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EFFECTIVE CODESCENT MORPHISMS IN LOCALLY PRESENTABLE CATEGORIES

B. Mesablishvili

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ABSTRACT. A necessary and sufficient condition for pure morphisms in locally presentable categories to be effective is given.

CONTENTS

The present article, conceived as the continuation of previous works on the problem of describing (effective) descent morphisms in various categories [11–18], deals with the problem of describing effective codescent morphisms in locally presentable categories. As background to the subject, we refer to S. MacLane [10] for generalities on category theory, and to G. Janelidze and W. Tholen [6–8] for descent theory.

1. Preliminaries from Category Theory

We write $\eta, \varepsilon \colon F \dashv U \colon \mathscr{A} \to \mathscr{B}$ to denote that $F : \mathscr{B} \to \mathscr{A}$ and $U : \mathscr{A} \to \mathscr{B}$ are functors, where F is left adjoint to U with unit $\eta : 1 \to UF$ and counit $\varepsilon : FU \to 1$.

A *comonad* $\mathbf{G}$ on a given category $\mathscr{A}$ is an endofunctor $G : \mathscr{A} \to \mathscr{A}$ equipped with natural transformations $\varepsilon : G \to 1$ and $\delta : G \to G^2$ satisfying

$$G\varepsilon \cdot \delta = \varepsilon G \cdot \delta = 1, \quad G\delta \cdot \delta = \delta G \cdot \delta.$$

For example, any adjunction $\eta, \epsilon \colon F \dashv U \colon \mathscr{A} \to \mathscr{B}$ determines its *associated comonad* $(FU, F\eta U, \varepsilon)$ on the category $\mathscr{A}$.

The *Eilenberg–Moore construction* associates to any comonad $\mathbf{G} = (G, \delta, \varepsilon)$ on $\mathscr{A}$ a category $\mathscr{A}^{\mathbf{G}}$ the objects of which are the $\mathbf{G}$-coalgebras (A, θ), where $A \in \mathscr{A}$ and $\theta : A \to G(A)$ is a morphism in $\mathscr{A}$ satisfying

$$\varepsilon_A \cdot \theta = 1, \quad \delta_A \cdot \theta = G(\theta) \cdot \theta.$$

The morphisms of $\mathscr{A}^{\mathbf{G}}$ from (A, θ) to (A', θ') are the morphisms $f : A \to A'$ in $\mathscr{A}$ for which

$$G(f) \cdot \theta = \theta' \cdot f.$$

Translated from Sovremennaya Matematika i Ee Prilozheniya (Contemporary Mathematics and Its Applications), Vol. 74, Proceedings of the International Conference “Modern Algebra and Its Applications” (Batumi, 2010), Part 1, 2011.

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The forgetful functor $U^{\mathbf{G}} : \mathscr{A}^{\mathbf{G}} \to \mathscr{A}$ sending (A, θ) to A admits as a right adjoint the *cofree* $\boldsymbol{G}$*-coalgebra functor* $F^{\mathbf{G}} : \mathscr{A} \to \mathscr{A}^{\mathbf{G}}$, which is defined on objects by

$$F^{\mathbf{G}}(A) = (G(A), \delta_A)$$

and on morphisms by

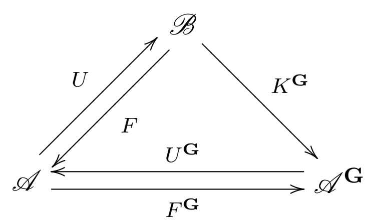
$$F^{\mathbf{G}}(f) = G(f).$$

The comonad associated to this adjunction is precisely the original $\mathbf{G}$.

Any adjunction $F \dashv U \colon \mathscr{A} \to \mathscr{B}$, with associated comonad $\mathbf{G} = (FU, \epsilon, F\eta U)$, admits a unique *comparison functor*

$$K^{\mathbf{G}} : \mathscr{B} \to \mathscr{A}^{\mathbf{G}},$$

making the diagram



commutative, i.e.,

$$U^{\mathbf{G}} \cdot K^{\mathbf{G}} = F, \quad K^{\mathbf{G}} \cdot U = F^{\mathbf{G}}.$$

Explicitly,

$$K^{\mathbf{G}}(B) = (F(B), F(\eta_B)) \quad \forall B \in \mathscr{B}.$$

One says (see [5]) that the functor F is *precomonadic* if $K^{\mathbf{G}}$ is full and faithful, and it is *comonadic* if $K^{\mathbf{G}}$ is an equivalence of categories.

