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ON A CLASS OF BOUNDED AND COMPACT HIGHER DIMENSIONAL KERNEL OPERATORS

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In this note boundedness/compactness criteria from $L^p(E)$ to $L^q_v(E)$ are presented for the operator with positive kernel

$$\mathcal{K}f(x) = \int\limits_{E_{r(x)}} k(x,y)f(y)dy, \quad x \in E, \tag{1}$$

where $1 < p, q < \infty$ or $0 < q \leq 1 < p < \infty$, $E_{r(x)}$ and E are certain cones in homogeneous groups and k satisfies the conditions which in one-dimensional case are similar to those of [12].

A full characterization of pairs of weights (v, w) governing the boundedness of integral operators with positive kernels from L^p_w to L^q_v, $1 < p < q < \infty$, have been established in [6] (see also [7]). Criteria guaranteeing the boundedness/compactness of the operator

$$\mathcal{R}_\alpha f(x) = \int\limits_0^x (x-t)^{\alpha-1} f(t)dt, \qquad x > 0,$$

from $L^p(\mathbb{R}_+)$ to $L^q_v(\mathbb{R}_+)$, $1 < p, q < \infty$, $1/p < \alpha < 1$ have been obtained in [11] and [17]. This result was generalized in [12] (see also [3], Ch. 2) for integral operators with positive kernels involving fractional integrals.

The two-weight problem for higher-dimensional Hardy-type operators defined on cones in $\mathbb{R}^n$ involving Oinarov [16] kernels was studied in [19], [8] (see also [18], for Hardy-type transforms on star-shaped regions).

In the present note we present also two-sided estimates of Schatten-von Neumann norms for the weighted integral operator with positive kernel

$$\mathcal{K}_u f(x) = u(x) \int\limits_{E_{r(x)}} k(x,y)f(y)dy, \quad x \in E, \tag{2}$$

where u is a measurable function on E.

A homogeneous group G is a simply connected nilpotent Lie group G on whose Lie algebra g is given one-parameter group of transformations $\delta_t = \exp(A \ \log \ t)$, $t > 0$, where A is a diagonalized linear operator on G with positive eigenvalues. For G the mappings $\exp \ o \ \delta_t \ o \ \exp^{-1}$, $t > 0$, are automorphisms on G, which will be denoted by δ_t. The number $Q = tr \ A$ is called homogeneous dimension of G. The symbol e will stand for the neutral element in G.

It is possible to equip G with a homogeneous norm $r : G \to [0, \infty)$ which is continuous function on G and smooth on $G \backslash \{e\}$ satisfying the following conditions:

(i) $r(x) = r(x^{-1})$ for every $x \in G$;

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(ii) $r(\delta_t x) = t \cdot r(x)$ for every $x \in G$ and $t > 0$;
(iii) $r(x) = 0$ if and only if $x = e$;
(iv) there exists $c_o \geq 1$ such that

$$r(xy) \leq c_o(r(x) + r(y)), \quad x, y \in G.$$

A ball in G, centered at x and with radius ρ , is defined as

$$B(x, \rho) = \{y \in \ G; \ r(xy^{-1}) < \rho\}.$$

It can be observed that $\delta_\rho B(e, 1) = B(e, \rho)$.

Let us fix a Haar measure $|\cdot|$ in G so that $|B(0, 1)| = 1$. Then $|\delta_t E| = t^Q |E|$, in particular, $|B(x, r)| = r^Q$ for $x \ \in G, \ r > 0$.

Examples of homogeneous groups are: Euclidean n-dimensional spaces, Heisenberg group, upper triangular groups etc (see [5] for the definition and basic properties of homogeneous groups).

Let S be the unit sphere in G i.e. $S = \{x \in G : \ r(x) = 1\}$, and let A be a measurable subset of S with positive measure. We denote by E a measurable cone in G defined by

$$E := \{x \in G : \ x = \delta_s \overline{x}, \ 0 < s < \infty, \ \overline{x} \in A\}.$$

We denote

$$E_t := \{y \in E : \ r(y) < t\}.$$

For the cones E and $E_{r(\cdot)}$ we define the kernel operator by (1), where $k(x, y)$ is non-negative function on $\{(x, y) \in E \times E : r(y) < r(x)\}$.

