous result for the maximal operator

$$Mf(x) = \sup \frac{1}{|Q|} \int_Q \left| f(y) \right| dy,$$

where the supremum is taken over all cubes Q containing $x \in \mathbb{R}^n$, was obtained in [6].

Finally we note that some optimal conditions for the weights v and w guaranteeing the boundedness of R_j from $L^p_w(\mathbb{R}^n)$ to $L^p_v(\mathbb{R}^n)$ were derived in [7] (see also [8]).

References

1. I. Gohberg and N. Krupnik, On the spectrum of singular integral operators on the spaces L_p. (Russian) *Studia Math.* **31**(1968), 347–362.

2. I. Gohberg and N. Krupnik, On the norm of the Hilbert transform on the space L_p. (Russian) *Funktsional Anal. i. Prilozh.* **2**(1968), No. 2, 91–92.

3. S. K. Pichorides, On the best values of the constants in the theorems of M. Riesz, Zygmund and Kolmogorov, *Studia Math.* **44**(1972), 165–179.

4. N. Krupnik and I. Verbitskiy, Exact constants in the theorems of K. I. Babenko and B. V. Khvedelidze on the boundedness of singular operators. (Russian) *Soobshch. AN Gruz. SSR* **85**(1977), No. 1, 21–24.

5. I. Feldman, N. Krupnik and I. Spitkovsky, Norms of the singular integral operators with Cauchy kernel along certain contours. *Integral Equations Operator Theory* **24**(1996), 68–80.

6. D. E. Edmunds and A. Meskhi, On a measure of non-compactness for maximal operators. *Math. Nachr.* (accepted for publication) 1999.

7. D. E. Edmunds and V. Kokilashvili, Two-weighted inequalities for singular integrals. *Canad. J. Math.* **38**(1995), No. 3, 295–303.

8. V. Kokilashvili and A. Meskhi, Boundedness and compactness criteria for some classical operators. *In: Lecture Notes in Pure and Applied Mathematics,* **213**, *"Function Spaces V", Proceedings of the Conference, Poznań, Poland* (*Ed. H. Hudzik and L. Skrzypczak*), 279–296, *New York, Bazel, Marcell Dekker,* 2000 (with V. Kokilashvili).

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