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The Fundamental Theorem for weak braided bimonads

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ABSTRACT

The theories of (Hopf) bialgebras and weak (Hopf) bialgebras have been introduced for vector space categories over fields and make heavily use of the tensor product. As first generalisations, these notions were formulated for monoidal categories, with braidings if needed. The present authors developed a theory of bimonads and Hopf monads H on arbitrary categories $\mathbb{A}$, employing distributive laws, allowing for a general form of the Fundamental Theorem for Hopf algebras. For τ-bimonads H, properties of braided (Hopf) bialgebras were captured by requiring a Yang–Baxter operator $\tau : HH \to HH$. The purpose of this paper is to extend the features of weak (Hopf) bialgebras to this general setting including an appropriate form of the Fundamental Theorem. This subsumes the theory of braided Hopf algebras (based on weak Yang–Baxter operators) as considered by Alonso Álvarez and others.

Contents

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Introduction

There are various generalisations of the notions of (weak) bialgebras and Hopf algebras in the literature, mainly for (braided) monoidal categories, and we refer to Böhm [6], the introductions to Alonso Álvarez e.a. [2], Böhm e.a. [9], and [13, Remarks 36.18] for more information about these.

Bimonads and Hopf monads on *arbitrary* categories were introduced in [23] and the purpose of the present paper is to develop a *weak version* of these notion, that is, the initial conditions on the behaviour of the involved distributive laws towards unit and counit are replaced by weaker conditions.

Recall that for a bialgebra $(H, m, e, \delta, \varepsilon)$ over a commutative ring k, there is a commutative diagram $(\otimes_k = \otimes)$

$$\begin{tikzcd} \mathbb{M} \arrow[rr, "-\otimes H"] \arrow[dr, "\phi_H"'] & & \mathbb{M}_H^H \arrow[dl, "U^{\widetilde{H}}"] \\ & \mathbb{M}_H & \end{tikzcd} ,\qquad \begin{tikzcd} M \arrow[r, mapsto] \arrow[dr, mapsto] & (M\otimes H, M\otimes m, M\otimes \delta) \arrow[d, mapsto] \\ & (M\otimes H, M\otimes m)\,, \end{tikzcd}$$

where $\mathbb{M}$ is the category of k-modules, $\mathbb{M}_H$ the category of right H-modules, and $\mathbb{M}_H^H$ denotes the category of mixed bimodules; the latter can also be considered as $(\mathbb{M}_H)^{\widetilde{H}}$, that is, the category of $\widetilde{H}$-comodules over $\mathbb{M}_H$ where $\widetilde{H}$ is the lifting of the comonad $-\otimes H$ to $\mathbb{M}_H$. H is a Hopf algebra provided the functor $-\otimes H$ is an equivalence of categories (Fundamental Theorem of Hopf modules).

Concentrating on the essential parts of this setting, we consider, for any category $\mathbb{A}$, the diagram

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "K"] \arrow[dr, "\phi_{\mathsf{T}}"'] & & (\mathbb{A}_{\mathsf{T}})^{\mathsf{G}} \arrow[dl, "U^{\mathsf{G}}"] \\ & \mathbb{A}_{\mathsf{T}}, & \end{tikzcd} \tag{0.1}$$

where T is some monad on $\mathbb{A}$, G is some comonad on the category $\mathbb{A}_{\mathsf{T}}$ of T-modules, ϕ_{T} and U^{G} denote the respective free and forgetful functors, and K is any functor making the diagram commutative.

Having such a diagram, one may ask when the functor K allows for a right adjoint $\overline{K}$. If such is the case, we have a monad P on $\mathbb{A}$, a monad morphism $\iota : P \to T$, the free functor $\phi_{\mathsf{P}} : \mathbb{A} \to \mathbb{A}_{\mathsf{P}}$, and the Eilenberg–Moore comparison functor $K_{\mathsf{P}} : (\mathbb{A}_{\mathsf{T}})^{\mathsf{G}} \to \mathbb{A}_{\mathsf{P}}$ for the monad P.

If $\mathbb{A}$ is Cauchy complete and P is a separable Frobenius monad, then the change-of-base-functor $\iota_! : \mathbb{A}_{\mathsf{P}} \to \mathbb{A}_{\mathsf{T}}$ exists (see Proposition 2.2). As a consequence, K_{P} has a left adjoint L_{P} (Proposition 3.6) leading to the commutative diagram

$$\begin{tikzcd} \mathbb{A} \arrow[r, "\phi_{\mathsf{P}}"'] \arrow[rr, bend left, "K"] \arrow[dr, "\phi_{\mathsf{T}}"'] & \mathbb{A}_{\mathsf{P}} \arrow[r, dotted, "L_{\mathsf{P}}"'] \arrow[d, dotted, "\iota_!"'] & (\mathbb{A}_{\mathsf{T}})^{\mathsf{G}} \arrow[dl, "U^{\mathsf{G}}"] \\ & \mathbb{A}_{\mathsf{T}} & \end{tikzcd} \quad . \tag{0.2}$$

Essentially, a Fundamental Theorem should describe the existence of a right (left) adjoint to the functor K and, eventually, an equivalence between the categories $\mathbb{A}_{\mathsf{P}}$ and $(\mathbb{A}_{\mathsf{T}})^{\mathsf{G}}$. This, evidently, requires to study the functors ϕ_{P} and K_{P}.

For a bimonad H on $\mathbb{A}$ with weak monad–comonad entwining $\omega : HH \to HH$, a comparison functor $K_\omega : \mathbb{A} \to \mathbb{A}^H_H(\omega)$ (the category of mixed bimodules) is considered. The existence of left and right adjoints for K_ω is described by equalisers and coequalisers of certain pairs of morphisms, respectively. *Weak braided bimonads* are defined by the existence of a monad–comonad entwining $\omega : HH \to HH$ as well as a comonad–monad entwining $\overline{\omega} : HH \to HH$. Referring to both of them, an antipode is defined and if it exists, one gets *weak braided Hopf monads*. For these monads, the Fundamental Theorem is proved.

After assembling preliminaries in Section 1 and properties of separable Frobenius monads in Section 2, the theory just sketched is outlined in Section 3.

Section 4 deals with the application of this to endofunctors H on a category $\mathbb{A}$ endowed with a monad $\underline{H}$ as well as a comonad structure $\overline{H}$, and a weak monad–comonad entwining $\omega : HH \to HH$. Exploiting ideas from [33] and Böhm [5], these data allow for the definition of a comonad G on $\mathbb{A}_H$ as well as a monad T on $\mathbb{A}^H$ (Propositions 4.3, 4.5). As a result, the category $\mathbb{A}^{\overline{\mathsf{H}}}_{\underline{\mathsf{H}}}(\omega)$ of mixed H-bimodules is isomorphic to the categories $(\mathbb{A}_H)^{\mathsf{G}}$ and $(\mathbb{A}^H)_{\mathsf{T}}$ (see Theorems 4.4, 4.6). If ω is a *compatible* entwining (i.e. $\delta \cdot m = Hm \cdot \omega H \cdot H\delta$, Section 4.1), there is a functor

$$K_\omega : \mathbb{A} \to \mathbb{A}^{\overline{\mathsf{H}}}_{\underline{\mathsf{H}}}(\omega), \quad a \ \mapsto \ (H(a), m_a, \delta_a),$$

leading to commutativity of the diagram corresponding to (0.1). Conditions for the existence of a right and a left adjoint functor for K_ω are investigated (Propositions 4.10, 4.11). These problems were considered in [23] for proper compatible entwinings ω.

In Section 5, we define *weak τ-bimonads*, also called *weak braided bimonads* (Definition 5.2), based on a *weak Yang–Baxter operator* $\tau : HH \to HH$ (Section 5.1). This type of operator was introduced by Alonso Álvarez e.a. in [1] for monoidal categories and is here adapted to the more general setting.

The conditions on weak braided bimonads induce a weak monad–comonad entwining $\omega : HH \to HH$ as well as a weak comonad–monad entwining $\overline{\omega} : HH \to HH$ and allow to refine the results in Section 4: the natural transformation $\overline{\xi} := H\varepsilon \cdot \overline{\omega} \cdot He : H \to H$ is idempotent and its splitting yields a separable Frobenius monad $H^{\overline{\xi}}$ (see Proposition 5.11). Then, if K_ω has a right adjoint functor $\overline{K}$, the induced monad P is just $H^{\overline{\xi}}$ and the diagram corresponding to (0.2) can be completed.

The *weak bialgebras* over a commutative ring k as considered by Böhm e.a. in [8] are weak braided bimonads in our sense (τ the ordinary twist, $\overline{\xi} = \varepsilon_s$) and for this case some of our results are shown there, including the Frobenius and separability property of $H^{\overline{\xi}}(= H_s)$ ([8, Proposition 4.4]).

Eventually, in Section 6, *weak braided Hopf monads* are defined as weak braided bimonads H with an *antipode* (Definition 6.1). In monoidal categories, these correspond to the weak braided Hopf algebras considered in [2,3].

We show that for a braided bimonad, in Cauchy complete categories, the existence of an antipode is equivalent to the functor K_ω having a left adjoint and a monadic right adjoint and this leads to an equivalence between the categories of $H^{\overline{\xi}}$-modules and $\mathbb{A}_{\underline{\mathsf{H}}}^{\overline{\mathsf{H}}}(\omega)$ (Fundamental Theorem 6.6).

Examples for our weak braided bimonads and weak braided Hopf monads are the *weak braided Hopf algebras* in strict monoidal Cauchy complete categories considered by Alonso Álvarez et al. in [1–3], the *weak bimonoids* and *weak Hopf monoids* in braided monoidal categories as defined by Pastro et. al. in [25] and also showing up in [9, Sections 3, 4]. These all subsume the braided Hopf algebras considered, e.g., by Takeuchi [31] and Schauenburg [26] and, of course, the weak Hopf algebras in vector space categories introduced by Böhm et al. in [10]. Moreover, we generalise the *bimonads* and *τ-Hopf monads* defined on arbitrary categories in [23] and these include, for example, bimonoids in duoidal categories (e.g. [24]).

Opmonoidal monads $\mathsf{T} = (T, m, e)$ on strict monoidal categories $(\mathbb{V}, \otimes, \mathbb{I})$ were also called *bimonads* by Bruguières et al. in [12] and these were generalised to *weak bimonads* in monoidal categories by Böhm et al. in [9]. As pointed out in [22, Section 5], the bimonads from [12] yield a special case of an entwining of the monad T with the comonad $- \otimes T(\mathbb{I})$, where $T(\mathbb{I})$ has a coalgebra structure derived from the opmonoidality of T. To transfer the structures from [9] to arbitrary categories, one has to consider *weak entwinings* between monads and comonads. It is planned to elaborate details for this in a subsequent article.

1. Preliminaries

1.1. Monads and comonads. Recall that a *monad* T on a category $\mathbb{A}$ (or shortly an $\mathbb{A}$-monad T) is a triple (T, m, e) where $T : \mathbb{A} \to \mathbb{A}$ is a functor with natural transformations $m : TT \to T$, $e : 1 \to T$, satisfying the usual associativity and unitality conditions. A T-*module* is an object $a \in \mathbb{A}$ with a morphism $h : T(a) \to a$ subject to associativity and unitality conditions. The (Eilenberg–Moore) category of T-modules is denoted by $\mathbb{A}_{\mathsf{T}}$ and there is an adjunction

$$e_{\mathsf{T}}, \varepsilon_{\mathsf{T}} : \phi_{\mathsf{T}} \dashv U_{\mathsf{T}} : \mathbb{A}_{\mathsf{T}} \to \mathbb{A},$$

with $\phi_{\mathsf{T}} : \mathbb{A} \to \mathbb{A}_{\mathsf{T}}$ and $U_{\mathsf{T}} : \mathbb{A}_{\mathsf{T}} \to \mathbb{A}$ given by the respective assignments

$$a \mapsto (T(a), m_a) \;\text{ and }\; (a, h) \mapsto a,$$

while $e_{\mathsf{T}} = e$ and $(\varepsilon_{\mathsf{T}})_{(a,\,h)} = h$ for each $(a, h) \in \mathbb{A}_{\mathsf{T}}$.

If $\mathsf{T} = (RF, R\varepsilon F, \eta)$ is the monad generated on $\mathbb{A}$ by an adjoint pair $\eta, \varepsilon \colon F \dashv R \colon \mathbb{B} \to \mathbb{A}$, then there is the *comparison functor* $K_{\mathsf{T}} : \mathbb{B} \to \mathbb{A}_{\mathsf{T}}$ which assigns to each object $b \in \mathbb{B}$ the T-algebra $(R(b), R(\varepsilon_b))$, and to each morphism $f : b \to b'$ the morphism $R(f) : R(b) \to R(b')$, satisfying $U_{\mathsf{T}} K_{\mathsf{T}} = R$ and $K_{\mathsf{T}} F = \phi_{\mathsf{T}}$. This situation is illustrated by the diagram

$$\begin{tikzcd}
\mathbb{B} \arrow[rr, "K_{\mathsf{T}}"] \arrow[dr, shift left, "R"] & & \mathbb{A}_{\mathsf{T}} \arrow[dl, shift right, "U_{\mathsf{T}}"'] \\
& \mathbb{A} \arrow[ul, shift left, "F"] \arrow[ur, shift right, "\phi_{\mathsf{T}}"'] &
\end{tikzcd}.$$

The functor R is called *monadic* (resp. *premonadic*) if the comparison functor K_{T} is an equivalence of categories (resp. full and faithful).

1.2. Theorem. (Beck [4]) *Let $\eta, \varepsilon \colon F \dashv R \colon \mathbb{B} \to \mathbb{A}$ be an adjunction, and $\mathsf{T} = (RF, R\varepsilon F, \eta)$ the corresponding monad on $\mathbb{A}$.*

(1) *The comparison functor $K_T : \mathbb{B} \to \mathbb{A}_T$ has a left adjoint $L_T : \mathbb{A}_T \to \mathbb{B}$ if and only if for each $(a, h) \in \mathbb{A}_T$, the pair of morphisms $(F(h), \varepsilon_{F(a)})$ has a coequaliser in $\mathbb{B}$.*
(2) *R is monadic if and only if it is conservative and, for any $(a, h) \in \mathbb{A}_T$, the pair of morphisms $(F(h), \varepsilon_{F(a)})$ has a coequaliser and this coequaliser is preserved by R.*

Dually, a *comonad* G on $\mathbb{A}$ (or shortly an $\mathbb{A}$-comonad G) is a triple (G, δ, ε) where $G : \mathbb{A} \to \mathbb{A}$ is a functor with natural transformations $\delta : G \to GG$, $\varepsilon : G \to 1$, and G-*comodules* are objects $a \in \mathbb{A}$ with morphisms $\theta : a \to G(a)$. Both notions are subject to coassociativity and counitality conditions. The (Eilenberg–Moore) category of G-comodules is denoted by $\mathbb{A}^{\mathsf{G}}$ and there is a cofree functor

$$\phi^{\mathsf{G}} : \mathbb{A} \to \mathbb{A}^{\mathsf{G}},\ a \mapsto (G(a), \delta_a),$$

which is right adjoint to the forgetful functor

$$U^{\mathsf{G}} : \mathbb{A}^{\mathsf{G}} \to \mathbb{A},\ (a, \theta) \mapsto a.$$

If $\eta, \varepsilon : F \dashv R \colon \mathbb{A} \to \mathbb{B}$ is an adjoint pair and $\mathsf{G} = (FR, F\eta R, \varepsilon)$ is the comonad on $\mathbb{A}$ associated to (R, F), then one has the comparison functor

$$K^{\mathsf{G}} : \mathbb{B} \to \mathbb{A}^{\mathsf{G}},\ b \to (F(b), F(\eta_b))$$

for which $U^{\mathsf{G}} \cdot K^{\mathsf{G}} = F$ and $K^{\mathsf{G}} \cdot R = \phi^{\mathsf{G}}$. One says that the functor F is *precomonadic* if K^{G} is full and faithful, and it is *comonadic* if K^{G} is an equivalence of categories.