The dual of Beck's monadicity theorem gives a necessary and sufficient condition for a left adjoint functor to be (pre)comonadic. Before stating this result, we need the following definitions (see [10]). An equalizer

$$B \xrightarrow{\ h\ } B' \underset{g}{\overset{f}{\rightrightarrows}} B''$$

in $\mathscr{B}$ is said to be *split* if there are morphisms $k : B' \to B$ and $l : B'' \to B'$ with

$$kh = 1, \quad lf = 1, \quad hk = lg.$$

Given a functor $F : \mathscr{B} \to \mathscr{A}$, a pair of morphisms $(f, g : B \rightrightarrows B')$ in $\mathscr{B}$ is F*-split* if the pair $(F(f), F(g))$ is part of a split equalizer in $\mathscr{A}$, and F preserves equalizers of F-split pairs if for any F-split pair $(f, g : B' \rightrightarrows B'')$ in $\mathscr{B}$, and any equalizer $h : B \to B'$ of f and g, $F(h)$ is an equalizer (necessarily split) of $F(f)$ and $F(g)$. A pair of morphisms $(f, g : B \rightrightarrows B')$ in $\mathscr{B}$ is *coreflexive* if f and g have a common left inverse.

1.1. Theorem (Beck [3])**.** *Let $\eta, \epsilon \colon F \dashv U \colon \mathscr{A} \to \mathscr{B}$ be an adjunction and $\boldsymbol{G} = (FU, F\eta U, \varepsilon)$ be the corresponding comonad on $\mathscr{A}$. Then*:

(1) *the functor $F : \mathscr{B} \to \mathscr{A}$ is precomonadic if and only if $\eta_B : B \to UF(B)$ is a regular monomorphism for each $B \in \mathscr{B}$;*

(2) *the functor $F : \mathscr{B} \to \mathscr{A}$ is comonadic if and only if F is* conservative (=isomorphism-reflecting)*, $\mathscr{B}$ has equalizers of coreflexive F-split pairs, and F preserves them.*

2. Pure Morphisms

In this section, we recall from [1] the basic facts about locally presentable categories, and the results we need on pure morphisms.

Throughout the paper, λ is a regular cardinal (i.e., an infinite cardinal that is not a sum of a smaller number of smaller cardinals). Recall that a nonempty partially ordered set is λ-*directed* if every subset of cardinality smaller that λ has an upper bound and a λ-*directed colimit* is a colimit of a functor whose domain is a λ-directed partially ordered set (considered as a category).

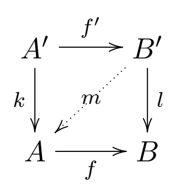
An object A of a category $\mathscr{A}$ is called λ-*presentable* if the hom-functor

$$\mathscr{A}(A,-) : \mathscr{A} \to \mathbf{Set}$$

preserves λ-directed colimits. For example, an object of the category **Set** is λ-presentable if and only if it has cardinality smaller than λ. An object of $\mathscr{A}$ is called *presentable* if it is λ-presentable for some regular cardinal λ.

A category $\mathscr{A}$ is called *locally* λ-*presentable* if it admits colimits and has a set of λ-presentable objects such that each object of $\mathscr{A}$ is a λ-directed colimit of objects from this set. A category is called locally presentable if there is some regular cardinal λ such that it is locally λ-presentable. Every locally presentable category is complete and co-well-powered. Moreover, in each locally λ-presentable category $\mathscr{A}$, λ-directed colimits commute with λ-small limits. In particular, λ-directed colimits commute with pullbacks in $\mathscr{A}$.

Let $\mathscr{A}$ be a locally λ-presentable category. Recall that a morphism $f : A \to B$ in $\mathscr{A}$ is λ-*pure* if for every commutative diagram



in which A' and B' are λ-presentable objects, the morphism k factors through f' (i.e., there is a morphism $m : B' \to A$ with $mf' = k$).