Let Ω be a measurable subset of G. A locally integrable almost everywhere positive function w on Ω we call a weight. Denote by $L^p_w(\Omega)$ $(0 < p < \infty)$ the weighted Lebesgue space, which is the space of all measurable functions $f : \Omega \to \mathbb{C}$ with finite norm (quasinorm if $0 < p < 1$)

$$\|f\|_{L^p_w(\Omega)} = \Bigg(\int\limits_{\Omega} |f(x)|^p w(x) dx \Bigg)^{\frac{1}{p}}.$$

Let H be a separable Hilbert space and let $\sigma_\infty(H)$ be the class of all compact linear operators $T : H \to H$, which form an ideal in the normed algebra B of all bounded linear operators on H. To construct a Schatten-von Neumann ideal $\sigma_p(H)$ $(0 < p \leq \infty)$ in $\sigma_\infty(H)$, the sequence of singular numbers $s_j(T) \equiv \lambda_j(|T|)$ is used, where the eigenvalues $\lambda_j(|T|)$ $(|T| \equiv (T^*T)^{1/2})$ are non-negative and are repeated according to their multiplicity and arranged in decreasing order. A Schatten-von Neumann quasinorm (norm if $1 \leq p \leq \infty$) is defined as follows:

$$\|T\|_{\sigma_p(H)} \equiv \Big(\sum_j s_j^p(T) \Big)^{1/p}, \quad 0 < p < \infty,$$

with the usual modification if $p = \infty$. Thus we have $\|T\|_{\sigma_\infty(H)} = \|T\|$ and $\|T\|_{\sigma_2(H)}$ is the Hilbert-Schmidt norm given by the formula

$$\|T\|_{\sigma_2(H)} = \Big(\int \int |a(x, y)|^2 dx dy \Big)^{1/2}$$

for the integral operator

$$Tf(x) = \int a(x, y) f(y) dy.$$

We refer, for example, to [1], [9] for more information concerning Schatten-von Neumann ideals.

For estimates of Schatten- von Neumann ideal norms for the Hardy-type transforms we refer to [15], [2], [4], [14]. The same problem for the operator with positive kernel involving one-sided potentials was studied in [13] (see also [3]).

Definition 1. Let k be a positive function on $\{(x,y) \in E \times E : r(y) < r(x)\}$ and let $1 < \lambda < \infty$. We say that the kernel $k \in V_\lambda$, if there exist positive constants c_1, c_2 and c_3 such that

(i) $k(x,y) \le c_1 k(x, \delta_{1/(2c_o)}x)$ for all x, y with $x, y \in E$, $r(y) < r(x)/(2c_o)$; $k(x,y) \ge c_2 k(x, \delta_{1/(2c_o)}x)$ for all x, y with $x, y \in E$, $r(x)/(2c_o) < r(y) < r(x)$;

(ii) $\int\limits_{E_{r(x)} \setminus E_{r(x)/(2c_0)}} k^{\lambda'}(x,y)dy \le c_3 r^Q(x) k^{\lambda'}(x, \delta_{1/(2c_o)}x)$, $\lambda' = \lambda/(\lambda-1)$, for all $x \in E$.

Example 1. Let $G = \mathbb{R}^n$, $r(xy^{-1}) = |x-y|$, $\delta_t x = tx$, $x, y \in \mathbb{R}^n$. Suppose that k is the potential kernel $k(x,y) = |x-y|^{\alpha - n}$. Then $k \in V_\lambda$ if $n/\lambda < \alpha \le n$.