1.3. Cauchy completeness. A morphism $e : A \to A$ in $\mathbb{A}$ is *idempotent* if $e^2 = e$ and $\mathbb{A}$ is said to be *Cauchy complete* if idempotents split in A in the sense that for every idempotent $e : a \to a$, there exists an object $\overline{a}$ and morphisms $p : a \to \overline{a}$ and $i : \overline{a} \to a$ such that $ip = e$ and $pi = 1_{\overline{a}}$. In this case, $(\overline{a}, i)$ is the equaliser of e and 1_a and $(\overline{a}, p)$ is the coequaliser of e and 1_a. Hence any category admitting either equalisers or coequalisers is Cauchy complete.

1.4. Proposition. *Let G be a comonad on $\mathbb{A}$. If $\mathbb{A}$ is Cauchy complete, then so is $\mathbb{A}^{\mathsf{G}}$. Moreover, the forgetful functor $U^{\mathsf{G}} : \mathbb{A}^{\mathsf{G}} \to \mathbb{A}$ preserves and creates splitting of idempotents. Explicitly, if $e : (a, \theta) \to (a, \theta)$ is an idempotent morphism in $\mathbb{A}^{\mathsf{G}}$ and if $a \xrightarrow{p} \overline{a} \xrightarrow{i} a$ is a splitting of e in $\mathbb{A}$, then $(\overline{a}, G(p) \cdot \theta \cdot i)$ is a G-comodule in such a way that p and i become morphisms of G-comodules. Similarly, if T is a monad on $\mathbb{A}$, then the forgetful functor $U_{\mathsf{T}} : \mathbb{A}_{\mathsf{T}} \to \mathbb{A}$ preserves and creates splitting of idempotents.*

Proof. The result follows from the fact that the forgetful functor $U^{\mathsf{G}} : \mathbb{A}^{\mathsf{G}} \to \mathbb{A}$ preserves and creates coequalisers, while the functor $U_{\mathsf{T}} : \mathbb{A}_{\mathsf{T}} \to \mathbb{A}$ preserves and creates equalisers. $\square$

1.5. Split (co)equalisers. Recall (e.g. from [19]) that a diagram

$$a \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} a' \xrightarrow[p]{\quad} x \tag{1.1}$$

with $p\partial_0 = p\partial_1$ is said to be *split* by a pair of morphisms $i : x \to a'$ and $s : a' \to a$ if $pi = 1_x$, $\partial_0 s = 1_{a'}$ and $\partial_1 s = ip$. Then p is an (absolute) coequaliser (preserved by any functor).

A pair of morphisms $(\partial_0, \partial_1 : a \rightrightarrows a')$ in $\mathbb{A}$ is called *split* if there exists a morphism $s : a' \to a$ with $\partial_0 s = 1$ and $\partial_1 s \partial_0 = \partial_1 s \partial_1$. In this case, $\partial_1 s : a' \to a'$ is an idempotent, and if we assume $\mathbb{A}$ to be Cauchy complete and if $a' \xrightarrow{p} x \xrightarrow{i} a'$ is a splitting of the idempotent $\partial_1 s$, then the diagram

$$\begin{tikzcd} a \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & a' \arrow[l, bend right, "s"'] \arrow[r, "p"'] & x \arrow[l, bend right, "i"'] \end{tikzcd}$$

is a split coequaliser. Conversely, if the above diagram is a split coequaliser, then s makes the pair $(\partial_0, \partial_1 : a \rightrightarrows a')$ split. Thus, when $\mathbb{A}$ is Cauchy complete, a pair $(\partial_0, \partial_1 : a \rightrightarrows a')$ is part of a split coequaliser diagram if and only if it is split.

If $F : \mathbb{A} \to \mathbb{B}$ is a functor, $(\partial_0, \partial_1 : a \rightrightarrows a')$ is called *F-split* if the pair of morphisms $(F(\partial_0), F(\partial_1) : F(a) \rightrightarrows F(a'))$ in $\mathbb{B}$ is split. The dual notions are those of *cosplit pairs* and *F-cosplit pairs.*

Given a monad T (resp. comonad G) on $\mathbb{A}$ and a category $\mathbb{X}$, one may consider the functor category $[\mathbb{X}, \mathbb{A}]$ and the induced monad $[\mathbb{X}, \mathsf{T}]$ (resp. comonad $[\mathbb{X}, \mathsf{G}]$) thereon, whose functor part sends a functor $F : \mathbb{X} \to \mathbb{A}$ to the composite $TF : \mathbb{X} \to \mathbb{A}$ (resp. $GF : \mathbb{X} \to \mathbb{A}$). Symmetrically, one has the induced monad $[\mathsf{T}, \mathbb{X}]$ (resp. comonad $[\mathsf{G}, \mathbb{X}]$) on $[\mathbb{A}, \mathbb{X}]$, whose functor-part sends $F' : \mathbb{A} \to \mathbb{X}$ to $F'T : \mathbb{A} \to \mathbb{X}$ (resp. $FG : \mathbb{A} \to \mathbb{X}$).

1.6. (Bi)module functors. Given a monad $\mathsf{T} = (T, m, e)$ on $\mathbb{A}$ and a category $\mathbb{X}$, a *left* T*-module with domain* $\mathbb{X}$ is an object of the Eilenberg–Moore category $[\mathbb{X}, \mathbb{A}]_{[\mathbb{X},\mathsf{T}]}$ of the monad $[\mathbb{X}, \mathsf{T}]$. Thus, a left T-module with domain $\mathbb{X}$ is a functor $M : \mathbb{X} \to \mathbb{A}$ together with a natural transformation $\varrho : TM \to M$, called the *action* (or T*-action*) on M, such that $\varrho \cdot eM = 1$ and $\varrho \cdot T\varrho = \varrho \cdot mM$. A morphism of left T-modules with domain $\mathbb{X}$ is a natural transformation in $[\mathbb{X}, \mathbb{A}]$ that commutes with the T-actions.

Similarly, for a category $\mathbb{Y}$, the category of *right* T*-modules with codomain* $\mathbb{Y}$ is defined as the Eilenberg–Moore category $[\mathbb{A}, \mathbb{Y}]_{[\mathsf{T},\mathbb{Y}]}$ of the monad $[\mathsf{T}, \mathbb{Y}]$.

Let S be another monad on $\mathbb{A}$. A (T, S)*-bimodule* is a functor $N : \mathbb{A} \to \mathbb{A}$ equipped with two natural transformations $\varrho_l : TN \to N$ and $\varrho_r : NS \to N$ such that $(N, \varrho_l) \in [\mathbb{X}, \mathbb{A}]_{[\mathbb{X},\mathsf{T}]}$, $(N, \varrho_r) \in [\mathbb{A}, \mathbb{Y}]_{[\mathsf{T},\mathbb{Y}]}$ and $\varrho_r \cdot \varrho_l S = \varrho_l \cdot T\varrho_r$. A morphism of (T, S)-bimodules is a morphism of left T-modules which is simultaneously a morphism of right S-modules. We write $[\mathbb{X}, \mathbb{A}]_{[\mathsf{S},\mathsf{T}]}$ for the corresponding category.

1.7. Canonical modules. Let $\mathsf{T} = (T, m, e)$ be an arbitrary monad on $\mathbb{A}$. Using the associativity and unitality of the multiplication m, we find that for any functor $M : \mathbb{X} \to \mathbb{A}$, the pair (TM, mM) is a left T-module. Moreover, if $\nu : M \to M'$ is a natural transformation, then $T\nu : TM \to TM'$ is a morphism of left T-modules.

Symmetrically, for any functor $N : \mathbb{A} \to \mathbb{Y}$, the pair (NT, Nm) is a right T-module, and if $\nu : N \to N'$ is a natural transformation, then $\nu T : NT \to NT'$ is a morphism of right T-modules. In particular, (T, m) can be regarded as a right as well as a left

T-module; again by the associativity of m, (T, m, m) is a (T, T)-bimodule. Moreover, if S is another monad on $\mathbb{A}$ and $\iota : \mathsf{S} \to \mathsf{T}$ is a monad morphism, then

(i) $(T, ST \xrightarrow{\iota T} TT \xrightarrow{m} T)$ is a left S-module;
(ii) $(T, TS \xrightarrow{T\iota} TT \xrightarrow{m} T)$ is a right S-module;
(iii) $(T, TT \xrightarrow{m} T, TS \xrightarrow{T\iota} TT \xrightarrow{m} T)$ is a (T, S)-bimodule;
(iv) $(T, ST \xrightarrow{\iota T} TT \xrightarrow{m} T, TT \xrightarrow{m} T)$ is an (S, T)-bimodule;
(v) $(T, ST \xrightarrow{\iota T} TT \xrightarrow{m} T, TS \xrightarrow{T\iota} TT \xrightarrow{m} T)$ is an (S, S)-bimodule;
(vi) $(\phi_{\mathsf{T}}, \varepsilon_{\mathsf{T}}\phi_{\mathsf{T}} \cdot \phi_{\mathsf{T}}\iota)$ is a right S-module;
(vii) $(U_{\mathsf{T}}, U_{\mathsf{T}}\varepsilon_{\mathsf{T}} \cdot \iota U_{\mathsf{T}})$ is a left S-module.

1.8. Tensor product of module functors. Let T be a monad on $\mathbb{A}$ and $\mathbb{X}$ and $\mathbb{Y}$ arbitrary categories. If (N, ρ) is a left T-module and (M, ϱ) a right T-module, then their *tensor product* (*over* T) is defined as the object part of the coequaliser

$$MTN \underset{M\rho}{\overset{\varrho N}{\rightrightarrows}} MN \xrightarrow{\mathrm{can}_{\mathsf{T}}^{M,N}} M \otimes_{\mathsf{T}} N \tag{1.2}$$

in $[\mathbb{X}, \mathbb{Y}]$, provided that such a coequaliser exists. We often abbreviate $\mathrm{can}_{\mathsf{T}}^{M,N}$ to $\mathrm{can}^{M,N}$, or even to can.

1.9. Proposition. *Let S, T be monads on a category $\mathbb{A}$. If the modules $(M, \varrho) \in [\mathbb{A}, \mathbb{X}]_{[\mathsf{T},\mathbb{X}]}$ and $(N, \rho_l, \rho_r) \in [\mathbb{A}, \mathbb{A}]_{[\mathsf{S},\mathsf{T}]}$ are such that the tensor product $M \otimes_{\mathsf{T}} N$ exists, then there is a natural transformation*

$$\zeta : (M \otimes_{\mathsf{T}} N)S \to M \otimes_{\mathsf{T}} N$$

making $M \otimes_{\mathsf{T}} N$ into a right S-module in such a way that the diagram

$$(MTN, MT\rho_r) \underset{M\rho_l}{\overset{\varrho N}{\rightrightarrows}} (MN, M\rho_r) \xrightarrow{can} (M \otimes_{\mathsf{T}} N, \zeta)$$

is a coequaliser in $[\mathbb{A}, \mathbb{X}]_{[\mathsf{S},\mathbb{X}]}$. The action ζ is uniquely determined by the property

$$\zeta \cdot can_{\mathsf{T}}^{M,N} S = can_{\mathsf{T}}^{M,N} \cdot M\rho_r.$$

Proof. Note first that both $(MTN, MT\rho_r)$ and $(MN, M\rho_r)$ are objects of the category $[\mathbb{A}, \mathbb{X}]_{[\mathsf{S},\mathbb{X}]}$, and that both ϱN and $M\rho_l$ can be seen as morphisms in $[\mathbb{A}, \mathbb{X}]_{[\mathsf{S},\mathbb{X}]}$ (the former by naturality of composition, the latter because $(N, \rho_l, \rho_r) \in [\mathbb{A}, \mathbb{A}]_{[\mathsf{S},\mathsf{T}]}$). Now, since the functor

$$[\mathsf{S}, \mathbb{X}] : [\mathbb{A}, \mathbb{X}] \to [\mathbb{A}, \mathbb{X}]$$

preserves all colimits (hence coequalisers), the result follows from [17, Proposition 2.3]. □

1.10. Comodule functors. Similarly as for monads, one defines comodule functors over a comonad G on a category $\mathbb{A}$. A *left* G*-comodule functor* is a pair (F, θ), where $F : \mathbb{X} \to \mathbb{A}$ is a functor and $\theta : F \to GF$ is a natural transformation inducing commutativity of the diagrams

$$\begin{tikzcd} F \arrow[r, "\theta"] \arrow[dr, equal] & GF \arrow[d, "\varepsilon F"] \\ & F, \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "\theta"] \arrow[d, "\theta"'] & GF \arrow[d, "\delta F"] \\ GF \arrow[r, "G\theta"'] & GGF. \end{tikzcd}$$

Symmetrically, *right* G*-comodule functors* $\mathbb{A} \to \mathbb{Y}$ are defined.

1.11. Cotensor product of comodule functors. Let G be a comonad on $\mathbb{A}$ and $\mathbb{X}$, $\mathbb{Y}$ arbitrary categories. If (C, θ) is a right G-comodule functor $\mathbb{A} \to \mathbb{X}$ and (D, ϑ) is a left G-comodule functor $\mathbb{Y} \to \mathbb{A}$, then their *cotensor product* (*over* G) is defined as the object part of the equaliser

$$C \otimes^{\mathsf{G}} D \xrightarrow{\operatorname{can}_{\mathsf{G}}^{C,D}} CD \underset{C\vartheta}{\overset{\theta D}{\rightrightarrows}} CGD \tag{1.3}$$

in $[\mathbb{Y}, \mathbb{X}]$, provided that such an equaliser exists. We often abbreviate $\operatorname{can}_{\mathsf{G}}^{C,D}$ to $\operatorname{can}^{C,D}$, or even to can.

Now recall the dual of [27, Lemma 21.1.5] and Dubuc's Adjoint Triangle Theorem [15].

1.12. Monads and adjunctions. For categories $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$, consider adjunctions $\eta, \varepsilon : F \dashv R : \mathbb{A} \to \mathbb{C}$ and $\eta', \varepsilon' : F' \dashv R' : \mathbb{B} \to \mathbb{C}$, with corresponding monads T and T', and let

$$\begin{tikzcd} \mathbb{A} \arrow[r, "K"] \arrow[dr, "R"'] & \mathbb{B} \arrow[d, "R'"] \\ & \mathbb{C} \end{tikzcd}$$

be a commutative diagram with some functor K. Write γ_K for the composite

$$F' \xrightarrow{F'\eta} F'RF = F'R'KF \xrightarrow{\varepsilon' KF} KF.$$

Then $t_K = R'\gamma_K$ is a monad morphism $\mathsf{T}' \to \mathsf{T}$ (see [15]).