2.1. Proposition (see [1, 2])**.** *λ-Pure morphisms have the following properties*:

(1) *λ-pure morphisms are closed under composition*;

(2) *λ-pure morphisms are left cancellative, i.e., if qp is a λ-pure morphism, then p is a λ-pure morphism*;

(3) *every λ-pure morphism is a regular monomorphism* (*i.e., an equalizer of a parallel pair of morphisms*);

(4) *every λ-pure morphisms p is a λ-directed colimit of split monomorphisms with the same domain as p in the category $\mathscr{A}^2$ of morphisms in $\mathscr{A}$* (*recall that objects of this category are morphisms in $\mathscr{A}$ and morphisms from $f : A \to B$ to $f' : A' \to B'$ in $\mathscr{A}$ are pairs (g,h), where $g : A \to A'$ and $h : B \to B'$ with $hf = f'g$*)*. More precisely, if $p : B \to E$ is λ-pure in $\mathscr{A}$, then there exists a λ-directed diagram of morphisms $(p_d : B \to E_d)_{d\in\mathcal{D}}$ in $\mathscr{A}^2$ with connecting morphisms $(1_B, e_{d,d'}) : p_d \to p_{d'}$ for $d \le d'$ such that each p_d is a split monomorphism and*

$$p = \varinjlim_{d\in\mathcal{D}}(p_d, (1_B, e_{d,d'}) : p_d \to p_{d'})$$

in $\mathscr{A}^2$. In such case, we say that the purity of p in $\mathscr{A}$ is presented by the λ-directed diagram of split monomorphisms $(p_d : B \to E_d)_{d\in\mathcal{D}}$ and that such a diagram is a presentation of the purity of p;

(5) *λ-pure morphisms are stable under pushout, that is, if*

$$\begin{array}{ccc} B & \xrightarrow{p} & E \\ \downarrow & & \downarrow \\ B' & \xrightarrow[p']{} & E' \end{array}$$

is a pushout, then p' is λ-pure, provided that p is λ-pure.

From (3) and (5) we obtain the following proposition.

2.2. Proposition. *In a locally λ-presentable category, λ-pure morphisms are pushout-stable regular monomorphisms.*

We know that the following is quite well known but we are not aware of a suitable reference.

2.3. Proposition. *Let $B \to X$ and $B \to Y \to Z$ be morphisms in a category with pushouts. If $Y \to Z$ is a pushout-stable (regular) monomorphism, then the induced morphism $X \sqcup_B Y \to X \sqcup_B Z$ is a (regular) monomorphism.*

Proof. The diagram

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ {\scriptstyle i_Y}\downarrow & & \downarrow{\scriptstyle i_Z} \\ X \sqcup_B Y & \longrightarrow & X \sqcup_B Z \end{array}$$

is easily seen to be a pushout and pushout-stability of the (regular) monomorphism $Y \to Z$. This implies that the morphism $X \sqcup_B Y \to X \sqcup_B Z$ is also a (regular) monomorphism. □

3. Codescent Morphisms

In this section, we collect some needed definitions and results from descent theory formulated, for convenience, in the dual form (see [6]). More details on descent theory can be found in [7, 8].

We begin by recalling that, for any object a of a category $\mathscr{A}$, one has the *coslice category* $A{\downarrow}\mathscr{A}$, an object of which is a morphism $\gamma : A \to X$ in $\mathscr{A}$, and a morphism $\gamma \to \gamma'$ in which is a morphism $f : X \to X'$ in $\mathscr{A}$ with $f\gamma = \gamma'$. Composition and identity morphisms are as in $\mathscr{A}$.

