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# Generalised bialgebras and entwined monads and comonads

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### Abstract

J.-L. Loday has defined generalised bialgebras and proved structure theorems in this setting which can be seen as general forms of the Poincaré–Birkhoff–Witt and the Cartier–Milnor–Moore theorems. It was observed by the present authors that parts of the theory of generalised bialgebras are special cases of results on entwined monads and comonads and the corresponding mixed bimodules. In this article the Rigidity Theorem of J.-L. Loday is extended to this more general categorical framework.

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### Introduction

The introduction of *entwining structures* between an algebra and a coalgebra by T. Brzeziński and S. Majid in [2] opened new perspectives in the mathematical treatment of quantum principal bundles. It turned out that these structures are special cases of distributive laws treated in J. Beck's paper [1]. The latter were also used by D. Turi and G. Plotkin [16] in the context of operational semantics.

These observations led to a revival of the investigation of various forms of distributive laws. In a series of papers [13], [14], [15] it was shown how they allow for formulating the theory of Hopf algebras and Galois extensions in a general categorical setting.

On the other hand, *generalised bialgebras* as defined in J.-L. Loday [8, Section 2.1], are vector spaces which are algebras over an *operad*  $\mathcal{A}$  and coalgebras over a *cooperad*  $\mathcal{C}$ . Moreover, the operad  $\mathcal{A}$  and the cooperad  $\mathcal{C}$  are required to be related by a distributive law. Since any operad  $\mathcal{A}$  yields a monad  $\mathcal{T}_{\mathcal{A}}$  and  $\mathcal{A}$ -algebras are nothing but  $\mathcal{T}_{\mathcal{A}}$ -modules, and similarly any cooperad  $\mathcal{C}$  yields a comonad  $\mathcal{G}_{\mathcal{C}}$  and  $\mathcal{C}$ -coalgebras are nothing but  $\mathcal{G}_{\mathcal{C}}$ -comodules, generalised bialgebras have interpretations in terms of bimodules over a bimonad in the sense of [14].

The purpose of the present paper is to make this relationship more precise (as proposed in [14, 2.3]). On the one hand, given a monad  $\mathcal{T}$  and a comonad  $\mathcal{G}$  on a category  $\mathbb{A}$  together with a mixed distributive law  $\lambda$  between them, we provide in Theorem 4.1 conditions for which a functor from  $\mathbb{A}$  to the category of  $TG$ -bimodules in  $\mathbb{A}$  is an equivalence of categories. In Theorem 5.8 we concentrate on the special case when the monad and the comonad share the same underlying functor. These are general theorems, which do not depend on the shape of the functors underlying the monads and comonads. On the other hand, when considering operads and cooperads the functors involved are analytic in the sense of A. Joyal [7] and, in particular, are graded and connected. In Section 6, we show that the general theory together with the grading lead to the Rigidity Theorem [8, 2.5.1] as a special case (without using idempotents as in [8]). In Corollary 6.9 we focus on the special case when the operad and cooperad share the same underlying analytic functor and prove (under mild conditions) that the category of generalised bialgebras is equivalent to the category of vector spaces.

## Section snippets

### Comodules and adjoint functors

In this section we provide basic notions and properties of comodule functors and adjoint pairs of functors. Throughout the paper  $\mathbb{A}$  and  $\mathbb{B}$  will denote any categories.

#### 2.1. Monads and comonads

Recall that a *monad*  $\mathcal{T}$  on  $\mathbb{A}$  is a triple  $(T, m, e)$  where  $T : \mathbb{A} \rightarrow \mathbb{A}$  is a functor with natural transformations  $m : TT \rightarrow T$ ,  $e : \mathbf{1} \rightarrow T$  satisfying associativity and unitality conditions. A  *$\mathcal{T}$ -module* is an object  $a \in \mathbb{A}$  with a morphism  $h : T(a) \rightarrow a$  subject to associativity and unitality conditions. The (Eilenberg–Moore) category of  $\mathcal{T}$ -modules is denoted by  $\mathbb{A}_{\mathcal{T}}$  and...

...