1.13. Theorem. *In the situation considered in 1.12, suppose that R' is a premonadic functor. Then K has a left adjoint $\underline{K}$ if and only if the following coequaliser (used as the definition of $\underline{K}$) exists in* $[\mathbb{B}, \mathbb{A}]$,

$$\begin{tikzcd} FR'F'R' \arrow[rr, "FR'\varepsilon'"] \arrow[dr, "Ft_K R' = FR'\gamma_K R'"'] & & FR' \arrow[r, "q"] & \underline{K} \\ & FR'KFR' = FRFR' \arrow[ur, "\varepsilon FR'"'] & & \end{tikzcd} .$$

When this is the case, the unit $\underline{\eta} : 1 \to K\underline{K}$ and counit $\underline{\varepsilon} : \underline{K}K \to 1$ of the adjunction $\underline{K} \dashv K$ are the unique such natural transformations yielding commutativity of the diagrams, respectively,

$$\begin{tikzcd} F'R' \arrow[r, "\varepsilon'"] \arrow[d, "\gamma_K R'"'] & 1 \arrow[d, "\underline{\eta}"] \\ KFR' \arrow[r, "Kq"'] & K\underline{K}, \end{tikzcd} \qquad \begin{tikzcd} FR = FR'K \arrow[r, "qK"] \arrow[dr, "\varepsilon"'] & \underline{K}K \arrow[d, "\underline{\varepsilon}"] \\ & 1. \end{tikzcd}$$

Precomposing the image of the last square under R' with $\eta R'$ and using the fact that $R'\gamma_K$ is a monad morphism $\mathsf{T}' \to \mathsf{T}$, we get the commutative diagram

$$\begin{tikzcd} R' \arrow[r, "\eta' R'"] \arrow[dr, "\eta R'"'] & R'F'R' \arrow[rr, "R'\varepsilon'"] \arrow[d, "R'\gamma_K R'"] & & R' \arrow[d, "R'\underline{\eta}"] \\ & RFR' = R'KFR' \arrow[rr, "R'Kq = Rq"] & & R'K\underline{K}. \end{tikzcd}$$

Since $R'\varepsilon' \cdot \eta' R' = 1$ by one of the triangular identities for $F' \dashv R'$, one gets

$$R'\underline{\eta} = Rq \cdot \eta R'. \tag{1.4}$$

1.14. Comonads and adjunctions. Again let $\eta, \varepsilon : F \dashv R : \mathbb{A} \to \mathbb{B}$ and $\eta', \varepsilon' : F' \dashv R' : \mathbb{A} \to \mathbb{C}$ be adjunctions and now consider the corresponding comonads G and G'. Let

$$\begin{tikzcd} \mathbb{B} \arrow[r, "K"] \arrow[dr, "F"'] & \mathbb{C} \arrow[d, "F'"] \\ & \mathbb{A} \end{tikzcd}$$

be a commutative diagram (no commutativity for R and R' is required) and define

$$\gamma_K : KR \xrightarrow{\eta' KR} R'F'KR = R'FR \xrightarrow{R'\varepsilon} R'.$$

Then $t_K = F'\gamma_K$ is a comonad morphism $\mathsf{G} \to \mathsf{G}'$.

1.15. Theorem. *In the situation described in 1.14, suppose that F' is precomonadic. Then K has a right adjoint $\overline{K}$ if and only if the following equaliser exists in $[\mathbb{C}, \mathbb{B}]$,*

$$\begin{tikzcd}
\overline{K} \arrow[r, "\iota"] & RF' \arrow[rr, "RF'\eta'"] \arrow[dr, "\eta RF'"'] & & RF'R'F' \\
& & RFRF' = RF'KRF' \arrow[ur, "Rt_K F' = RF'\gamma_K F'"'] &
\end{tikzcd} .$$

When this equaliser exists, the unit $\overline{\eta} : 1 \to \overline{K}K$ and counit $\overline{\varepsilon} : K \dashv \overline{K}$ of the adjunction $K \dashv \overline{K}$ are the unique such natural transformations yielding commutativity of the diagrams, respectively,

$$\begin{tikzcd}
1 \arrow[r, "\overline{\eta}"] \arrow[dr, "\eta"'] & \overline{K}K \arrow[d, "\iota K"] \\
& RF = RF'K,
\end{tikzcd}
\qquad
\begin{tikzcd}
K\overline{K} \arrow[r, "K\iota"] \arrow[d, "\overline{\varepsilon}"'] & KRF' \arrow[d, "\gamma_K F'"] \\
1 \arrow[r, "\eta'"'] & R'F'.
\end{tikzcd}$$

Moreover, one has

$$F'\overline{\varepsilon} = \varepsilon F' \cdot F\iota. \tag{1.5}$$

1.16. The restriction- and change-of-base functors. Any morphism

$$\iota : \mathsf{S} = (S, m^{\mathsf{S}}, e^{\mathsf{S}}) \to \mathsf{T} = (T, m^{\mathsf{T}}, e^{\mathsf{T}})$$

of monads on $\mathbb{A}$ (that is, a natural transformation $\iota : S \to T$ such that $\iota \cdot e^{\mathsf{S}} = e^{\mathsf{T}}$ and $\iota \cdot m^{\mathsf{S}} = m^{\mathsf{T}} \cdot (\iota\iota)$) gives rise (see [4]) to the functor

$$\iota^* : \mathbb{A}_{\mathsf{T}} \to \mathbb{A}_{\mathsf{S}}, \quad (a, h) \mapsto (a, h \cdot \iota_a),$$

called the *restriction-of-base functor*. It is clear that ι^* makes the diagram

$$\begin{tikzcd}
\mathbb{A}_{\mathsf{T}} \arrow[r, "\iota^*"] \arrow[dr, "U_{\mathsf{T}}"'] & \mathbb{A}_{\mathsf{S}} \arrow[d, "U_{\mathsf{S}}"] \\
& \mathbb{A}
\end{tikzcd}$$

commute. Since the forgetful functor $U_\mathsf{S} : \mathbb{A}_\mathsf{S} \to \mathbb{A}$ is clearly monadic, it follows from Theorem 1.13 that ι^* has a left adjoint $\iota_! : \mathbb{A}_\mathsf{S} \to \mathbb{A}_\mathsf{T}$ if and only if the pair of natural transformations

$$\phi_\mathsf{T} S U_\mathsf{S} = \phi_\mathsf{T} U_\mathsf{S} \phi_\mathsf{S} U_\mathsf{S} \underset{\varrho U_\mathsf{S}}{\overset{\phi_\mathsf{T} U_\mathsf{S}\, \varepsilon_\mathsf{S}}{\rightrightarrows}} \phi_\mathsf{T} U_\mathsf{S} \overset{q}{\dashrightarrow} \iota_!\,, \tag{1.6}$$

where $\varrho : \phi_\mathsf{T} S \to \phi_\mathsf{T}$ is the composite $\phi_\mathsf{T} S \xrightarrow{\phi_\mathsf{T}\iota} \phi_\mathsf{T} T = \phi_\mathsf{T} U_\mathsf{T} \phi_\mathsf{T} \xrightarrow{\varepsilon_\mathsf{T}\phi_\mathsf{T}} \phi_\mathsf{T}$, has a coequaliser $q : \phi_\mathsf{T} U_\mathsf{S} \to \iota_!$ in $[\mathbb{A}_\mathsf{S}, \mathbb{A}_\mathsf{T}]$. This $\iota_! : \mathbb{A}_\mathsf{S} \to \mathbb{A}_\mathsf{T}$, when it exists, is called the *change-of-base functor*. Recalling that for any S-algebra (a, g), $U_\mathsf{S}((\varepsilon_\mathsf{S})_{(a,g)}) = g$ and $(\varepsilon_\mathsf{T}\phi_\mathsf{T} U_\mathsf{S})_{(a,g)} = m^\mathsf{T}_a$, one finds that $\iota_!$ sends an S-algebra (a, g) to the object $\iota_!(a, g)$ in the coequaliser diagram in $\mathbb{A}_\mathsf{T}$,

$$\begin{tikzcd} \phi_\mathsf{T} S(a) \arrow[rr, "\phi_\mathsf{T}(g)"] \arrow[dr, "\phi_\mathsf{T}(\iota_a)"'] & & \phi_\mathsf{T}(a) \arrow[r, "q_{(a,g)}"] & \iota_!(a,g) \\ & \phi_\mathsf{T} T(a) \arrow[ur, "m^\mathsf{T}_a"'] & & \end{tikzcd} \tag{1.7}$$

.

1.17. Remark. (1) Since $(\phi_\mathsf{T}, \varepsilon_\mathsf{T}\phi_\mathsf{T} \cdot \phi_\mathsf{T}\iota)$ is a right S-module by Section 1.7(vi), the diagram

$$\phi_\mathsf{T} SS \underset{\varepsilon_\mathsf{T}\phi_\mathsf{T} S \cdot\, \phi_\mathsf{T}\iota S}{\overset{\phi_\mathsf{T} m^\mathsf{S}}{\rightrightarrows}} \phi_\mathsf{T} S \xrightarrow{\phi_\mathsf{T}\iota} \phi^\mathsf{T} T \xrightarrow{\varepsilon_\mathsf{T}\phi_\mathsf{T}} \phi_\mathsf{T}$$

is a coequaliser in $[\mathbb{A}, \mathbb{A}_\mathsf{T}]$. Observing that the pair $\phi_\mathsf{T} SS \underset{\varepsilon_\mathsf{T}\phi_\mathsf{T} S \cdot\, \phi_\mathsf{T}\iota S}{\overset{\phi_\mathsf{T} m^\mathsf{S}}{\rightrightarrows}} \phi_\mathsf{T} S$ is just the pair $\phi_\mathsf{T} S U_\mathsf{S} \phi_\mathsf{S} \underset{\varrho U_\mathsf{S} \phi_\mathsf{S}}{\overset{\phi_\mathsf{T} U_\mathsf{S}\, \varepsilon_\mathsf{S}\phi_\mathsf{S}}{\rightrightarrows}} \phi_\mathsf{T} U_\mathsf{S} \phi_\mathsf{S}$, it follows that $q\phi_\mathsf{S}$ is $\phi_\mathsf{T} S \xrightarrow{\phi_\mathsf{T}\iota} \phi_\mathsf{T} T \xrightarrow{\varepsilon_\mathsf{T}\phi_\mathsf{T}} \phi_\mathsf{T}$.

(2) It is easy to see that $U_\mathsf{S}\, \varepsilon_\mathsf{S}\iota^*\phi_\mathsf{T}$ is the composite $ST \xrightarrow{\iota T} TT \xrightarrow{m^\mathsf{T}} T$ and since

- $U_\mathsf{S}\iota^*\phi_\mathsf{T} = U_\mathsf{T}\phi_\mathsf{T} = T$,
- $U_\mathsf{T}\varepsilon_\mathsf{T}\phi_\mathsf{T} = m^\mathsf{T}$,
- $U_\mathsf{T}\varrho = TS \xrightarrow{T\iota} TT \xrightarrow{m^\mathsf{T}} T$,

we may identify the pairs

$$(U_\mathsf{T}\phi_\mathsf{T} U_\mathsf{S}\, \varepsilon_\mathsf{S}\iota^*\phi_\mathsf{T},\ U_\mathsf{T}\varrho U_\mathsf{S}\iota^*\phi_\mathsf{T}), \qquad (Tm^\mathsf{T} \cdot T\iota T,\ m^\mathsf{T} T \cdot T\iota T).$$

Thus, if the coequaliser diagram (1.6) exists and if its image under the functor $[\mathbb{A}_\mathsf{S}, U_\mathsf{T}] : [\mathbb{A}_\mathsf{S}, \mathbb{A}_\mathsf{T}] \to [\mathbb{A}_\mathsf{S}, \mathbb{A}]$ is again a coequaliser, we get a coequaliser

$$TST \underset{m^{\mathsf{T}}T\cdot T\iota T}{\overset{Tm^{\mathsf{T}}\cdot T\iota T}{\rightrightarrows}} TT \xrightarrow{U_{\mathsf{T}}q\iota^*\phi_{\mathsf{T}}} U_{\mathsf{T}}\iota_!\iota^*\phi_{\mathsf{T}}.$$

Since $(T, m^{\mathsf{T}}\cdot T\iota)$ is a right S-module, while $(T, m^{\mathsf{T}}\cdot \iota T)$ is a left S-module (see Section 1.7), one concludes that $U_{\mathsf{T}}\iota_!\iota^*\phi_{\mathsf{T}} = T\otimes_{\mathsf{S}}T$ and $U_{\mathsf{T}}q\iota^*\phi_{\mathsf{T}} = \mathrm{can}_{\mathsf{S}}^{T,T}$.

1.18. Proposition. *If $\iota : \mathsf{S} \to T$ is a morphism of $\mathbb{A}$-monads such that the change-of-base functor $\iota_! : \mathbb{A}_{\mathsf{S}} \to \mathbb{A}_T$ exists, then the diagram*

$$\begin{tikzcd} \mathbb{A} \arrow[r, "\phi_{\mathsf{S}}"] \arrow[dr, "\phi_T"'] & \mathbb{A}_{\mathsf{S}} \arrow[d, "\iota_!"] \\ & \mathbb{A}_T \end{tikzcd}$$

commutes (up to isomorphism).

Proof. Since $U_{\mathsf{S}} \cdot \iota^* = U_{\mathsf{T}}$ (see Section 1.16), $\phi_{\mathsf{S}} \dashv U_{\mathsf{S}}$ and $\iota_! \dashv \iota^*$, the result follows by uniqueness of left adjoints. □

2. Separable Frobenius monads

The crucial role of separable Frobenius functors (e.g. [28]) in the theory of weak bimonads was pointed out by Szlachányi in [30] and such functors are used by Böhm et al. in [9] as an integral part of their definition of weak bimonads on monoidal categories. In this section we show that in our approach separable Frobenius monads S are of interest since they imply the existence of the change-of-base functor for monad morphisms $S \to T$.

2.1. Definition. A *Frobenius $\mathbb{A}$-monad* is an endofunctor $S : \mathbb{A} \to \mathbb{A}$ which carries an $\mathbb{A}$-monad structure $\underline{\mathsf{S}} = (S, m^{\mathsf{S}}, e^{\mathsf{S}})$ and an $\mathbb{A}$-comonad structure $\overline{\mathsf{S}} = (S, \delta^{\mathsf{S}}, \varepsilon^{\mathsf{S}})$ such that the following diagram commutes

$$\begin{tikzcd} SS \arrow[rr, "\delta^{\mathsf{S}}S"] \arrow[dr, "m^{\mathsf{S}}"] \arrow[dd, "S\delta^{\mathsf{S}}"'] & & SSS \arrow[dd, "Sm^{\mathsf{S}}"] \\ & S \arrow[dr, "\delta^{\mathsf{S}}"] & \\ SSS \arrow[rr, "m^{\mathsf{S}}S"'] & & SS. \end{tikzcd} \tag{2.1}$$

$\mathsf{S} = (S, m^{\mathsf{S}}, e^{\mathsf{S}}, \delta^{\mathsf{S}}, \varepsilon^{\mathsf{S}})$ is called *separable Frobenius* if, in addition, $m^{\mathsf{S}} \cdot \delta^{\mathsf{S}} = 1$.