3.1. Proposition. *The underlying object functor $A{\downarrow}\mathscr{A} \to \mathscr{A}$ is conservative and preserves and reflects (finite) limits that exists in $\mathscr{A}$. That is, if $\mathscr{A}$ has (finite) limits, the category $A{\downarrow}\mathscr{A}$ also has (finite) limits, formed as in $\mathscr{A}$. In particular, a morphism is a regular monomorphism in $A{\downarrow}\mathscr{A}$ if and only if it is so in $\mathscr{A}$. Moreover, if $\mathscr{A}$ is (finitely) cocomplete, then $A{\downarrow}\mathscr{A}$ is (finitely) cocomplete as well.*

Any morphism $p : B \to E$ in $\mathscr{A}$ induces a functor

$$p^! : E{\downarrow}\mathscr{A} \to B{\downarrow}\mathscr{A}$$

sending $\gamma : E \to X$ to $\gamma p : B \to X$; and when $\mathscr{A}$ has pushouts, this has a left adjoint $p^* : B{\downarrow}\mathscr{A} \to E{\downarrow}\mathscr{A}$ (known as the *change-of-cobase* functor) given by pushing out along p. Thus, for an object

$\sigma : B \to Y$ of $B{\downarrow}\mathscr{A}$, $p^*(\sigma)$ is defined by the pushout

$$\begin{array}{ccc} B & \xrightarrow{\;p\;} & E \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle p^*(\sigma)} \\ Y & \xrightarrow[\;i_Y\;]{} & E \sqcup_B Y\,. \end{array}$$

The unit of this adjunction has

$$i_Y : \sigma \to p^*(\sigma) \cdot p = p^!(p^*(\sigma))$$

as its σ-component.

We henceforth suppose that $\mathscr{A}$ admits pushouts.

For a morphism $p : B \to E$ in $\mathscr{A}$, a *codescent datum* on $(X, \gamma) \in E{\downarrow}\mathscr{A}$ (*with respect to* $p : B \to E$) is given by a morphism $\theta : X \to E \sqcup_B X$ in $\mathscr{A}$ such that the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{\;\theta\;} & E \sqcup_B X \\ {\scriptstyle\gamma}\uparrow & \nearrow{\scriptstyle i_E} & \\ E & & \end{array} \qquad \begin{array}{ccc} E \sqcup_B X & \xrightarrow{\;\langle\gamma, 1_X\rangle\;} & X \\ {\scriptstyle\theta}\uparrow & \nearrow\!\!= & \\ X & & \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\;\theta\;} & E \sqcup_B X \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle E\sqcup_B\theta} \\ E \sqcup_B X & \xrightarrow[\;E\sqcup_B i_X\;]{} & E \sqcup_B (E \sqcup_B X), \end{array}$$

where

$$i_E : E \to E \sqcup_B X, \quad i_X : X \to E \sqcup_B X$$

are the inclusions of the pushout diagram

$$\begin{array}{ccc} B & \xrightarrow{\;p\;} & E \\ {\scriptstyle p}\downarrow & & \\ E & & \downarrow{\scriptstyle i_E} \\ {\scriptstyle\gamma}\downarrow & & \\ X & \xrightarrow[\;i_X\;]{} & E \sqcup_B X. \end{array}$$

One sometimes refers to the commutativity of the second diagram as the *unit condition*, and to that of the third as the *cocycle condition*.

Descent data (with respect to $p : B \to E$) form the category $\mathfrak{Codes}_{\mathscr{A}}(p)$: a morphism from $((X, \gamma), \theta)$ to $((X', \gamma'), \theta')$ is a morphism $f : X \to X'$ in $E{\downarrow}\mathscr{A}$ that commutes with the descent data in the sense that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & X' \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta'} \\ E \sqcup_B X & \xrightarrow[\;E\sqcup_B f\;]{} & E \sqcup_B X' \end{array}$$

commutes.

If $(Y, \sigma) \in B{\downarrow}\mathscr{A}$, then $(E \sqcup_B Y, i_E : E \to E \sqcup_B Y)$ comes equipped with *canonical descent data* given by the morphism $E \sqcup_B i_Y : E \sqcup_B Y \to E \sqcup_B (E \sqcup_B Y)$. Thus, one has a functor $K_p : B{\downarrow}\mathscr{A} \to \mathfrak{Codes}_{\mathscr{A}}(p)$

yielding commutativity in the diagram

$$\begin{array}{ccc} & \mathfrak{Codes}_{\mathscr{A}}(p) & \\ {\scriptstyle K_p}\nearrow & & \searrow{\scriptstyle U} \\ B{\downarrow}\mathscr{A} & \xrightarrow[p^*]{\qquad\qquad} & E{\downarrow}\mathscr{A}, \end{array}$$

where U is the evident forgetful functor.