## Distributive laws

Distributive laws were introduced by J. Beck in [1]. Here we are mainly interested in the following case (e.g. [6] or [17, 5.3]).

### 3.1. Mixed distributive laws

Let  $\mathcal{T} = (T, m, e)$  be a monad and  $\mathcal{G} = (G, \delta, \varepsilon)$  a comonad on the category  $\mathbb{A}$ . A natural transformation  $\lambda : TG \rightarrow GT$  is said to be a *mixed distributive law* or a (*mixed*) *entwining* provided it induces commutative diagrams

Recall (for example, from [18]) that if  $\mathcal{T}$  is a monad and  $\mathcal{G}$  is a comonad on a category  $\mathbb{A}$ , then there are bijective correspondences between

- mixed distributive laws  $\lambda : TG \rightarrow GT$ ;...

...

...

## Grouplike morphisms

Let  $\mathcal{G} = (G, \delta, \varepsilon)$  be a comonad on a category  $\mathbb{A}$ . By [13, Definition 3.1], a natural transformation  $g : \mathbf{1} \rightarrow G$  is called a *grouplike morphism* provided it is a comonad morphism from the identity comonad to  $\mathcal{G}$ , that is, it induces commutative diagrams

The dual notion is that of an *augmentation of a monad*  $\mathcal{T} = (T, m, e)$  on  $\mathbb{A}$ , that is, a monad morphism  $T \rightarrow \mathbf{1}$ .

Let  $\mathcal{T} = (T, m, e)$  and  $\mathcal{G} = (G, \delta, \varepsilon)$  be given on  $\mathbb{A}$  with an entwining  $\lambda : TG \rightarrow GT$ . If  $\mathcal{G}$  has a grouplike morphism  $g : \mathbf{1} \rightarrow G$ , then the above conditions guarantee that the morphisms ( $g$ ...

## Compatible entwining

Let  $\underline{\mathcal{H}} = (H, m, e)$  be a monad,  $\overline{\mathcal{H}} = (H, \delta, \varepsilon)$  a comonad on  $\mathbb{A}$ , and let  $\lambda : HH \rightarrow HH$  be an entwining from the monad  $\underline{\mathcal{H}}$  to the comonad  $\overline{\mathcal{H}}$ . The datum  $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$  is called a *monad-comonad triple*. The objects of the category  $\mathbb{A}_{\underline{\mathcal{H}}, \overline{\mathcal{H}}}^{\lambda}$  are called (*mixed*)  $\lambda$ -*bimodules*.

### 5.1. Lemma

The triple  $(H(a), m_a, \delta_a)$  is a  $\lambda$ -bimodule for all  $a \in \mathbb{A}$  if and only if we have a commutative diagram In this case, there is a functor  $K : \mathbb{A} \rightarrow (\mathbb{A}_{\mathcal{H}})^{\widehat{\mathcal{H}}}$ ,  $a \mapsto ((H(a), m_a), \delta_a)$ , satisfying  $\phi_{\mathcal{H}} = U^{\widehat{\mathcal{H}}} K \dots$

## 5.2. Definitions

Given a monad–comonad triple  $(\mathcal{H}, \overline{\mathcal{H}}, \lambda)$ , the entwining  $\lambda : HH \rightarrow HH$  is said to...

...

## Generalised bialgebras

In this section, we apply our results in the context of operads to recover results of J.-L. Loday on generalised bialgebras in [8]. The Leitmotiv of the section is that a (co)operad is a particular type of (co)monad. Let  $\mathbf{k}$  denote a field and  $\mathbb{A}$  the category of  $\mathbf{k}$ -vector spaces. For more details on operads, we refer to [10].

### 6.1. Schur functors

An  $\mathbb{S}$ -module  $\mathcal{M}$  in  $\mathbb{A}$  (or *vector species*) is a collection of objects  $\mathcal{M}(n)$ , for  $n \geq 0$ , together with an action of the symmetric group  $\mathbb{S}_n$ . To an  $\mathbb{S}$ -module  $\mathcal{M}$  one associates the functor  $F_{\mathcal{M}} \dots$

...

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