2.2. Proposition. *Let $\mathsf{S} = (S, m^{\mathsf{S}}, e^{\mathsf{S}}, \delta^{\mathsf{S}}, \varepsilon^{\mathsf{S}})$ be a Frobenius separable monad on a Cauchy complete category $\mathbb{A}$. Then for any morphism $\iota : \underline{\mathsf{S}} \to \mathsf{T}$ of monads, the change-of-base functor $\iota_! : \mathbb{A}_{\underline{\mathsf{S}}} \to \mathbb{A}_{\mathsf{T}}$ exists.*

Proof. We claim that, under our assumptions, (1.6) is a split pair, a splitting morphism being the composite

$$\pi : \ \phi_{\mathsf{T}} U_{\underline{\mathsf{S}}} \xrightarrow{\phi_{\mathsf{T}} e^{\mathsf{S}} U_{\underline{\mathsf{S}}}} \phi_{\mathsf{T}} S U_{\underline{\mathsf{S}}} \xrightarrow{\phi_{\mathsf{T}} \delta^{\mathsf{S}} U_{\underline{\mathsf{S}}}} \phi_{\mathsf{T}} SS U_{\underline{\mathsf{S}}} \xrightarrow{\varrho S U_{\underline{\mathsf{S}}}} \phi_{\mathsf{T}} S U_{\underline{\mathsf{S}}} .$$

Indeed, that $\varrho \cdot \pi = 1$ follows from commutativity of the diagram

$$\begin{tikzcd}
\phi_{\mathsf{T}} U_{\underline{\mathsf{S}}} \arrow[r, "\phi_{\mathsf{T}} e^{\mathsf{S}} U_{\underline{\mathsf{S}}}"] \arrow[drrr, bend right, "1"'] & \phi_{\mathsf{T}} S U_{\underline{\mathsf{S}}} \arrow[r, "\phi_{\mathsf{T}} \delta^{\mathsf{S}} U_{\underline{\mathsf{S}}}"] \arrow[dr, equal] & \phi_{\mathsf{T}} SS U_{\underline{\mathsf{S}}} \arrow[r, "\varrho S U_{\underline{\mathsf{S}}}"] \arrow[d, "\phi_{\mathsf{T}} m^{\mathsf{S}} U_{\underline{\mathsf{S}}}"] & \phi_{\mathsf{T}} S U_{\underline{\mathsf{S}}} \arrow[d, "\varrho U_{\underline{\mathsf{S}}}"] \\
 & & \phi_{\mathsf{T}} S U_{\underline{\mathsf{S}}} \arrow[r, "\varrho U_{\underline{\mathsf{S}}}"] & \phi_{\mathsf{T}} U_{\underline{\mathsf{S}}} .
\end{tikzcd}$$

Here the square and the curved region commute since $(\phi_{\mathsf{T}}, \varrho)$ is a right $\underline{\mathsf{S}}$-module by Section 1.7(vii), while the triangle commutes by separability of the monad $\underline{\mathsf{S}}$.

Next, to show that $\phi_{\mathsf{T}} U_{\underline{\mathsf{S}}} \varepsilon_{\underline{\mathsf{S}}} \cdot \pi \cdot \varrho U_{\underline{\mathsf{S}}} = \phi_{\mathsf{T}} U_{\underline{\mathsf{S}}} \varepsilon_{\underline{\mathsf{S}}} \cdot \pi \cdot \phi_{\mathsf{T}} U_{\underline{\mathsf{S}}} S \varepsilon_{\underline{\mathsf{S}}}$, consider the diagram

$$\begin{tikzcd}
\phi_{\mathsf{T}} S U_{\underline{\mathsf{S}}} \arrow[r, "\varrho U_{\underline{\mathsf{S}}}"] \arrow[dd, "\phi_{\mathsf{T}} \delta^{\mathsf{S}} U_{\underline{\mathsf{S}}}"'] \arrow[dr, "\phi_{\mathsf{T}} S e^{\mathsf{S}} U_{\underline{\mathsf{S}}}"'] & \phi_{\mathsf{T}} U_{\underline{\mathsf{S}}} \arrow[r, "\phi_{\mathsf{T}} e^{\mathsf{S}} U_{\underline{\mathsf{S}}}"] & \phi_{\mathsf{T}} S U_{\underline{\mathsf{S}}} \arrow[r, "\phi_{\mathsf{T}} \delta^{\mathsf{S}} U_{\underline{\mathsf{S}}}"] & \phi_{\mathsf{T}} SS U_{\underline{\mathsf{S}}} \arrow[r, "\varrho S U_{\underline{\mathsf{S}}}"] & \phi_{\mathsf{T}} S U_{\underline{\mathsf{S}}} \arrow[d, "\phi_{\mathsf{T}} U_{\underline{\mathsf{S}}} \varepsilon_{\underline{\mathsf{S}}}"] \\
 & \phi_{\mathsf{T}} SS U_{\underline{\mathsf{S}}} \arrow[ur, "\varrho S U_{\underline{\mathsf{S}}}"'] \arrow[r, "\phi_{\mathsf{T}} S \delta^{\mathsf{S}} U_{\underline{\mathsf{S}}}"'] \arrow[d, "\phi_{\mathsf{T}} \delta^{\mathsf{S}} S U_{\underline{\mathsf{S}}}"'] & \phi_{\mathsf{T}} SSS U_{\underline{\mathsf{S}}} \arrow[ur, "\varrho SS U_{\underline{\mathsf{S}}}"'] \arrow[r, "\phi_{\mathsf{T}} m^{\mathsf{S}} S U_{\underline{\mathsf{S}}}"'] & \phi_{\mathsf{T}} SS U_{\underline{\mathsf{S}}} \arrow[ur, "\varrho S U_{\underline{\mathsf{S}}}"'] & \phi_{\mathsf{T}} U_{\underline{\mathsf{S}}} \\
\phi_{\mathsf{T}} SS U_{\underline{\mathsf{S}}} \arrow[r, "\phi_{\mathsf{T}} SS e^{\mathsf{S}} U_{\underline{\mathsf{S}}}"'] & \phi_{\mathsf{T}} SSS U_{\underline{\mathsf{S}}} \arrow[urr, bend right, "\phi_{\mathsf{T}} S m^{\mathsf{S}} U_{\underline{\mathsf{S}}}"'] & & &
\end{tikzcd} ,$$

in which the curved region commutes since S is assumed to be Frobenius, the right-hand parallelogram commutes since ϱ is a morphism of right $\underline{\mathsf{S}}$-modules, while the other regions commute by naturality of composition. Thus the whole diagram is commutative, implying – since $m^{\mathsf{S}} \cdot S e^{\mathsf{S}} = 1$ – that

$$\phi_{\mathsf{T}} U_{\underline{\mathsf{S}}} \varepsilon_{\underline{\mathsf{S}}} \cdot \pi \cdot \varrho U_{\underline{\mathsf{S}}} = \phi_{\mathsf{T}} U_{\underline{\mathsf{S}}} \varepsilon_{\underline{\mathsf{S}}} \cdot \varrho S U_{\underline{\mathsf{S}}} \cdot \phi_{\mathsf{T}} \delta^{\mathsf{S}} U_{\underline{\mathsf{S}}}.$$

In a similar manner one proves that

$$\phi_{\mathsf{T}} U_{\underline{\mathsf{S}}} \varepsilon_{\underline{\mathsf{S}}} \cdot \pi \cdot \phi_{\mathsf{T}} U_{\underline{\mathsf{S}}} \varepsilon_{\underline{\mathsf{S}}} = \phi_{\mathsf{T}} U_{\underline{\mathsf{S}}} \varepsilon_{\underline{\mathsf{S}}} \cdot \varrho S U_{\underline{\mathsf{S}}} \cdot \phi_{\mathsf{T}} \delta^{\mathsf{S}} U_{\underline{\mathsf{S}}}.$$

So $\phi_{\mathsf{T}} U_{\underline{\mathsf{S}}} \varepsilon_{\underline{\mathsf{S}}} \cdot \pi \cdot \varrho U_{\underline{\mathsf{S}}} = \phi_{\mathsf{T}} U_{\underline{\mathsf{S}}} \varepsilon_{\underline{\mathsf{S}}} \cdot \pi \cdot \phi_{\mathsf{T}} U_{\underline{\mathsf{S}}} \varepsilon_{\mathsf{S}}$. Therefore, the pair (1.6) splits by the morphism π. Since $\mathbb{A}$ is assumed to be Cauchy complete, $\mathbb{A}_{\mathsf{T}}$ (hence the functor category $[\mathbb{A}_{\underline{\mathsf{S}}}, \mathbb{A}_{\mathsf{T}}]$) is also Cauchy complete (see Proposition 1.4). It then follows that the pair (1.6) admits a (split) coequaliser. Thus the extension-of-base functor $\iota_! : \mathbb{A}_{\underline{\mathsf{S}}} \to \mathbb{A}_{\mathsf{T}}$ exists. □

Dually, we have:

2.3. Proposition. *Let S be a separable Frobenius comonad on a Cauchy complete category $\mathbb{A}$. Then for any morphism $f : \overline{\mathsf{S}} \to \mathsf{G}$ of comonads, the change-of-cobase functor $f_! : \mathbb{A}^{\mathsf{G}} \to \mathbb{A}^{\overline{\mathsf{S}}}$ exists.*

3. Comparison functors

Given a comonad G on $\mathbb{A}$ and a category $\mathbb{B}$, one has the induced comonads $[\mathbb{B}, \mathsf{G}]$ on $[\mathbb{B}, \mathbb{A}]$ and $[\mathsf{G}, \mathbb{B}]$ on $[\mathbb{A}, \mathbb{B}]$.

3.1. Comodules and adjoint functors. Consider a comonad $\mathsf{G} = (G, \delta, \varepsilon)$ on $\mathbb{A}$ and an adjunction $\eta, \sigma : F \dashv R : \mathbb{B} \to \mathbb{A}$.

There exist bijective correspondences (e.g. [16]) between

- functors $K : \mathbb{B} \to \mathbb{A}^{\mathsf{G}}$ with $U^{\mathsf{G}} K = F$;
- left G-comodule structures $\theta : F \to GF$ on F (i.e., $(F, \theta) \in [\mathbb{B}, \mathbb{A}]^{[\mathbb{B}, \mathsf{G}]}$);
- comonad morphisms from the comonad generated by $F \dashv R$ to the comonad G;
- right G-comodule structures $\vartheta : R \to RG$ on R (i.e., $(R, \vartheta) \in [\mathbb{A}, \mathbb{B}]^{[\mathsf{G}, \mathbb{B}]}$).

These bijections are constructed as follows. If $U^{\mathsf{G}} K = F$, then $K(b) = (F(b), \theta_b)$ for some morphism $\alpha_b : F(b) \to GF(b)$, and the collection $\{\theta_b,\, b \in \mathbb{B}\}$ constitutes a natural transformation $\theta : F \to GF$ making F a left G-comodule.

Conversely, if $(F, \theta) \in [\mathbb{B}, \mathbb{A}]^{[\mathbb{B}, \mathsf{G}]}$, then $K : \mathbb{B} \to \mathbb{A}^{\mathsf{G}}$ is defined by $K(b) = (F(b), \theta_b)$.

Next, for any left G-comodule structure $\theta : F \to GF$, the composite

$$t_K : FR \xrightarrow{\theta R} GFR \xrightarrow{G\sigma} G$$

is a morphism from the comonad generated by $F \dashv R$ to the comonad G. Then the corresponding right G-comodule structure $\vartheta : R \to RG$ on R is $R \xrightarrow{\eta R} RFR \xrightarrow{Rt_K} RG$.

Conversely, for $(R, \vartheta) \in [\mathbb{A}, \mathbb{B}]^{[\mathsf{G}, \mathbb{B}]}$, the corresponding comonad morphism $t_K : FR \to \mathsf{G}$ is the composite

$$FR \xrightarrow{F\vartheta} FRG \xrightarrow{\sigma G} G,$$

while the corresponding left G-comodule structure $\theta : F \to GF$ on F is the composite $F \xrightarrow{F\eta} FRF \xrightarrow{t_K F} GF$.

3.2. Theorem. *Let $\mathsf{G} = (G, \delta, \varepsilon)$ be a comonad on $\mathbb{A}$ and $\eta, \sigma : F \dashv R : \mathbb{B} \to \mathbb{A}$ an adjunction. For a functor $K : \mathbb{B} \to \mathbb{A}^{\mathsf{G}}$ with $U^{\mathsf{G}} K = F$, the following are equivalent:*

(a) *K is an equivalence of categories;*
(b) *F is comonadic and the comonad morphism $t_K : FR \to \mathsf{G}$ is an isomorphism;*
(c) *F is comonadic and the composite*

$$\gamma_K : KR \xrightarrow{\eta^{\mathsf{G}} KR} \phi^{\mathsf{G}} U^{\mathsf{G}} KR = \phi^{\mathsf{G}} FR \xrightarrow{\phi^{\mathsf{G}} \sigma} \phi^{\mathsf{G}}$$

is an isomorphism.

Proof. (a) and (b) are equivalent by [20, Theorem 4.4]; (b) and (c) are equivalent since $U^{\mathsf{G}} \gamma_K = t_K$ by [15] and U^{G} reflects isomorphisms. □

3.3. Right adjoint of *K*. Now fix a functor $K : \mathbb{B} \to \mathbb{A}^{\mathsf{G}}$ with commutative diagram

$$\begin{tikzcd} \mathbb{B} \arrow[r, "K"] \arrow[dr, "F"'] & \mathbb{A}^{\mathsf{G}} \arrow[d, "U^{\mathsf{G}}"] \\ & \mathbb{A}. \end{tikzcd} \tag{3.1}$$

Then γ_K is the composite $KR \xrightarrow{\eta^{\mathsf{G}} KR} \phi^{\mathsf{G}} U^{\mathsf{G}} KR = \phi^{\mathsf{G}} FR \xrightarrow{\phi^{\mathsf{G}} \varepsilon} \phi^{\mathsf{G}}$ and using the fact from Section 1.14 that $U^{\mathsf{G}} \gamma_K$ is just the comonad morphism $t_K : FR \to G$ induced by the triangle, an easy calculation shows that

$$\beta U^{\mathsf{G}} = RU^{\mathsf{G}} \gamma_K U^{\mathsf{G}} \cdot \eta RU^{\mathsf{G}},$$

where $\beta : R \to RG$ is the right G-module structure on R corresponding to the triangle (3.1). Thus, when the right adjoint $\overline{K}$ of K exists, it is determined by the equaliser diagram

$$\overline{K} \xrightarrow{\iota} RU^{\mathsf{G}} \underset{\beta U^{\mathsf{G}}}{\overset{RU^{\mathsf{G}} \eta^{\mathsf{G}}}{\rightrightarrows}} RGU^{\mathsf{G}} = RU^{\mathsf{G}} \phi^{\mathsf{G}} U^{\mathsf{G}}. \tag{3.2}$$

It is easy to see that for any $(a, \theta) \in \mathbb{A}^{\mathsf{G}}$, the (a, θ)-component of (3.2) is the equaliser diagram

$$\overline{K}(a, \theta) \xrightarrow{\iota_{(a,\theta)}} R(a) \underset{\beta_a}{\overset{R(\theta)}{\rightrightarrows}} RG(a) \tag{3.3}$$

and, referring to (1.5), for the counit $\overline{\sigma} : K\overline{K} \to 1$ of $K \dashv \overline{K}$, one gets

$$U^{\mathsf{G}}(\overline{\sigma}_{(a,\,\theta)}) = \sigma_a \cdot F(\iota_{(a,\,\theta)})\,. \tag{3.4}$$

Suppose now that $\overline{K}$ exists, write P for the monad on $\mathbb{B}$ generated by the adjunction $K \dashv \overline{K}$, and consider the corresponding comparison functor $K_{\mathsf{P}} : \mathbb{A}^{\mathsf{G}} \to \mathbb{B}_{\mathsf{P}}$. Then $K_{\mathsf{P}}(a,\theta) = (\overline{K}(a,\theta), \overline{K}(\overline{\sigma}_{(a,\,\theta)}))$ for any $(a,\theta) \in \mathbb{A}^{\mathsf{G}}$. Moreover, $K_{\mathsf{P}}K = \phi_{\mathsf{P}}$ and $U_{\mathsf{P}}K_{\mathsf{P}} = \overline{K}$. The situation may be pictured as

$$\begin{tikzcd}
\mathbb{B} \arrow[rr, bend right=20, "\phi_{\mathsf{P}}"'] \arrow[rrrr, bend right=40, "K"'] \arrow[drr, "F"'] & & \mathbb{B}_{\mathsf{P}} \arrow[ll, "U_{\mathsf{P}}"'] & & \mathbb{A}^{\mathsf{G}} \arrow[ll, "K_{\mathsf{P}}"'] \arrow[llll, bend right=30, "\overline{K}"'] \arrow[dll, "U^{\mathsf{G}}"] \\
& & \mathbb{A} & &
\end{tikzcd} \tag{3.5}$$

In order to proceed, we need the following (see [27, Lemma 21.2.7]).