We call

$$K_p : B{\downarrow}\mathscr{A} \to \mathfrak{Codes}_{\mathscr{A}}(p)$$

a *comparison functor* associated to the morphism p. The arrow $p : B \to E$ is said to be (an *effective*) *codescent* if the functor K_p is (an equivalence) fully faithful.

There is another way of representing descent data that involves coalgebras over the comonad associated to the adjunction $p^* \dashv p^!$.

The adjunction

$$B{\downarrow}\mathscr{A} \underset{p^!}{\overset{p^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} E{\downarrow}\mathscr{A}$$

generates a comonad $\mathbf{G}_p$ on the category $E{\downarrow}\mathscr{A}$, whose Eilenberg–Moore category of coalgebras is isomorphic to the codescent category $\mathfrak{Codes}_{\mathscr{A}}(p)$ (and thus, a codescent datum θ on (X, γ) is nothing but a $\mathbf{G}_p$-coalgebra structure on (X, γ)). This allows us to identify, modulo this isomorphism, the comparison functor $K_p : B{\downarrow}\mathscr{A} \to \mathfrak{Codes}_{\mathscr{A}}(p)$ with the comparison functor $K_{\mathbf{G}_p} : B{\downarrow}\mathscr{A} \to (E{\downarrow}\mathscr{A})_{\mathbf{G}_p}$ corresponding to the comonad $\mathbf{G}_p$. Accordingly we conclude that p is a codescent (respectively, an (effective) codescent) morphism if and only if the functor p^* is precomonadic (respectively, comonadic).

As for every adjunction, the comparison functor K_p is full and faithful if and only if the components of the unit of the adjunction $p^* \dashv p^!$ are regular monomorphisms. Since these components are given as

$$\eta_{(Y,\sigma)} = i_Y : (Y, \sigma) \to (E \sqcup_B Y, i_E p)$$

and since i_Y is a pushout of p, the morphism i_Y is a regular monomorphism in $\mathscr{A}$, and hence in $B{\downarrow}\mathscr{A}$ (recall that, by Proposition 3.1, the forgetful functor $B{\downarrow}\mathscr{A} \to \mathscr{A}$ preserves and reflects regular monomorphisms), if and only if p is a pushout-stable regular monomorphism. Hence we have the following assertion.

3.2. Proposition (see [5])**.** *A morphism in a category with pushouts is a codescent morphism if and only if it is a pushout-stable regular monomorphism.*

4. Main Result

Now we fix a locally λ-presentable category $\mathscr{A}$, where λ is a regular cardinal and consider a λ-pure morphism $p : B \to E$ in $\mathscr{A}$. Suppose that a λ-directed diagram of split monomorphisms $(p_d : B \to E_d)_{d \in \mathcal{D}}$ is a presentation of the purity of p. For any coreflexive p^*-split pair of morphisms

$$Y \underset{h}{\overset{g}{\rightrightarrows}} Z$$

in $B{\downarrow}\mathscr{A}$ and any $d \in \mathcal{D}$, write (V_d, f_d) for an equalizer of the pair of morphisms $(E_d \sqcup_B g, E_d \sqcup_B h)$.

4.1. Theorem. *A λ-pure morphism $p : B \to E$ in $\mathscr{A}$ is an effective codesent morphism if and only if for any λ-directed diagram of split monomorphisms $(p_d : B \to E_d)_{d \in \mathcal{D}}$ that represents the purity of p in $\mathscr{A}$, and for any coreflexive p^*-split pair of morphisms*

$$Y \underset{h}{\overset{g}{\rightrightarrows}} Z$$

in $B{\downarrow}\mathscr{A}$, the morphism

$$E \sqcup_B f_d : E \sqcup_B V_d \to E \sqcup_B (E_d \sqcup_B Y)$$

is a monomorphism for every $d \in \mathcal{D}$.

