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ON A WEIGHTED STRICHARTZ ESTIMATE FOR INHOMOGENEOUS WAVE EQUATIONS

In this note we look for sufficient condition on a weight pair (V, W) governing the two-weight Strichartz estimate

$$\|V(t-|x|,t+|x|)\omega\|_{L^q(t\geq|x|)} \leq$$
$$\leq C\|W(t-|x|,t+|x|)F\|_{L^{q'}(t\geq|x|)}, \quad q' = \frac{q}{q-1}, \tag{1}$$

for the solution of inhomogeneous wave equation

$$\begin{cases} \Box\omega(t,x) = F(t,x), \quad (t,x) \in R_+^{1+n} \\ 0 = \omega(0,\cdot) = \partial_t\omega(0,\cdot), \end{cases} \tag{2}$$

Here $\Box = \frac{\partial^2}{\partial t^2} - \Delta_x$ denotes the D'Alemberian and n is odd.

Two-weight Strichartz estimates with power-type weights has been established in [G], [GLS], [KO]. In these papers existence of global weak solution for the semilinear wave equation

$$\begin{cases} \Box\omega = |u|^p, & (t,x) \in R_+^{1+n}, \\ u(0,x) = \varepsilon f(x), & \partial_t u(0,x) = \varepsilon g(x), \end{cases}$$

where ε is small and p is more that critical exponent p_c in the sense of Strauss (see [S1-S2], [J]) have been proved.

To formulate our main results we need the following.

Definition. We say that the weight $\rho(s,\tau)$ defined on $R_+^2 := (0,\infty)\times(0,\infty)$ satisfies the doubling condition in the first variable uniformly to another one ($\rho \in DC(s)$) if there exists a positive constant c such that for all $t, \tau > 0$ the inequality

$$\int_0^{2t} \rho(s,\tau)ds \leq c\int_0^t \rho(s,\tau)ds$$

holds. Analogously can be defined the class $DC(\tau)$.

Theorem 1. *Let n be odd and let $\frac{2n}{n-1} < q \leq \frac{2(n+1)}{n-1}$. Suppose that F is spherically symmetric and* supp $F \subset \{(t,x) \in R_+^{1+n} : |x| < t\}$. *Assume that two-dimensional weights V and W are increasing in each variable uniformly with respect to another one. In addition, suppose that $W^{-q} \in DC(s)\cap DC(\tau)$ or $W(s,\tau) = W_1(s)W_2(\tau)$. If ω solves*

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(2), *then the condition*

$$\sup_{a,b>0}\Big(\int_a^\infty\int_b^\infty \frac{V^q(s,\tau)}{(s\tau)^{q(n-1)(1/2-1/q)}}dsd\tau\Big)\times$$
$$\times\Big(\int_0^a\int_0^b W^{-q}(s,\tau)dsd\tau\Big)<\infty \tag{3}$$

implies the inequality (1) *with the constant* C *depending only on* V, W, q *and* n.

Theorem 2. *Let* n *be odd and let* $2<q<\frac{2n}{n-1}$. *Suppose that* F *is spherically symmetric and* supp $F\subset\{(t,x)\in R_+^{1+n}:|x|<t\}$. *Assume that two-dimensional weight* W *is increasing in each variable uniformly with respect to another one. In addition, suppose that* $W^{-q}\in DC(s)\cap DC(\tau)$ *or* $W(s,\tau)=W_1(s)W_2(\tau)$. *Then if* ω *solves* (2), *condition* (3) *implies the inequality* (1) *with the constant* C *depending only on* V, W, q *and* n.

The *proofs* of these statements are based on the integral representation of the solution ω for equation (2)

$$\omega(t,r)=r^{-(n-1)/2}\int_0^t\int_{|t-r-s|}^{t+r-s}P_m(\mu)F(s,\sigma)\sigma^{(n-1)/2}d\sigma ds, \tag{4}$$

where $P_m(\mu)$ are Legendre polynomials of degree $m=(n-3)/2$ and $\mu=(r^2+\sigma^2-(t-s)^2)/2r\sigma$ satisfies $-1\le\mu\le 1$ in the domain of integration (see e.g. [LS]), and weighted boundedness criterion for the Riemann-Liouville operator with product kernels

$$R_{\alpha,\beta}f(x,y)=\int_0^x\int_0^y\frac{f(t,\tau)}{(x-t)^{1-\alpha}(y-\tau)^{1-\beta}}dtd\tau$$

(for some two-weight inequalities for this operator see [KM1-KM3]).