3.4. Proposition. *Let $\eta, \sigma : F \dashv R : \mathbb{A} \to \mathbb{C}$ and $\eta', \sigma' : F' \dashv R' : \mathbb{B} \to \mathbb{C}$ be adjunctions with corresponding monads T and T', respectively, and let*

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "K"] & \mathbb{B} \\
& \mathbb{C} \arrow[ul, "F"] \arrow[u, "F'"']
\end{tikzcd}$$

be a commutative diagram of categories and functors. Then the composite

$$T = RF \xrightarrow{\eta' RF} R'F'RF = R'KFRF \xrightarrow{R'K\sigma F} R'KF = R'F' = T'$$

is a monad morphism $\mathsf{T} \to \mathsf{T}'$.

Suppose again that $\overline{K}$ exists and consider the natural transformation $\iota : P \to RF$, where $\iota_b = \iota_{K(b)}$ for all $b \in \mathbb{B}$.

3.5. Proposition. *$\iota : P \to RF$ is a monad morphism from the monad P to the monad generated by the adjunction $F \dashv R$.*

Proof. Applying Proposition 3.4 to the diagram

$$\begin{tikzcd}
\mathbb{A}^{\mathsf{G}} \arrow[rr, "U^{\mathsf{G}}"] \arrow[dr, "\overline{K}"] & & \mathbb{A} \arrow[dl, "R"'] \\
& \mathbb{B} \arrow[ul, bend left, "K"] \arrow[ur, bend right, "F"'] &
\end{tikzcd}$$

in which $U^{\mathsf{G}}K = F$, gives that the natural transformation,

$$P = \overline{K}K \xrightarrow{\eta\overline{K}K} RF\overline{K}K = RU^{\mathsf{G}}K\overline{K}K \xrightarrow{RU^{\mathsf{G}}\overline{\sigma}K} RU^{\mathsf{G}}K = RF,$$

is a monad morphism from the monad P to the monad generated by the adjunction $F \dashv R$. Since for any $(a, \theta) \in \mathbb{A}^{\mathsf{G}}$, $U^{\mathsf{G}}(\overline{\sigma}_{(a,\,\theta)}) = \sigma_a \cdot F(\iota_{(a,\,\theta)})$ (by equation (3.4)), it follows that, for each $b \in \mathbb{B}$, the b-component of the above natural transformation is the composite

$$P(b) \xrightarrow{\eta_{P(b)}} UFP(b) \xrightarrow{UF(\iota_b)} UFUF(b) \xrightarrow{U\sigma_{F(b)}} UF(b),$$

which is easily verified to be just the morphism $\iota_b : P(b) \to UF(b)$. This completes the proof. □

We are mainly interested in the case where the functor F is monadic. So, our standard situation of interest, and our standard notation, will henceforth be as follows. We consider a monad $\mathsf{T} = (T, m^{\mathsf{T}}, e^{\mathsf{T}})$ on $\mathbb{A}$, a comonad G on $\mathbb{A}_{\mathsf{T}}$, and an adjunction $\overline{\eta}, \overline{\sigma} : K \dashv \overline{K} : (\mathbb{A}_{\mathsf{T}})^{\mathsf{G}} \to \mathbb{A}$, where $K : \mathbb{A} \to (\mathbb{A}_{\mathsf{T}})^{\mathsf{G}}$ is a functor with $U_{\mathsf{G}}K = \phi_{\mathsf{T}}$. Write $\mathsf{P} = (P, m^{\mathsf{P}}, e^{\mathsf{P}})$ for the monad on $\mathbb{A}$ generated by the adjunction $K \dashv \overline{K}$ and write $\iota : P \to T$ for the induced morphism of monads. This is pictured in the diagram

$$\begin{tikzcd}
\mathbb{A} \arrow[r, bend right, "\phi_{\mathsf{P}}"'] \arrow[rr, bend right=40, "K"'] \arrow[dr, "\phi_{\mathsf{T}}"'] & \mathbb{A}_{\mathsf{P}} \arrow[l, "U_{\mathsf{P}}"'] \arrow[r, dashed, bend right, "L_{\mathsf{P}}"'] & (\mathbb{A}_{\mathsf{T}})^{\mathsf{G}} \arrow[l, "K_{\mathsf{P}}"'] \arrow[ll, bend right=30, "\overline{K}"'] \arrow[dl, "U^{\mathsf{G}}"] \\
& \mathbb{A}_{\mathsf{T}} &
\end{tikzcd} , \tag{3.6}$$

in which $K_{\mathsf{P}} : (\mathbb{A}_{\mathsf{T}})^{\mathsf{G}} \to \mathbb{A}_{\mathsf{P}}$ is the Eilenberg–Moore comparison functor for the monad P, and thus $K_{\mathsf{P}}K = \phi_{\mathsf{P}}$ and $U_{\mathsf{P}}K_{\mathsf{P}} = \overline{K}$.

3.6. Proposition. *In the situation above, the functor $K_{\mathsf{P}} : (\mathbb{A}_T)^G \to \mathbb{A}_{\mathsf{P}}$ admits a left adjoint $L_{\mathsf{P}} : \mathbb{A}_{\mathsf{P}} \to (\mathbb{A}_T)^G$ if and only if the restriction-of-base functor $\iota^* : \mathbb{A}_T \to \mathbb{A}_{\mathsf{P}}$ admits a left adjoint, i.e., the change-of-base functor $\iota_! : \mathbb{A}_{\mathsf{P}} \to \mathbb{A}_T$ exists. Moreover, when this is the case, $\iota_! = U_{\mathsf{G}} L_{\mathsf{P}}$.*

Proof. According to Section 1.16, $\iota_! : \mathbb{A}_{\mathsf{P}} \to \mathbb{A}_{\mathsf{T}}$ exists if and only if for each $(a, g) \in \mathbb{A}_{\mathsf{P}}$, the pair of morphisms $(\phi_{\mathsf{T}}(g), m_a^{\mathsf{T}} \cdot \phi_{\mathsf{T}}(\iota_a))$ has a coequaliser in $\mathbb{A}_{\mathsf{T}}$, while by Proposition 1.2(1), $L_{\mathsf{P}} : \mathbb{A}_{\mathsf{P}} \to (\mathbb{A}_{\mathsf{T}})^{\mathsf{G}}$ exists if and only if the pair of morphisms $(K(g), \overline{\sigma}_{K(a)})$ has a coequaliser in $(\mathbb{A}_{\mathsf{T}})^{\mathsf{G}}$.

Since the functor $U^{\mathsf{G}} : (\mathbb{A}_{\mathsf{T}})^{\mathsf{G}} \to \mathbb{A}_{\mathsf{T}}$ preserves and creates coequalisers, it suffices to show that the image of the pair $(K(g), \overline{\sigma}_{K(a)})$ under U^{G} is just the pair $(\phi_{\mathsf{T}}(g), m_a^{\mathsf{T}} \cdot \phi_{\mathsf{T}}(\iota_a))$. That $U^{\mathsf{G}} K(g) = \phi_{\mathsf{T}}(g)$ follows from the equality $U^{\mathsf{G}} K = \phi_{\mathsf{T}}$. Next, by (3.4), $U^{\mathsf{G}}(\overline{\sigma}_{K(a)}) = (\varepsilon_{\mathsf{T}})_{U^{\mathsf{G}} K(a)} \cdot \phi_{\mathsf{T}}(\iota_{K(a)})$. But since $(\varepsilon_{\mathsf{T}})_{U^{\mathsf{G}} K(a)} = (\varepsilon_{\mathsf{T}})_{\phi_{\mathsf{T}}(a)} = m_a^{\mathsf{T}}$, we see that $U^{\mathsf{G}}(\overline{\sigma}_{K(a)}) = m_a^{\mathsf{T}} \cdot \phi_{\mathsf{T}}(\iota_a)$. Hence

$$U^{\mathsf{G}}(K(g), \overline{\sigma}_{K(a)}) = (\phi_{\mathsf{T}}(g), m_a^{\mathsf{T}} \cdot \phi_{\mathsf{T}}(\iota_a))$$

and thus the result follows. $\square$

Now assume that the change-of-base functor $\iota_! : \mathbb{A}_{\mathsf{P}} \to \mathbb{A}_{\mathsf{T}}$ exists, that is, $K_{\mathsf{P}} : (\mathbb{A}_{\mathsf{T}})^{\mathsf{G}} \to \mathbb{A}_{\mathsf{P}}$ admits a left adjoint $L_{\mathsf{P}} : \mathbb{A}_{\mathsf{P}} \to (\mathbb{A}_{\mathsf{T}})^{\mathsf{G}}$. Thus, for any $(a, g) \in \mathbb{A}_{\mathsf{P}}$, $\iota_!(a, g)$ is given by be the coequaliser

$$\begin{tikzcd} TP(a) \arrow[r, "T(\iota_a)"'] \arrow[rr, bend left, "T(g)"] & TT(a) \arrow[r, "m_a"'] & T(a) \arrow[r, "q_{(a,\, g)}"] & \iota_!(a, g) \end{tikzcd} .$$

Since $\iota_! = U_{\mathsf{G}} L_{\mathsf{P}}$ by Proposition 3.6 and $\iota_! \cdot \phi_{\mathsf{P}} = \phi_{\mathsf{T}}$ by Proposition 1.18, both triangles in the diagram

$$\begin{tikzcd} \mathbb{A} \arrow[r, "\phi_{\mathsf{P}}"'] \arrow[rr, bend left, "K"] \arrow[dr, "\phi_{\mathsf{T}}"'] & \mathbb{A}_{\mathsf{P}} \arrow[r, "L_{\mathsf{P}}"'] \arrow[d, "\iota_!"] & (\mathbb{A}_{\mathsf{T}})^{\mathsf{G}} \arrow[dl, "U^{\mathsf{G}}"] \\ & \mathbb{A}_{\mathsf{T}} & \end{tikzcd} \tag{3.7}$$

commute. Write G_1 (respectively G_2) for the $\mathbb{A}_{\mathsf{T}}$-comonad generated by the adjunction $\phi_{\mathsf{T}} \dashv U_{\mathsf{T}}$ (respectively $\iota_! \dashv \iota^*$), and consider the related comonad morphism $t_{\phi_{\mathsf{P}}} : \mathsf{G}_1 \to \mathsf{G}_2$ (respectively $t_{L_{\mathsf{P}}} : \mathsf{G}_2 \to \mathsf{G}$) corresponding to the left (respectively right) triangle in the above diagram (see Sections 1.14 and 3.1). Since $U_{\mathsf{P}} K_{\mathsf{P}} = \overline{K}$ and ϕ_{P} (respectively L_{P})

has a right adjoint U_{P} (respectively K_{P}), it follows – by uniqueness of right adjoints – that $L_{\mathsf{P}} \cdot \phi_{\mathsf{P}} = K$. Thus we may apply [22, Proposition 1.21] to obtain the equality

$$t_K = t_{L_{\mathsf{P}}} \cdot t_{\phi_{\mathsf{P}}}. \tag{3.8}$$

Recall from Section 1.16 that $\iota_!$ can be obtained as the coequaliser of diagram (1.6).

3.7. Proposition. *If $K_{\mathsf{P}} : (\mathbb{A}_T)^G \to \mathbb{A}_{\mathsf{P}}$ admits a left adjoint, then $t_{\phi_{\mathsf{P}}} = q\iota^*$.*

Proof. Applying the results of Section 1.14 to the left triangle in diagram (3.7) gives that $t_{\phi_{\mathsf{P}}} = \iota_!\gamma$, with the composite

$$\gamma : \phi_{\mathsf{P}} U_{\mathsf{T}} \xrightarrow{\underline{\eta}\,\phi_{\mathsf{P}} U_{\mathsf{T}}} \iota^*\iota_!\phi_{\mathsf{P}} U_{\mathsf{T}} = \iota^*\phi_{\mathsf{T}} U_{\mathsf{T}} \xrightarrow{\iota^*\varepsilon_{\mathsf{T}}} \iota^*.$$

Here $\underline{\eta}$ is the unit of the adjunction $\iota_! \dashv \iota^*$, which (applying Theorem 1.15 to the diagram $U_{\mathsf{P}} \cdot \iota^* = U_{\mathsf{T}}$) is the unique natural transformation making the diagram

$$\begin{tikzcd}
\phi_{\mathsf{P}} U_{\mathsf{P}} \arrow[r, "\varepsilon_{\mathsf{P}}"] \arrow[d, "\gamma' U_{\mathsf{P}}"'] & 1 \arrow[d, "\underline{\eta}"] \\
\iota^*\phi_{\mathsf{T}} U_{\mathsf{P}} \arrow[r, "\iota^* q"] & \iota^*\iota_!
\end{tikzcd}$$

commute with the composite

$$\gamma' : \phi_{\mathsf{P}} \xrightarrow{\phi_{\mathsf{P}}\,\eta_{\mathsf{T}}} \phi_{\mathsf{P}} U_{\mathsf{T}}\phi_{\mathsf{T}} = \phi_{\mathsf{P}} U_{\mathsf{P}}\iota^*\phi_{\mathsf{T}} \xrightarrow{\varepsilon_{\mathsf{P}}\,\iota^*\phi_{\mathsf{T}}} \iota^*\phi_{\mathsf{T}}.$$

The equations $U_{\mathsf{P}}\gamma' = \iota$ and $\iota \cdot e^{\mathsf{P}} = e^{\mathsf{T}}$ imply commutativity of the diagram

$$\begin{tikzcd}
U_{\mathsf{P}} \arrow[r, "e^{\mathsf{P}} U_{\mathsf{P}}"] \arrow[dr, "e^{\mathsf{T}} U_{\mathsf{P}}"'] & PU_{\mathsf{P}} = U_{\mathsf{P}}\phi_{\mathsf{P}} U_{\mathsf{P}} \arrow[r, "U_{\mathsf{P}}\,\varepsilon_{\mathsf{P}}"] \arrow[d, "\iota U_{\mathsf{P}} = U_{\mathsf{P}}\gamma' U_{\mathsf{P}}"] & U_{\mathsf{P}} \arrow[d, "U_{\mathsf{P}}\underline{\eta}"] \\
& TU_{\mathsf{P}} = U_{\mathsf{P}}\iota^*\phi_{\mathsf{T}} U_{\mathsf{P}} \arrow[r, "U_{\mathsf{T}} q = U_{\mathsf{P}}\iota^* q"] & U_{\mathsf{P}}\iota^*\iota_!.
\end{tikzcd}$$

Since $U_{\mathsf{P}}\varepsilon_{\mathsf{P}} \cdot e^{\mathsf{P}} U_{\mathsf{P}} = 1$ (triangular identity for $\phi_{\mathsf{P}} \dashv U_{\mathsf{P}}$), it follows that $U_{\mathsf{P}}\underline{\eta}$ is the composite

$$U_{\mathsf{P}} \xrightarrow{e^{\mathsf{T}} U_{\mathsf{P}}} TU_{\mathsf{P}} = U_{\mathsf{P}}\iota^*\phi_{\mathsf{T}} U_{\mathsf{P}} \xrightarrow{U_{\mathsf{P}}\,\iota^* q} U_{\mathsf{P}}\iota^*\iota_!.$$

In particular, $U_{\mathsf{P}}\underline{\eta}\phi_{\mathsf{P}}$ is the composite

$$P \xrightarrow{e^{\mathsf{T}} P} TP = U_{\mathsf{P}}\iota^*\phi_{\mathsf{T}} U_{\mathsf{P}}\phi_{\mathsf{P}} \xrightarrow{U_{\mathsf{P}}\,\iota^* q\,\phi_{\mathsf{P}}} U_{\mathsf{P}}\iota^*\iota_!\phi_{\mathsf{P}} = U_{\mathsf{P}}\iota^*\phi_{\mathsf{T}} = U_{\mathsf{T}}\phi_{\mathsf{T}}.$$

By Remark 1.17(1), $q\phi_{\mathsf{P}}$ is the composite

$$\phi_{\mathsf{T}}P \xrightarrow{\phi_{\mathsf{T}}\iota} \phi_{\mathsf{T}}T \xrightarrow{\varepsilon_{\mathsf{T}}\phi_{\mathsf{T}}} \phi_{\mathsf{T}},$$

and $U_{\mathsf{P}}\underline{\eta}\phi_{\mathsf{P}}$ is the composite

$$P \xrightarrow{e^{\mathsf{T}}P} TP \xrightarrow{T\iota} TT \xrightarrow{m^{\mathsf{T}}} T.$$

Since $T\iota \cdot e^{\mathsf{T}}P = e^{\mathsf{T}}T \cdot \iota$ (by naturality) and $m^{\mathsf{T}} \cdot e^{\mathsf{T}} = 1$, one concludes that $U_{\mathsf{P}}\underline{\eta}\phi_{\mathsf{P}} = \iota$. For any $(a,h) \in \mathbb{A}_{\mathsf{T}}$, $(\varepsilon_{\mathsf{T}})_{(a,\,h)} = h$, and it turns out that $\gamma_{(a,h)}$ is just $P(a) \xrightarrow{\iota_a} T(a) \xrightarrow{h} a$.