Proof. Assume that $(p_d : B \to E_d)_{d \in \mathcal{D}}$ is a λ-directed diagram of morphisms with connecting morphisms $(1_B, e_{d,d'}) : p_d \to p_{d'}$ for $d \leq d'$, representing the purity of p is $\mathscr{A}$. Then each p_d is a split monomorphism (say, with splitting q_d) and p is a colimit of this diagram, say, with colimit morphisms,

$$\begin{array}{ccc} B & \xrightarrow{p_d} & E_d \\ \| & & \downarrow{\scriptstyle \kappa_d} \\ B & \xrightarrow[p]{} & E \end{array}, \quad d \in \mathcal{D}.$$

Now, if p is is an effective codesent morphism, then the functor $p^* : B{\downarrow}\mathscr{A} \to E{\downarrow}\mathscr{A}$ is comonadic, and according to Theorem 1.1, it preserves equalizers of coreflexive p^*-split pairs. If $g, h : Y \to Z$ is such a pair, then it is easy to see that

$$E_d \sqcup_B Y \underset{E_d \sqcup_B h}{\overset{E_d \sqcup_B g}{\rightrightarrows}} E_d \sqcup_B Z$$

is also a coreflexive p^*-split pair. It follows that the morphism $E \sqcup_B f_d$ is an equalizer of the pair of morphisms $(E \sqcup_B (E_d \sqcup_B g), E \sqcup_B (E_d \sqcup_B h))$. Hence, in particular, $E \sqcup_B f_d$ is a monomorphism in $E{\downarrow}\mathscr{A}$ and hence in $\mathscr{A}$. This proves the necessity.

For the sufficiency, note first that, because p is a pushout-stable regular monomorphism by Proposition 2.2, $p^* : B{\downarrow}\mathscr{A} \to E{\downarrow}\mathscr{A}$ reflects isomorphisms (see, e.g., [9]). So in order to be able to apply the comonadicity criterion of Theorem 1.1 we still have to show that an equalizer of a coreflexive pair of morphisms

$$Y \underset{h}{\overset{g}{\rightrightarrows}} Z$$

in $B{\downarrow}\mathscr{A}$ is preserved by p^*, whenever an equalizer of

$$E \sqcup_B Y \underset{E \sqcup_B h}{\overset{E \sqcup_B g}{\rightrightarrows}} E \sqcup_B Z$$

in $E{\downarrow}\mathscr{A}$ is split. Since split equalizers are absolute (in the sense that they are preserved by any functor) and since, by Proposition 3.1, the underlying object functor $A{\downarrow}\mathscr{A} \to \mathscr{A}$ preserves and reflects finite limits, it is enough to show that if

$$X \xrightarrow{f} Y \underset{h}{\overset{g}{\rightrightarrows}} Z \tag{4.1}$$

is an equalizer in $\mathscr{A}$ such that

$$E \sqcup_B Y \underset{E \sqcup_B h}{\overset{E \sqcup_B g}{\rightrightarrows}} E \sqcup_B Z$$

has a split equalizer in $\mathscr{A}$, then

$$E \sqcup_B X \xrightarrow{E\sqcup_B f} E \sqcup_B Y \underset{E\sqcup_B h}{\overset{E\sqcup_B g}{\rightrightarrows}} E \sqcup_B Z$$

is also an equalizer in $\mathscr{A}$. Assume that (4.1) is such an equalizer and that the diagram

$$V \xrightarrow{\quad \overline{f} \quad} E \sqcup_B Y \underset{E\sqcup_B h}{\overset{E\sqcup_B g}{\rightrightarrows}} E \sqcup_B Z$$

is a split equalizer. Then we have the following commutative diagram:

$$\begin{array}{ccccc}
X & \xrightarrow{\quad f \quad} & Y & \underset{h}{\overset{g}{\rightrightarrows}} & Z \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle p\sqcup_B Y} & & \downarrow{\scriptstyle p\sqcup_B Z} \\
V & \xrightarrow[\quad \overline{f} \quad]{} & E \sqcup_B Y & \underset{E\sqcup_B h}{\overset{E\sqcup_B g}{\rightrightarrows}} & E \sqcup_B Z
\end{array} \tag{4.2}$$

for some $s : X \to V$.