Example 2. It is easy to see that if

(i) $k(\delta_t \bar{x}, \delta_\tau \bar{y}) \le c_1 k(\delta_t \bar{x}, \delta_s \bar{z})$ for all t, τ, s, $\bar{x}$, $\bar{y}$, $\bar{z}$ with $0 < \tau < s < t$; $\bar{x}, \bar{y}, \bar{z} \in A$;

(ii) $\int_{t/(2c_o)}^{t} k^{\lambda'}(\delta_t \bar{x}, \delta_\tau \bar{y}) \tau^{Q-1} d\tau \le c_2 t^Q \cdot k^{\lambda'}(\delta_t \bar{x}, \delta_{t/(2c_o)} \bar{x})$, $t > 0$, $\bar{x} \in A$, then $k \in V_\lambda$.

Example 3. Let $k(x,y) = \overline{k}(r(x), r(y))$ be a radial kernel and let there exist positive constants c_1 and c_2 such that

(i) $\bar{k}(s,l) \le c_1 \bar{k}(s,t)$, $0 < l < t < s$,

(ii) $\int_{t/(2c_0)}^{t} \bar{k}^{\lambda'}(t,s) s^{Q-1} ds \le c_2 t^Q \bar{k}^{\lambda'}(t, t/(2c_0))$, $1 < \lambda < \infty$. Then $k \in V_\lambda$.

Theorem 1. *Let $1 < p \le q < \infty$ and let v be a weight on E. Suppose that $k \in V_p$. Then $\mathcal{K}$ is bounded from $L^p(E)$ to $L^q_v(E)$ if and only if*

$$B \equiv \sup_{j \in \mathbb{Z}} B(j) \equiv \sup_{j \in \mathbb{Z}} \Big(\int\limits_{F_{2^{j+1}} \setminus E_{2^j}} k^q(x, \delta_{1/(2c_o)}x) v(x) dx \Big)^{1/q} \big(2^j\big)^{Q/p'} < \infty.$$

Theorem 2. *Let $1 < p \le q < \infty$. Suppose that $k \in V_p$. Then $\mathcal{K}$ is compact from $L^p(E)$ to $L^q_v(E)$ if and only if*

(a) $B < \infty$;

(b) $\lim\limits_{j \to -\infty} B(j) = \lim\limits_{j \to +\infty} B(j) = 0.$

For $q < p$ we have

Theorem 3. *Let $0 < q < p < \infty$ and let $p > 1$. Suppose that $k \in V_p$. Then the following conditions are equivalent:*

(i) *$\mathcal{K}$ is bounded from $L^p(E)$ to $L^q_v(E)$;*

(ii) *$\mathcal{K}$ is compact from $L^p(E)$ to $L^q_v(E)$;*

(iii) $\Bigg(\int\limits_E \Bigg(\int\limits_{E \setminus E_{r(x)}} k^q(y, \delta_{1/(2c_o)}y) v(y) dy \Bigg)^{p/(p-q)} r(x)^{Qp(q-1)/(p-q)} dx \Bigg)^{(p-q)/pq} < \infty.$

Let $k_0(x) := r^Q(x) k^2(x, \delta_{1/(2c_o)}x)$ and let $l^p\big(L^2_{k_0}(E)\big)$ be the set of all measurable functions $g : E \to \mathbb{R}$ for which

$$\|g\|_{l^p(L^2_{k_0}(E))} = \Big(\sum_{n \in \mathbb{Z}} \Big(\int\limits_{E_{2^{n+1}} \setminus E_{2^n}} |g(x)|^2 k_0(x) dx \Big)^{p/2} \Big)^{1/p} < \infty.$$

Now we give two-sided estimates of Schatten von-Neumann norms for the operator (2).

Theorem 4. *Let $2 \leq p < \infty$ and let k in V_2. Then $\mathcal{K}_u \in \sigma_p(L^2(E))$ if and only if $u \in l^p(L^2_{k_0}(E))$. Moreover, there exist positive constants b_1 and b_2 such that*

$$b_1 \|u\|_{l^p(L^2_{k_0}(E))} \leq \|\mathcal{K}_u\|_{\sigma_p(L^2(E))} \leq b_2 \|u\|_{l^p(L^2_{k_0}(E))}.$$

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