Now, by Remark 1.17,

$$\begin{tikzcd} TPP(a) \arrow[r, shift left, "T(m_a^{\mathsf{P}})"] \arrow[r, shift right, "m_a^{\mathsf{T}} \cdot T(\iota_{P(a)})"'] & TP(a) \arrow[r, "T(\iota_a)"] & TT(a) \arrow[r, "m_a^{\mathsf{T}}"] & T(a) \end{tikzcd}$$

is the coequaliser defining $\iota_!(P(a), m_a^{\mathsf{P}}) = \iota_!(\phi_{\mathsf{P}}(a))$, and it follows that $\iota_!(\gamma_{(a,\,h)}) = (t_{\phi_{\mathsf{P}}})_{(a,\,h)}$ is the unique morphism leading to commutativity of the diagram

$$\begin{tikzcd} TP(a) \arrow[r, "T(\iota_a)"] \arrow[d, "T(\iota_a)"'] & TT(a) \arrow[r, "m_a^{\mathsf{T}}"] & T(a) \arrow[dd, "\iota_!(\gamma_{(a,h)})"] \\ TT(a) \arrow[d, "T(h)"'] & & \\ T(a) \arrow[rr, "q_{\iota^*(a,h)}"] & & \iota_!(\iota^*(a,h)). \end{tikzcd}$$

Since $\iota_a \cdot e_a^{\mathsf{P}} = e_a^{\mathsf{T}}$ and $m_a^{\mathsf{T}} \cdot T(e_a^{\mathsf{T}}) = 1 = h \cdot T(e_a^{\mathsf{T}})$, it follows from this diagram that $(t_{\phi_{\mathsf{P}}})_{(a,\,h)} = \iota_!(\gamma_{(a,h)}) = q_{\iota^*(a,\,h)}$, as claimed. $\square$

4. Weak entwinings

Let H be an endofunctor on any category $\mathbb{A}$, admitting both a monad $\underline{\mathsf{H}} = (H, m, e)$ and a comonad $\overline{\mathsf{H}} = (H, \delta, \varepsilon)$ structure, and define

$$\begin{aligned} \sigma &: HH \xrightarrow{\delta H} HHH \xrightarrow{Hm} HH, \\ \overline{\sigma} &: HH \xrightarrow{H\delta} HHH \xrightarrow{mH} HH. \end{aligned} \tag{4.1}$$

The class $\mathbf{Nat}(H,H)$ of all natural transformations from H to itself allows for the structure of a monoid by defining the (convolution) product of any two $\varphi, \varphi' \in \mathbf{Nat}(H,H)$ as the composite $\varphi * \varphi' = m \cdot \varphi\varphi' \cdot \delta$. The identity for this product is $e \cdot \varepsilon : H \to H$.

Recall that weak entwinings of tensor functors were defined by Caenepeel and De Groot in [14] and a more general theory was formulated by Böhm (e.g. [5, Example 5.2]).

4.1. Weak monad comonad entwinings. For a natural transformation $\omega : HH \to HH$, define the natural transformations

$$\begin{aligned}
\xi :&\quad H \xrightarrow{eH} HH \xrightarrow{\omega} HH \xrightarrow{\varepsilon H} H,\\
\kappa :&\quad HH \xrightarrow{eHH} HHH \xrightarrow{\omega H} HHH \xrightarrow{Hm} HH,\\
\kappa' :&\quad HH \xrightarrow{HHe} HHH \xrightarrow{H\omega} HHH \xrightarrow{mH} HH.
\end{aligned}$$

$(\underline{H}, \overline{H}, \omega)$ is called a *weak entwining* (from the monad $\underline{\mathsf{H}}$ to the comonad $\overline{\mathsf{H}}$) provided

$$\begin{aligned}
&\text{(i)} \qquad \omega \cdot mH = Hm \cdot \omega H \cdot H\omega, \quad \delta H \cdot \omega = H\omega \cdot \omega H \cdot H\delta,\\
&\text{(ii)} \qquad \omega \cdot eH = H\xi \cdot \delta, \quad \varepsilon H \cdot \omega = m \cdot H\xi,
\end{aligned} \tag{4.2}$$

and is said to be *compatible* if

$$\delta \cdot m = Hm \cdot \omega H \cdot H\delta. \tag{4.3}$$

It is easily checked that

$$\begin{aligned}
&\kappa \cdot He = \omega \cdot eH, \ \ \varepsilon H \cdot \kappa = m \cdot \xi H, && \text{always hold,}\\
&\xi * \xi = \xi, \quad \kappa \cdot \kappa = \kappa, \quad \kappa \cdot \omega = \omega, && \text{follow by (4.2)(i),}\\
&\kappa \cdot \delta = \delta, \quad \kappa \cdot \sigma = \sigma, \quad \xi * 1 = 1, && \text{follow by (4.3).}
\end{aligned} \tag{4.4}$$

4.2. Mixed bimodules. We write $\mathbb{A}^{\overline{\mathsf{H}}}_{\underline{\mathsf{H}}}(\omega)$ for the category of mixed H-bimodules, whose objects are triples (a, h, θ), where $(a, h) \in \mathbb{A}_{\underline{\mathsf{H}}}$, $(a, \theta) \in \mathbb{A}^{\overline{\mathsf{H}}}$ with commutative diagram

$$\begin{tikzcd}
H(a) \arrow[r, "h"] \arrow[d, "H(\theta)"'] & a \arrow[r, "\theta"] & H(a) \\
HH(a) \arrow[rr, "\omega_a"] & & HH(a), \arrow[u, "H(h)"']
\end{tikzcd}$$

and whose morphisms are those in $\mathbb{A}$ which are $\underline{\mathsf{H}}$-module as well as $\overline{\mathsf{H}}$-comodule morphisms.

The following is a particular case of [5, Proposition 5.7].

4.3. Proposition. *Let $\mathsf{H} = (\underline{H}, \overline{\mathsf{H}}, \omega)$ be a weak entwining on $\mathbb{A}$. Then the composite*

$$\Gamma : HU_{\underline{H}} \xrightarrow{eHU_{\underline{H}}} HHU_{\underline{H}} \xrightarrow{\omega U_{\underline{H}}} HHU_{\underline{H}} = HU_{\underline{H}}\phi_{\underline{H}}U_{\underline{H}} \xrightarrow{HU_{\underline{H}}\varepsilon_{\underline{H}}} HU_{\underline{H}}$$

is an idempotent, and if

$$\begin{tikzcd} HU_{\underline{H}} \arrow[rr, "\Gamma"] \arrow[dr, "p"'] & & HU_{\underline{H}} \\ & G \arrow[ur, "i"'] & \end{tikzcd}$$

is its splitting, then there is a comonad $\mathsf{G} = (\widetilde{G}, \widetilde{\delta}, \widetilde{\varepsilon})$ on $\mathbb{A}_{\underline{H}}$, whose functor part takes an arbitrary $(a,h) \in \mathbb{A}_{\underline{H}}$ to

$$(G(a,h),\, p_{(a,h)} \cdot H(h) \cdot \omega_a \cdot H(i_{(a,h)}) : HG(a,h) \to G(a,h)),$$

and whose comultiplication $\widetilde{\delta}$ and counit $\widetilde{\varepsilon}$ evaluated at (a,h) are the composites, respectively,

$$G(a,h) \xrightarrow{i_{(a,h)}} H(a) \xrightarrow{\delta_a} HH(a) \xrightarrow{H(p_{(a,h)})} HG(a,h) \xrightarrow{p_{G(a,h)}} GG(a,h),$$

$$G(a,h) \xrightarrow{i_{(a,h)}} H(a) \xrightarrow{h} a.$$

We call G the *comonad induced by* $\mathsf{H} = (\underline{\mathsf{H}}, \overline{\mathsf{H}}, \omega)$. Obviously, $U_{\underline{\mathsf{H}}}\widetilde{G} = G$.

4.4. Theorem. *Let $H = (\underline{H}, \overline{H}, \omega)$ be a weak entwining on a Cauchy complete category $\mathbb{A}$ and G the induced comonad on $\mathbb{A}_{\underline{H}}$. Then there is an isomorphism of categories*

$$\Phi : \mathbb{A}^{\overline{H}}_{\underline{H}}(\omega) \to (\mathbb{A}_{\underline{H}})^{G}, \quad (a,h,\theta) \;\mapsto\; ((a,h), p_{(a,h)} \cdot \theta),$$

with the inverse given by $\Phi^{-1}((a,h),\zeta) = (a,h,i_{(a,h)} \cdot \zeta)$.

Proof. Since $p\,i = 1$, it is clear that $\Phi\Phi^{-1} = 1$. To show that $\Phi^{-1}\Phi = 1$, consider an arbitrary object $(a,h,\theta) \in \mathbb{A}^{\overline{\mathsf{H}}}_{\underline{\mathsf{H}}}(\omega)$. In the diagram

$$\begin{tikzcd} a \arrow[r, "e_a"] \arrow[d, "\theta"'] & H(a) \arrow[r, "h"] \arrow[d, "H(\theta)"'] & a \arrow[drr, "\theta"] & & \\ H(a) \arrow[r, "e_{H(a)}"] \arrow[rrr, bend right, "\Gamma_{(a,h)} = i_{(a,h)} \cdot p_{(a,h)}"'] & HH(a) \arrow[r, "\omega_a"] & HH(a) \arrow[r, "H(h)"] & H(a) \end{tikzcd}$$

the square commutes by naturality of e, while the trapezium commutes since $(a,h,\theta) \in \mathbb{A}^{\overline{\mathsf{H}}}_{\underline{\mathsf{H}}}(\omega)$. Since $h \cdot e = 1$, this means

$$\theta = \Gamma_{(a.h)} \cdot \theta = i_{(a,h)} \cdot p_{(a,h)} \cdot \theta.$$

Thus $\Phi^{-1}\Phi(a,h,\theta) = (a,h,\theta)$, that is, $\Phi^{-1}\Phi = 1$. $\square$

Again by [5, Proposition 5.7], we get as counterpart of Proposition 4.3:

4.5. Proposition. *Let $\mathsf{H} = (\underline{\mathsf{H}}, \overline{\mathsf{H}}, \omega)$ be a weak entwining on $\mathbb{A}$. Then*

$$\Gamma' : HU^{\overline{\mathsf{H}}} \xrightarrow{HU_{\overline{\mathsf{H}}}\eta^{\overline{\mathsf{H}}}} HU^{\overline{\mathsf{H}}}\phi^{\overline{\mathsf{H}}}U^{\overline{\mathsf{H}}} = HHU^{\overline{\mathsf{H}}} \xrightarrow{\omega U^{\overline{\mathsf{H}}}} HHU^{\overline{\mathsf{H}}} \xrightarrow{\varepsilon HU^{\overline{\mathsf{H}}}} HU^{\overline{\mathsf{H}}}$$

is an idempotent, and if

$$\begin{tikzcd} HU^{\overline{\mathsf{H}}} \arrow[rr, "\Gamma'"] \arrow[dr, "p'"'] & & HU^{\overline{\mathsf{H}}} \\ & T \arrow[ur, hook, "i'"'] & \end{tikzcd}$$

is its splitting, then there is a comonad $\mathsf{T} = (\widetilde{T}, \widetilde{m}, \widetilde{e})$ on $\mathbb{A}^{\overline{\mathsf{H}}}$, whose functor part takes an arbitrary $(a, \theta) \in \mathbb{A}^{\overline{\mathsf{H}}}$ to

$$(T(a,\theta), H(p'_{(a,\theta)}) \cdot \omega_a \cdot H(\theta) \cdot i'_{(a,\theta)} : T(a,\theta) \to HT(a,\theta)),$$

and whose multiplication $\widetilde{m}$ and unit $\widetilde{e}$, evaluated at an $\overline{\mathsf{H}}$-comodule (a, θ), are the composites, respectively,

$$TT(a,\,\theta) \xrightarrow{i'_{T(a,\theta)}} HT(a,\theta) \xrightarrow{H(i'_{(a,\theta)})} HH(a) \xrightarrow{m_a} H(a) \xrightarrow{p'_{G(a,\theta)}} T(a,\theta),$$
$$a \xrightarrow{\theta} H(a) \xrightarrow{p'_{(a,\theta)}} T(a,\theta).$$

We call T the *monad induced by* $\mathsf{H} = (\underline{\mathsf{H}}, \overline{\mathsf{H}}, \omega)$. Obviously, $U^{\overline{\mathsf{H}}}\widetilde{T} = T$.

4.6. Theorem. *Let $\mathsf{H} = (\underline{\mathsf{H}}, \overline{\mathsf{H}}, \omega)$ be a weak entwining on a Cauchy complete category $\mathbb{A}$ and T the induced monad on $\mathbb{A}^{\overline{\mathsf{H}}}$. Then there is an isomorphism of categories*

$$\Phi' : \mathbb{A}^{\overline{\mathsf{H}}}_{\underline{\mathsf{H}}}(\omega) \to (\mathbb{A}^{\overline{\mathsf{H}}})_{\mathsf{T}}, \quad (a, h, \theta) \;\mapsto\; ((a,\theta), h \cdot i'_{(a,\theta)}),$$

with the inverse given by $(\Phi')^{-1}((a,\theta), g) = (a, g \cdot i'_{(a,h)}, \theta)$.