It is not hard to see that

$$(f_d, (v_{d,d'}, e_{d,d'} \sqcup_B Z) : f_d \to f_{d'})_{d \leq d' \in \mathcal{D}}$$

is a λ-directed diagram in $\mathscr{A}^2$, where connecting morphisms $v_{d,d'} : V_d \to V_{d'}$ $(d \leq d')$ are the comparison morphisms induced by the universal property of equalizers:

$$\begin{array}{ccccc}
V_d & \xrightarrow{\quad f_d \quad} & E_d \sqcup_B Y & \underset{E_d\sqcup_B h}{\overset{E_d\sqcup_B g}{\rightrightarrows}} & E_d \sqcup_B Z \\
{\scriptstyle v_{d,d'}}\downarrow & & \downarrow{\scriptstyle e_{d,d'}\sqcup_B Y} & & \downarrow{\scriptstyle e_{d,d'}\sqcup_B Z} \\
V_{d'} & \xrightarrow[\quad f_{d'} \quad]{} & E_{d'} \sqcup_B Y & \underset{E_{d'}\sqcup_B h}{\overset{E_{d'}\sqcup_B g}{\rightrightarrows}} & E_{d'} \sqcup_B Z.
\end{array}$$

For each $d \in \mathcal{D}$, write $\iota_d : V_d \to V$ for the comparison morphism making the diagram

$$\begin{array}{ccccc}
V_d & \xrightarrow{\quad f_d \quad} & E_d \sqcup_B Y & \underset{E_d\sqcup_B h}{\overset{E_d\sqcup_B g}{\rightrightarrows}} & E_d \sqcup_B Z \\
{\scriptstyle \iota_d}\downarrow & & \downarrow{\scriptstyle \kappa_d\sqcup_B Y} & & \downarrow{\scriptstyle \kappa_d\sqcup_B Z} \\
V & \xrightarrow[\quad \overline{f} \quad]{} & E \sqcup_B Y & \underset{E\sqcup_B h}{\overset{E\sqcup_B g}{\rightrightarrows}} & E \sqcup_B Z
\end{array}$$

commutative. Since E is a λ-directed colimit of E_d's and since λ-directed colimits commute with equalizers, taking λ-directed colimit in the last diagram gives

$$\overline{f} = \varinjlim_{d \in \mathcal{D}} (f_d, (v_{d,d'}, e_{d,d'} \sqcup_B Z) : f_d \to f_{d'}).$$

Now consider the following commutative diagram:

$$\begin{array}{ccccc}
X & \xrightarrow{\quad f \quad} & Y & \underset{h}{\overset{g}{\rightrightarrows}} & Z \\
{\scriptstyle s_d}\downarrow\uparrow{\scriptstyle t_d} & & {\scriptstyle p_d\sqcup_B Y}\downarrow\uparrow{\scriptstyle q_d\sqcup_B Y} & & {\scriptstyle p_d\sqcup_B Z}\downarrow\uparrow{\scriptstyle q_d\sqcup_B Z} \\
V_d & \xrightarrow[\quad f_d \quad]{} & E_d \sqcup_B Y & \underset{E_d\sqcup_B h}{\overset{E_d\sqcup_B g}{\rightrightarrows}} & E_d \sqcup_B Z,
\end{array}$$

in which s_d and t_d are the comparison morphisms induced by the universal property of equalizers. This gives rise to a λ-directed diagram

$$(s_d, (1_X, v_{d,d'}) : s_d \to s_{d'})_{d \le d' \in \mathcal{D}}.$$

Since $\varinjlim_{d\in\mathcal{D}} f_d = \overline{f}$ and since λ-directed colimits commute with equalizers in $\mathscr{A}$, taking the λ-directed colimit in the last diagram, we obtain

$$s = \varinjlim_{d\in\mathcal{D}} (s_d, (1_X, v_{d,d'}) : s_d \to s_{d'}).$$

Since $q_d p_d = 1$, $t_d s_d = 1$, and hence each s_d is a split monomorphism, implying that s is a λ-pure morphism. Looking now at the left square in Diagram 4.2 and using that the morphism $\overline{f}$, being a split monomorphism, is λ-pure, we conclude by Proposition 2.1(1), (2) that f is also λ-pure.