Theorem 3. *Let* n *be odd and let* $\frac{2n}{n-1}<q\le\frac{2(n+1)}{n-1}$. *Suppose that* F *is spherically symmetric and* supp $F\subset\{(t,x)\in R_+^{1+n}:|x|<t\}$. *Assume that two-dimensional weights* V *and* W *are increasing in each variable uniformly with respect to another one. In addition, suppose that* $W^{-q}\in DC(s)$ *and*

$$\int_{2^k}^{2^{k+1}}V^q(s,\tau)ds\le c\int_{2^{k-1}}^{2^k}V^q(s,\tau)ds$$

for all $k\in Z$ *and* $\tau>0$. *If* ω *solves* (2), *then the condition*

$$\sup_{\substack{a,k,\\ a>2^k,k\in Z}}\Big(\int_a^\infty\Big(\int_{2^k}^{2^{k+1}}\frac{V^q(s,\tau)}{s^{q(n-1)(1/2-1/q)}}ds\Big)(\tau-2^k)^{q(n-1)(1/2-1/q)}d\tau\Big)\times$$
$$\times\Big(\int_{2^k}^a\Big(\int_0^{2^k}W^{-q}(s,\tau)ds\Big)d\tau\Big)<\infty$$

implies inequality (1) *with the constant* C *depending only on* V, W, q *and* n.

The proof of the latter theorem follows from the integral representation (4) of the solution of equation (2) and the following

Theorem 4. *Let $1 < p \le q < \infty$ and let $0 < \alpha, \beta < 1/p$. Suppose that the two-dimensional weight functions v and w are increasing in each variable uniformly to another ones. Suppose also that $w^{1-p'}(s,\tau) \in DC(s)$ and*

$$\int_{2^k}^{2^{k+1}} v(s,\tau)ds \le c \int_{2^{k-1}}^{2^k} v(s,\tau)ds.$$

for all $k \in Z$ and $\tau > 0$. Then the two-weight inequality

$$\left[\iint_{y<x} v(y,x)\left(\int_0^y \int_y^x \frac{f(\tau,t)d\tau dt}{(x-\tau)^{1-\alpha}(y-\tau)^{1-\beta}}\right)^q dydx\right]^{1/q} \le$$

$$\le c\left(\iint_{y<x} w(y,x)(f(y,x))^p dydx\right)^{1/p}$$

holds with the positive constant c independent of $f \in L^p_w(y<x)$, $f \ge 0$, if and only if

$$\sup_{\substack{a,k,\\ a>2^k, k\in Z}} \left(\int_a^\infty \left(\int_{2^k}^{2^{k+1}} \frac{v(s,\tau)}{s^{(1-\beta)q}}ds\right)(\tau-2^k)^{(\alpha-1)q}d\tau\right)^{1/q} \times$$

$$\times\left(\int_{2^k}^a \left(\int_0^{2^k} w^{1-p'}(s,\tau)ds\right)d\tau\right)^{1/p'} < \infty.$$

Now we give some corollaries of the statements formulated above:

Corollary 1 [GLS]. *Let n be odd and let $2 < q \le \frac{2(n+1)}{(n-1)}$. Suppose that* supp $F \subset \{(t,x) \in R^{1+n}_+ : |x| < t\}$. *If ω solves (2), then*

$$\|(t^2-|x|^2)^{-\alpha}\omega\|_{L^q(R^{1+n}_+)} \le C_\gamma \|(t^2-|x|^2)^\beta F\|_{L^{q'}(R^{1+n}_+)},$$

where $\beta < 1/q$, $\alpha+\beta+\gamma = 2/q$, $\gamma = (n-1)(1/2-1/q)$.

Corollary 2. *Let n be odd and let $q = \frac{2(n+1)}{n-1}$. Suppose that F is spherically symmetric and supported in the light cone $\{(t,x) \in R^{1+n}_+ : |x| < t\}$. Then the inequality*

$$\left\|(t^2-|x|^2)^{\gamma-1/q}\log^\beta \frac{4T^2}{t^2-|x|^2}\omega\right\|_{L^q(t+|x|\le T)} \le$$

$$\le C\left\|(t^2-|x|^2)^{1/q}\log^\lambda \frac{4T^2}{t^2-|x|^2}F\right\|_{L^{q'}(t+|x|\le T)}$$

holds, where $\beta = \lambda - 4/q$, $\lambda > 3/q$ and $\gamma = (n-1)(1/2-1/q)$.

From this corollary we have

Proposition 1. *Let n be odd and let $T \ge 2$, $q = \frac{2(n+1)}{n-1}$. Assume that F is spherically symmetric and* supp $F \subset \{(t,x) : t^2-|x|^2 \ge 1\}$. *Then the inequality*

$$\|(t^2-|x|^2)^{1/q}\omega\|_{L^q(\{|x|<t<T/2\})} \le c(\log T)^{4/q}\|(t^2-|x|^2)^{1/q}F\|_{L^{q'}}$$

holds, where the constant c does not depend on T.

Proposition 2. *Let n be odd and let $T > 1$. Suppose that $q = \frac{2(n+1)}{n-1}$. Assume that F is spherically symmetric and* supp $F \subset \{(t,x) : t-|x| > 1\}$. *Then the inequality*

$$\|(t-|x|)^{1/q}\omega\|_{L^q(\{t-|x|<T\})} \le c(\log T)^{2/q}\|(t-|x|)^{1/q}F\|_{L^{q'}(\{t-|x|<T\})}$$

holds and the constant c does not depend on T.

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