4.7. Comparison functors. Let $\mathsf{H} = (\underline{\mathsf{H}}, \overline{\mathsf{H}}, \omega)$ be a compatible weak entwining on $\mathbb{A}$. By (4.3), there is a functor (e.g. [18, Lemma 5.1])

$$K_\omega : \mathbb{A} \to \mathbb{A}^{\overline{\mathsf{H}}}_{\underline{\mathsf{H}}}(\omega), \quad a \;\mapsto\; (H(a), m_a, \delta_a).$$

Precomposing K_ω with Φ and Φ' gives functors

$$K : \mathbb{A} \to (\mathbb{A}_{\underline{\mathsf{H}}})^{\mathsf{G}}, \quad a \;\mapsto\; ((H(a), m_a),\, p_{(H(a),\, m_a)} \cdot \delta_a),$$
$$K' : \mathbb{A} \to (\mathbb{A}^{\overline{\mathsf{H}}})_{\mathsf{T}}, \quad a \;\mapsto\; ((H(a), \delta_a),\, m_a \cdot i'_{(H(a),\, m_a)}),$$

leading to commutative diagrams

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, "K"] \arrow[d, "\phi_{\underline{\mathsf{H}}}"'] & (\mathbb{A}_{\underline{\mathsf{H}}})^{\mathsf{G}} \arrow[dl, "U^{\mathsf{G}}"] \\
\mathbb{A}_{\underline{\mathsf{H}}} &
\end{tikzcd} ,
\quad
\begin{tikzcd}
\mathbb{A} \arrow[r, "K'"] \arrow[d, "\phi^{\overline{\mathsf{H}}}"'] & (\mathbb{A}^{\overline{\mathsf{H}}})_{\mathsf{T}} \arrow[dl, "U_{\mathsf{T}}"] \\
\mathbb{A}^{\overline{\mathsf{H}}} &
\end{tikzcd} ,
\quad
\begin{tikzcd}
\mathbb{A} \arrow[r, "K"] \arrow[d, "K'"'] \arrow[dr, "K_\omega"] & (\mathbb{A}_{\underline{\mathsf{H}}})^{\mathsf{G}} \\
(\mathbb{A}^{\overline{\mathsf{H}}})_{\mathsf{T}} & \mathbb{A}^{\overline{\mathsf{H}}}_{\underline{\mathsf{H}}}(\omega) . \arrow[l, "\Phi'"] \arrow[u, "\Phi"']
\end{tikzcd}
\tag{4.5}
$$

We will use that the splittings of Γ, Γ' (from 4.3, 4.5) lead to splittings of κ, κ' (see 4.1),

$$
\begin{tikzcd}
HH \arrow[rr, "\kappa"] \arrow[dr, two heads, "\overline{p}=p\,\phi_{\underline{\mathsf{H}}}"'] & & HH \\
& \overline{G}=G\phi_{\underline{\mathsf{H}}} \arrow[ur, tail, "\overline{i}=i\phi_{\underline{\mathsf{H}}}"'] &
\end{tikzcd} ,
\quad
\begin{tikzcd}
HH \arrow[rr, "\kappa'"] \arrow[dr, two heads, "\overline{p}\,'=p'\,\phi_{\underline{\mathsf{H}}}"'] & & HH \\
& \overline{T}=T\phi_{\underline{\mathsf{H}}} \arrow[ur, tail, "\overline{i}\,'=i'\phi_{\underline{\mathsf{H}}}"'] &
\end{tikzcd} .
\tag{4.6}
$$

4.8. Proposition. *In the situation described above, consider the comonad morphism* $t : \phi_{\underline{H}}U_{\underline{H}} \to \mathsf{G}$ *induced by the left triangle in (4.5) (see Section 3.1).*

(1) *For any $\underline{H}$-module (a,h), the (a,h)-component for t is the composite*

$$t_{(a,h)} : H(a) \xrightarrow{\delta_a} HH(a) \xrightarrow{H(h)} H(a) \xrightarrow{p_{(a,h)}} G(a,h).$$

(2) *For any $a \in \mathbb{A}$, the $\phi_{\underline{H}}(a)$-component for t,*

$$t_{\phi_{\underline{H}}(a)} : HH(a) \xrightarrow{\delta_{H(a)}} HHH(a) \xrightarrow{Hm_a} HH(a) \xrightarrow{\overline{p}_a} \overline{G}(a),$$

is the unique morphism leading to commutativity of the diagram

$$
\begin{tikzcd}
HH(a) \arrow[r, "t_{\phi_{\underline{H}}(a)}"] \arrow[dr, "\sigma_a"'] & \overline{G}(a) \arrow[d, "\overline{i}_a"] \\
& HH(a).
\end{tikzcd}
$$

Proof. (1) Since $K(a) = ((H(a), m_a), p_{(H(a),\, m_a)} \cdot \delta_a)$, the left G-comodule structure $\alpha : \phi_{\underline{\mathsf{H}}} \to \mathsf{G}\phi_{\underline{\mathsf{H}}}$ on $\phi_{\underline{\mathsf{H}}}$ corresponding to the left triangle in (4.5), has for its a-component $\alpha_a = p_{(H(a),\, m_a)} \cdot \delta_a$. It then follows from Section 3.1 that, for any $(a,h) \in \mathbb{A}_{\underline{\mathsf{H}}}$, the (a,h)-component $t_{(a,h)}$ is the composite

$$t_{(a,h)} : H(a) \xrightarrow{\delta_a} HH(a) \xrightarrow{p_{(H(a),\, m_a)}} G(H(a), m_a) \xrightarrow{G(h)} G(a,h),$$

which, by naturality of composition, is the same as

$$t_{(a,h)} : H(a) \xrightarrow{\delta_a} HH(a) \xrightarrow{H(h)} H(a) \xrightarrow{p_{(a,h)}} G(a,h).$$

Then, in particular, $t_{\phi_{\underline{\mathsf{H}}}(a)} = p_{\phi_{\underline{\mathsf{H}}}(a)} \cdot H(m_a) \cdot \delta_{H(a)} = \overline{p}_a \cdot \sigma_a$.

(2) Since $\overline{G} \xrightarrow{\overline{i}} HH \overset{\kappa}{\underset{1}{\rightrightarrows}} HH$ is an equaliser diagram and $\kappa \cdot \sigma = \sigma$ (see (4.4)), there is a unique morphism $j : HH \to \overline{G}$ such that $\overline{i} \cdot j = \sigma$. Then $t_{\phi_{\underline{\mathsf{H}}}(a)} = \overline{p}_a \cdot \sigma_a = \overline{p}_a \cdot \overline{i}_a \cdot j_a = j_a$ and the result follows. □

Symmetrically, we have:

4.9. Proposition. *In the situation described in 4.7, the monad morphism $t : \mathsf{T} \to \phi^{\overline{H}} U^{\overline{H}}$ induced by the right triangle in (4.5) (see Section 3.1), has for its (a,θ)-component*

$$t_{(a,\theta)} : T(a,\theta) \xrightarrow{i'_{(a,\theta)}} H(a) \xrightarrow{H(\theta)} HH(a) \xrightarrow{m_a} H(a).$$

Our general results from Section 1 now yield:

4.10. Proposition. *Let $\mathsf{H} = (\underline{\mathsf{H}}, \overline{\mathsf{H}}, \omega)$ be a compatible weak entwining on $\mathbb{A}$. Then the functor $K : \mathbb{A} \to (\mathbb{A}_{\underline{\mathsf{H}}})^{\mathsf{G}}$ (and hence also $K_\omega : \mathbb{A} \to \mathbb{A}^{\overline{H}}_{\underline{H}}(\omega)$) has a right adjoint if and only if, for any $(a,h,\theta) \in \mathbb{A}^{\overline{H}}_{\underline{H}}(\omega)$, the pair of morphisms*

$$\begin{tikzcd} a \arrow[r, "e_a"'] \arrow[rrr, bend left, "\theta"] & H(a) \arrow[r, "\delta_a"'] & HH(a) \arrow[r, "H(h)"'] & H(a) \end{tikzcd} \tag{4.7}$$

has an equaliser in $\mathbb{A}$.

Proof. Since the functor $U^{\mathsf{G}} : (\mathbb{A}_{\underline{\mathsf{H}}})^{\mathsf{G}} \to \mathbb{A}$ is clearly (pre)comonadic, it follows from Theorem 1.15 that the functor $K : \mathbb{A} \to (\mathbb{A}_{\underline{\mathsf{H}}})^{\mathsf{G}}$ admits a right adjoint if and only if for any $((a,h),\nu) \in (\mathbb{A}_{\underline{\mathsf{H}}})^{\mathsf{G}}$, the pair

$$a \overset{\nu}{\underset{\beta_{(a,\,h)}}{\rightrightarrows}} G(a,h),$$

where $\beta : U_{\underline{\mathsf{H}}} \to U_{\underline{\mathsf{H}}} G$ is the right G-comodule structure on $U_{\underline{\mathsf{H}}} : \mathbb{A}_{\underline{\mathsf{H}}} \to \mathbb{A}$ induced by the triangle (4.5) (see Section 3.1), has an equaliser in $\mathbb{A}$, which – since $i : G \to HU_{\underline{\mathsf{H}}}$ is a (split) monomorphism – is the case if and only if the pair

$$a \overset{i_{(a,\,h)} \cdot \nu}{\underset{i_{(a,\,h)} \cdot \beta_{(a,\,h)}}{\rightrightarrows}} H(a)$$

has one. According to Propositions 4.8 and 3.1, $\beta_{(a,\,h)}$ is the composite

$$a \xrightarrow{e_a} H(a) \xrightarrow{\delta_a} HH(a) \xrightarrow{p_{(H(a),\,m_a)}} G(H(a), m_a) \xrightarrow{G(h)} G(a, h).$$

Since $\kappa \cdot \delta = \delta$ by (4.4) and $\Gamma = i \cdot p$, it follows by naturality of i that the diagram

$$\begin{tikzcd}
a \arrow[r, "e_a"'] \arrow[rrrr, bend left, "\beta_{(a,\,h)}"] & H(a) \arrow[r, "\delta_a"] \arrow[rrd, bend right, "\delta_a"'] & HH(a) \arrow[r, "p_{(H(a),m_a)}"] \arrow[rd, "\kappa_a = \Gamma_{(H(a),m_a)}"'] & G(H(a), m_a) \arrow[r, "G(h)"'] \arrow[d, "i_{(H(a),m_a)}"] & G(a, h) \arrow[d, "i_{(a,\,h)}"] \\
 & & & HH(a) \arrow[r, "H(h)"'] & H(a).
\end{tikzcd}$$

is commutative. So we have

$$i_{(a,\,h)} \cdot \beta_{(a,\,h)} = H(h) \cdot \delta_a \cdot e_a.$$

Thus, the functor $K : \mathbb{A} \to (\mathbb{A}_{\underline{\mathsf{H}}})^{\mathsf{G}}$ has a right adjoint if and only if for any $((a, h), \nu) \in (\mathbb{A}_{\underline{\mathsf{H}}})^{\mathsf{G}}$, the pair of morphisms

$$\begin{tikzcd}
a \arrow[r, shift left, "i_{(a,h)} \cdot \nu"] \arrow[r, shift right, "H(h) \cdot \delta_a \cdot e_a"'] & H(a)
\end{tikzcd}$$

has an equaliser. Recalling that $\Phi : \mathbb{A}_{\underline{\mathsf{H}}}^{\overline{\mathsf{H}}}(\omega) \to (\mathbb{A}_{\underline{\mathsf{H}}})^{\mathsf{G}}$ is an isomorphism of categories and $\Phi^{-1}((a, h), \nu) = (a, h, i_{(a,\,h)} \cdot \nu)$ gives the desired result. $\square$

Symmetrically, we have:

4.11. Proposition. *Let $H = (\underline{H}, \overline{H}, \omega)$ be a compatible weak entwining on $\mathbb{A}$. Then the functor $K' : \mathbb{A} \to (\mathbb{A}^{\overline{H}})_T$ (and hence also $K_\omega : \mathbb{A} \to \mathbb{A}_{\underline{H}}^{\overline{H}}(\omega)$) has a left adjoint if and only if for any $(a, h, \theta) \in \mathbb{A}_{\underline{H}}^{\overline{H}}(\omega)$, the pair of morphisms*

$$\begin{tikzcd}
H(a) \arrow[r, "H(\theta)"'] \arrow[rrr, bend left, "h"] & HH(a) \arrow[r, "m_a"'] & H(a) \arrow[r, "\varepsilon_a"'] & a
\end{tikzcd} \tag{4.8}$$

has a coequaliser in $\mathbb{A}$.

Symmetric to 4.1 one may consider

4.12. Weak comonad monad entwinings. For a natural transformation $\overline{\omega}: \overline{H}\underline{H} \to \underline{H}\overline{H}$, define the natural transformation

$$\overline{\xi}: \quad H \xrightarrow{He} HH \xrightarrow{\overline{\omega}} HH \xrightarrow{H\varepsilon} H.$$

$(\overline{H}, \underline{H}, \overline{\omega})$ is called a *weak entwining* (from comonad $\overline{\mathsf{H}}$ to monad $\underline{\mathsf{H}}$) provided

$$\begin{aligned} &\text{(i)} \quad \overline{\omega} \cdot Hm = mH \cdot H\overline{\omega} \cdot \overline{\omega}H, \quad H\delta \cdot \overline{\omega} = \overline{\omega}H \cdot H\overline{\omega} \cdot \delta H, \\ &\text{(ii)} \quad \overline{\omega} \cdot He = \overline{\xi}H \cdot \delta, \quad H\varepsilon \cdot \overline{\omega} = m \cdot \overline{\xi}H, \end{aligned} \tag{4.9}$$

and is said to be compatible if

$$\delta \cdot m = mH \cdot H\overline{\omega} \cdot \delta H. \tag{4.10}$$

Here we get

$$\overline{\xi} * \overline{\xi} = \overline{\xi}, \quad 1 * \overline{\xi} = 1\,. \tag{4.11}$$

Certainly, the theory for this notion will be similar to that for monad comonad entwinings. However, the mixed bimodules (as in 4.2) do not play the same role here but are to be replaced by liftings to Kleisli categories. Nevertheless, comonad monad entwinings will enter the picture in the next section.

5. Weak braided bimonads

In the theory of Hopf algebras H over a field k, the twist map for k-vector spaces M, N, $\mathsf{tw}_{M,N}: M \otimes_k N \to N \otimes_k M$, plays a crucial part. In particular it helps to commute $H \otimes_k -$ with itself by $\mathsf{tw}_{H,H}: H \otimes_k H \to H \otimes_k H$. Generalising this to monoidal categories, often a *braiding* is required, that is, a condition on the whole category. It was observed (e.g. in [23]) that it can be enough to have such a twist only for the functor H under consideration, that is, a natural isomorphism $\tau: HH \to HH$ satisfying the Yang–Baxter equation. For the study of *weak braided Hopf algebras*, Alonso Álvarez e.a. suggested in [1, Definition 1.2] to consider, for any object D in a monoidal category, a *weak Yang–Baxter operator* $t_{D,D}: D \otimes D \to D \otimes D$, which is not necessarily invertible but only regular. Here we take up this notion and formulate it for any functor on an arbitrary category.

5.1. Weak Yang Baxter operator. Given an endofunctor $H: \mathbb{A} \to \mathbb{A}$, a pair of natural transformations $\tau, \tau': HH \to HH$ is said to be a *weak YB-pair* provided the following equalities hold:

$$\tau \cdot \tau' \cdot \tau = \tau, \quad \tau' \cdot \tau \cdot \tau' = \tau', \quad \tau \cdot \tau' = \tau' \cdot \tau, \tag{5.1}$$

$$\begin{aligned} H\tau \cdot \tau H \cdot H\tau &= \tau H \cdot H\tau \cdot \tau H, \\ H\tau' \cdot \tau' H \cdot H\tau' &= \tau' H \cdot H\tau' \cdot \tau H', \end{aligned} \tag{5.2}$$

and for $\nabla := \tau \cdot \tau'$,

$$\begin{aligned} \tau H \cdot H\nabla = H\nabla \cdot \tau H, &\quad H\tau \cdot \nabla H = H\tau \cdot \nabla H, \\ \tau' H \cdot H\nabla = H\nabla \cdot \tau' H, &\quad H\tau' \cdot \nabla H = H\tau' \cdot \nabla H. \end{aligned} \tag{5.3}$$

The conditions in (5.2) are the usual Yang–Baxter equations for τ and τ', respectively. The equations in (5.1) and (5.3) are obviously satisfied if $\tau' = \tau^{-1}$ and in this case τ is known as *Yang–Baxter operator*.