Next, since λ-pure morphisms are stable under pushout, the morphism $E \sqcup_B f$ in the following commutative diagram:

$$\begin{array}{ccc} E \sqcup_B X & \xrightarrow{E\sqcup_B f} & E \sqcup_B Y \\ {\scriptstyle E\sqcup_B s_d}\big\downarrow \big\uparrow{\scriptstyle E\sqcup_B t_d} & & {\scriptstyle E\sqcup_B (p_d \sqcup_B Y)}\big\downarrow \big\uparrow{\scriptstyle E\sqcup_B (q_d \sqcup_B Y)} \\ E \sqcup_B V_d & \xrightarrow[E\sqcup_B f_d]{} & E \sqcup_B (E_d \sqcup_B Y), \end{array}$$

is also λ-pure and thus a (regular) monomorphism (see Proposition 2.1). Now, since the morphism $E \sqcup_B f_d$ is also a monomorphism by hypothesis, it follows from [4, Lemma 10] that the square

$$\begin{array}{ccc} E \sqcup_B X & \xrightarrow{E\sqcup_B f} & E \sqcup_B Y \\ {\scriptstyle E\sqcup_B s_d}\big\downarrow & & \big\downarrow{\scriptstyle E\sqcup_B (p_d \sqcup_B Y)} \\ E \sqcup_B V_d & \xrightarrow[E\sqcup_B f_d]{} & E \sqcup_B (E_d \sqcup_B Y) \end{array}$$

is a pullback. Since

$$\varinjlim_{d\in\mathcal{D}} s_d = s, \quad \varinjlim_{d\in\mathcal{D}} f_d = \overline{f}$$

and hence

$$\varinjlim_{d\in\mathcal{D}} (E \sqcup_B s_d) = E \sqcup_B s, \quad \varinjlim_{d\in\mathcal{D}} (E \sqcup_B f_d) = E \sqcup_B \overline{f},$$

and since pullbacks commute with λ-directed colimits in $\mathscr{A}$, the diagram

$$\begin{array}{ccc} E \sqcup_B X & \xrightarrow{E\sqcup_B f} & E \sqcup_B Y \\ {\scriptstyle E\sqcup_B s}\big\downarrow & & \big\downarrow{\scriptstyle E\sqcup_B (p \sqcup_B Y)} \\ E \sqcup_B V & \xrightarrow[E\sqcup_B \overline{f}]{} & E \sqcup_B (E \sqcup_B Y) \end{array}$$

is also a pullback. Applying now [19, Lemma 2.3] to the commutative diagram

$$\begin{array}{ccccc}
E \sqcup_B X & \xrightarrow{E\sqcup_B f} & E \sqcup_B Y & \overset{E\sqcup_B g}{\underset{E\sqcup_B h}{\rightrightarrows}} & E \sqcup_B Z \\
\Big\downarrow{\scriptstyle E\sqcup_B s} & & \Big\downarrow{\scriptstyle E\sqcup_B (p\sqcup_B Y)} & & \Big\downarrow{\scriptstyle E\sqcup_B (p\sqcup_B Z)} \\
E \sqcup_B V & \xrightarrow[E\sqcup_B \overline{f}]{} & E \sqcup_B (E \sqcup_B Y) & \overset{E\sqcup_B (E\sqcup_B g)}{\underset{E\sqcup_B (E\sqcup_B h)}{\rightrightarrows}} & E \sqcup_B (E \sqcup_B Z),
\end{array}$$

in which the bottom row is a (split) equalizer, one concludes that the top row is also an equalizer diagram, as desired. Thus the functor p^* is comonadic, and hence p is an effective codescent morphism.

□

We call a morphism $p : B \to E$ in $\mathscr{A}$ *weakly flat* if the change-of-cobase functor $p^* = E \sqcup_B - : B{\downarrow}\mathscr{A} \to E{\downarrow}\mathscr{A}$ takes regular monomorphisms into monomorphisms.

As an immediate consequence of Theorem 4.1, we obtain the following assertion.

4.2. Theorem. *In a locally λ-presentable category, any λ-pure and weakly flat morphism is an effective codesent morphism.*

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B. Mesablishvili
A. Razmadze Mathematical Institute,
I. Javakhishvili Tbilisi State University,
Tbilisi Centre for Mathematical Sciences, Tbilisi, Georgia
E-mail: `bachi@rmi.acnet.ge`