5.2. Definition. Let $\underline{\mathsf{H}} = (H, m, e)$ a monad, $\overline{\mathsf{H}} = (H, \delta, \varepsilon)$ a comonad on $\mathbb{A}$, and $\tau, \tau' : HH \to HH$ a weak YB-pair with $\nabla := \tau \cdot \tau'$. The triple $\mathsf{H} = (\underline{\mathsf{H}}, \overline{\mathsf{H}}, \tau)$ is called a *weak braided bimonad* (or *weak τ-bimonad*) provided

(1) $m \cdot \nabla = m$, $\nabla \cdot \delta = \delta$;
(2) $\nabla \cdot He = \tau \cdot eH$, $H\varepsilon \cdot \nabla = \varepsilon H \cdot \tau$, $\nabla \cdot eH = \tau \cdot He$, $\varepsilon H \cdot \nabla = H\varepsilon \cdot \tau$;
(3) $\delta H \cdot \tau = H\tau \cdot \tau H \cdot H\delta$, $\tau \cdot mH = Hm \cdot \tau H \cdot H\tau$;
(4) $H\delta \cdot \tau = \tau H \cdot H\tau \cdot \delta H$, $\tau \cdot Hm = mH \cdot H\tau \cdot \tau H$;
(5) $\delta \cdot m = mm \cdot H\tau H \cdot \delta\delta$;
(6) $\varepsilon\varepsilon \cdot mm \cdot H\delta H \;=\; \varepsilon \cdot m \cdot mH \;=\; \varepsilon\varepsilon \cdot mm \cdot H\tau' H \cdot H\delta H$;
(7) $HmH \cdot \delta\delta \cdot ee \;=\; \delta H \cdot \delta \cdot e \;=\; HmH \cdot H\tau' H \cdot \delta\delta \cdot ee$.

For vector space categories and (finite dimensional) tensor functors $H \otimes -$ with τ the twist map, these conditions were introduced in [10, Definition 1]. For monoidal categories and monoidal functors the conditions are those for a *weak braided bialgebra* introduced and studied by Alonso Álvarez e.a. [1,2] and we can – and will – freely use essential parts of their results in our situation. Note that if ∇ is the identity of H, the conditions (1)–(4) in the definition describe the invertible double entwinings considered in [23].

The following observations provide the key to apply our previous results.

5.3. Related entwinings. Given the data from Definition 5.2, define

$$\omega : HH \xrightarrow{\delta H} HHH \xrightarrow{H\tau} HHH \xrightarrow{mH} HH,$$

$$\overline{\omega} : HH \xrightarrow{H\delta} HHH \xrightarrow{\tau H} HHH \xrightarrow{Hm} HH.$$

From the Sections 4.1 and 4.12 we get the natural transformations

$$\xi : H \xrightarrow{eH} HH \xrightarrow{\omega} HH \xrightarrow{\varepsilon H} H, \quad \overline{\xi} : H \xrightarrow{He} HH \xrightarrow{\overline{\omega}} HH \xrightarrow{H\varepsilon} H,$$

and the obvious equalities

$$\omega \cdot He = \delta = \overline{\omega} \cdot eH, \qquad \xi \cdot e = e = \overline{\xi} \cdot e.$$

Recalling σ and $\overline{\sigma}$ from 4.1, we define

$$\chi : H \xrightarrow{eH} HH \xrightarrow{\sigma} HH \xrightarrow{H\varepsilon} H, \quad \overline{\chi} : H \xrightarrow{He} HH \xrightarrow{\overline{\sigma}} HH \xrightarrow{\varepsilon H} H.$$

With these notions we collect the basic identities proved in [2].

5.4. Proposition. *Let $\mathsf{H} = (\underline{H}, \overline{H}, \tau)$ be a weak braided bimonad on $\mathbb{A}$. Then*

(1) *ξ, $\overline{\xi}$, χ and $\overline{\chi}$ are idempotent (w.r.t. composition) and respect unit and counit of H;*
(2) *$\xi \cdot m \cdot H\xi = \xi \cdot m$ and $\overline{\xi} \cdot m \cdot \overline{\xi}H = \overline{\xi} \cdot m$;*
(3) *$H\xi \cdot \delta \cdot \xi = \delta \cdot \xi$ and $\overline{\xi}H \cdot \delta \cdot \overline{\xi} = \delta \cdot \overline{\xi}$;*
(4) *$\sigma \cdot eH = \chi H \cdot \delta$ and $\overline{\sigma} \cdot He = H\overline{\chi} \cdot \delta$;*
(5) *$H\varepsilon \cdot \sigma = m \cdot H\chi$ and $\varepsilon H \cdot \overline{\sigma} = m \cdot \overline{\chi}H$;*
(6) *$\xi \cdot \chi = \xi$, $\xi \cdot \overline{\chi} = \overline{\chi}$, $\chi \cdot \xi = \chi$, $\overline{\chi} \cdot \xi = \xi$, $\overline{\xi} \cdot \chi = \chi$, $\overline{\xi} \cdot \overline{\chi} = \overline{\xi}$, $\chi \cdot \overline{\xi} = \overline{\xi}$, $\overline{\chi} \cdot \overline{\xi} = \overline{\xi}$.*

Proof. (1) is shown in [2, Proposition 2.9]; (2), (3) are from [2, Proposition 2.14]; (4) is shown in [2, Proposition 2.6], (5) in [2, Proposition 2.4], and (6) in [2, Proposition 2.10]. □

The following shows the way to apply results from the preceding section.

5.5. Proposition. *Let $\mathsf{H} = (\underline{H}, \overline{H}, \tau)$ be a weak braided bimonad. Then*

- *$(\underline{H}, \overline{H}, \omega)$ is a compatible weak (monad comonad) entwining;*
- *$(\overline{H}, \underline{H}, \overline{\omega})$ is a compatible weak (comonad monad) entwining.*

Proof. As easily seen, condition (5) in 5.2 yield the equalities (4.3) and (4.10) and also implies (4.2(i)) and (4.9(i)) for ω and $\overline{\omega}$, respectively (e.g. [7,22]). Now Propositions 2.3 and 2.5 in [2] show the equations in (4.2)(ii) and (4.9)(ii). □

Direct inspection yields the technical observation:

5.6. Lemma. *Suppose that $f, g : X \to X$ are idempotents in an arbitrary category such that $fg = g$ and $gf = f$. If $X \xrightarrow{p_f} X_f \xrightarrow{i_f} X$ (resp. $X \xrightarrow{p_g} X_g \xrightarrow{i_g} X$) is a splitting of the idempotent f (resp. g), then*

$$X_g \xrightarrow{i_g} X \underset{1}{\overset{f}{\rightrightarrows}} X \qquad (resp.\ \ X_f \xrightarrow{i_f} X \underset{1}{\overset{g}{\rightrightarrows}} X)$$

is a (split) equaliser diagram, while

$$X \underset{1}{\overset{f}{\rightrightarrows}} X \xrightarrow{p_g} X_g \qquad (\textit{resp.}\ \ X \underset{1}{\overset{g}{\rightrightarrows}} X \xrightarrow{p_f} X_f)$$

is a (split) coequaliser diagram.

Henceforth we work over a Cauchy complete category $\mathbb{A}$, with a fixed a splitting of $\overline{\xi}$,

$$\begin{tikzcd} H \arrow[rr, "\overline{\xi}"] \arrow[dr, "q^{\overline{\xi}}"'] & & H \\ & H^{\overline{\xi}} \arrow[ur, "\iota^{\overline{\xi}}"'] & \end{tikzcd} . \tag{5.4}$$

5.7. Proposition. *In the situation of Proposition 4.10, the diagram*

$$\begin{tikzcd} H^{\overline{\xi}} \arrow[r, "\iota^{\overline{\xi}}"] & H \arrow[r, "eH"'] \arrow[rrr, bend left, "\delta"] & HH \arrow[r, "\delta H"'] & HHH \arrow[r, "Hm"'] & HH \end{tikzcd} \tag{5.5}$$

is a (split) equaliser in $[\mathbb{A}, \mathbb{A}]$.

Proof. Since $Hm \cdot \delta H \cdot eH = \chi H \cdot \delta$ by Proposition 5.4 (4), we have to show that the diagram

$$\begin{tikzcd} H^{\overline{\xi}} \arrow[r, "\iota^{\overline{\xi}}"] & H \arrow[r, "\delta"'] \arrow[rr, bend left, "\delta"] & HH \arrow[r, "\chi H"'] & HH \end{tikzcd}$$

is a split equaliser. Let us first show that the pair

$$\begin{tikzcd} H \arrow[r, "\delta"'] \arrow[rr, bend left, "\delta"] & HH \arrow[r, "\chi H"'] & HH \end{tikzcd} \tag{5.6}$$

is cosplit by the morphism $H\varepsilon : HH \to H$. Indeed, since $H\varepsilon \cdot \delta = 1$ and $H\varepsilon \cdot \chi H \cdot \delta = \chi \cdot H\varepsilon \cdot \delta = \chi$, it remains to show that $\chi H \cdot \delta \cdot \chi = \delta \cdot \chi$. For this, consider the diagram

$$\begin{tikzcd}
H \arrow[rrr, "\chi"] \arrow[d, "eH"] \arrow[dddd, bend right=40, "\chi H\cdot\delta"'] \arrow[drrr, phantom, "(1)"] & & & H \arrow[d, "eH"'] \arrow[ddd, bend left=40, "\chi H\cdot\delta"] \\
HH \arrow[rrr, "H\chi"] \arrow[dd, "\delta H"] \arrow[dr, "\delta H"] \arrow[drrr, phantom, "(2)"] & & & HH \arrow[d, "\delta H"'] \\
& HHH \arrow[rr, "HH\chi"] \arrow[d, "H\delta H"] \arrow[dl, phantom, "(3)"] \arrow[drr, phantom, "(4)"] & & HHH \arrow[d, "Hm"'] \\
HHH \arrow[r, "\delta HH"'] \arrow[d, "Hm"] \arrow[dr, phantom, "(5)"] & HHHH \arrow[r, "HHm"] & HHH \arrow[r, "HH\varepsilon"] \arrow[d, phantom, "(6)"] & HH \\
HH \arrow[urr, "\delta H"'] \arrow[rr, "H\varepsilon"'] & & H \arrow[ur, "\delta"'] &
\end{tikzcd}$$

in which the

- regions (1), (2), (5) and (6) commute by naturality of composition;
- region (3) commutes by coassociativity of δ;
- region (4) commutes by Proposition 5.4(5);
- the curved regions commute by Proposition 5.4(4).

Hence the whole diagram commutes, implying

$$\chi H \cdot \delta \cdot \chi = \delta \cdot H\varepsilon \cdot \chi H \cdot \delta = \delta \cdot \chi.$$

So the pair (5.6) is cosplit by the morphism $H\varepsilon$ and hence one finds its equaliser by splitting the idempotent $\chi = H\varepsilon \cdot \chi H \cdot \delta$. But since $\overline{\xi} \cdot \chi = \chi$ and $\chi \cdot \overline{\xi} = \overline{\xi}$ by Proposition 5.4 (6) and $\iota^{\overline{\xi}} \cdot q^{\overline{\xi}}$ is the splitting of the idempotent $\overline{\xi}$ (see (5.4)), it follows from Lemma 5.6 that (5.5) is a (split) equaliser diagram. □

Symmetrically, we have:

5.8. Proposition. *In the situation of Proposition 4.11, the diagram*

$$\begin{tikzcd}
HH \arrow[r, "H\delta"'] \arrow[rrr, bend left, "m"] & HHH \arrow[r, "mH"'] & HH \arrow[r, "\varepsilon H"'] & H \arrow[r, "q^{\overline{\xi}}"] & H^{\overline{\xi}}
\end{tikzcd}$$

is a (split) coequaliser diagram in $[\mathbb{A}, \mathbb{A}]$.

Since $Hm \cdot \delta H \cdot eH = \chi H \cdot \delta$ and $\varepsilon H \cdot mH \cdot H\delta = m \cdot \overline{\chi} H$ by Proposition 5.4(4) and (5), Propositions 5.7 and 5.8 immediately yield:

5.9. Corollary. *Let $\mathsf{H} = (\underline{H}, \overline{H}, \tau)$ be a weak braided bimonad $\mathbb{A}$. Then*

$$\begin{tikzcd} H^{\overline{\xi}} \arrow[r, "\iota^{\overline{\xi}}"] & H \arrow[l, bend left, dashed, "q^{\overline{\xi}}"] \arrow[r, "\delta"] \arrow[rr, bend left, "\delta"] & HH \arrow[r, "\chi H"] & HH \arrow[ll, bend left, dashed, "H\varepsilon"] \end{tikzcd}$$

is a (split) equaliser yielding the monad

$$\underline{H}^{\overline{\xi}} = (H^{\overline{\xi}}, m^{\overline{\xi}}, e^{\overline{\xi}}) \text{ with } m^{\overline{\xi}} = q^{\overline{\xi}} \cdot m \cdot (\iota^{\overline{\xi}} \iota^{\overline{\xi}}) \text{ and } e^{\overline{\xi}} = q^{\overline{\xi}} \cdot e,$$

and

$$\begin{tikzcd} HH \arrow[r, "\overline{\chi} H"] \arrow[rr, bend left, "m"] & HH \arrow[r, "m"] & H \arrow[ll, bend left, dashed, "He"] \arrow[r, "q^{\overline{\xi}}"] & H^{\overline{\xi}} \arrow[l, bend left, dashed, "\iota^{\overline{\xi}}"] \end{tikzcd}$$

is a (split) coequaliser yielding the comonad

$$\overline{H}_{\overline{\xi}} = (H^{\overline{\xi}}, \delta_{\overline{\xi}}, \varepsilon_{\overline{\xi}}) \text{ with } H_{\overline{\xi}} = H^{\overline{\xi}},\ \delta_{\overline{\xi}} = (q^{\overline{\xi}} q^{\overline{\xi}}) \cdot \delta \cdot \iota^{\overline{\xi}} \text{ and } \varepsilon_{\overline{\xi}} = \varepsilon \cdot \iota^{\overline{\xi}}.$$

The next result provides the technical data to show Frobenius separability.

5.10. Lemma. *Let $\mathsf{H} = (\underline{H}, \overline{H}, \tau)$ be a weak braided bimonad and consider the composite*

$$\upsilon : 1 \xrightarrow{e} H \xrightarrow{\delta} HH \xrightarrow{q^{\overline{\xi}} q^{\overline{\xi}}} H^{\overline{\xi}} H^{\overline{\xi}}.$$

Then $m^{\overline{\xi}} \cdot \upsilon = e^{\overline{\xi}}$ and one has commutativity of the diagrams

$$\begin{tikzcd} H^{\overline{\xi}} \arrow[rr, "\upsilon H^{\overline{\xi}}"] \arrow[d, "\iota^{\overline{\xi}}"'] \arrow[drr, dotted, "\delta_{\overline{\xi}}"] & & H^{\overline{\xi}} H^{\overline{\xi}} H^{\overline{\xi}} \arrow[d, "H^{\overline{\xi}} m^{\overline{\xi}}"] \\ H \arrow[r, "\delta"'] & HH \arrow[r, "q^{\overline{\xi}} q^{\overline{\xi}}"'] & H^{\overline{\xi}} H^{\overline{\xi}}, \end{tikzcd} \qquad \begin{tikzcd} H^{\overline{\xi}} \arrow[rr, "H^{\overline{\xi}} \upsilon"] \arrow[d, "\iota^{\overline{\xi}}"'] \arrow[drr, dotted, "\delta_{\overline{\xi}}"] & & H^{\overline{\xi}} H^{\overline{\xi}} H^{\overline{\xi}} \arrow[d, "m^{\overline{\xi}} H^{\overline{\xi}}"] \\ H \arrow[r, "\delta"'] & HH \arrow[r, "q^{\overline{\xi}} q^{\overline{\xi}}"'] & H^{\overline{\xi}} H^{\overline{\xi}}. \end{tikzcd}$$

Proof. The diagram

$$
\begin{tikzcd}
1 \arrow[r, "e"] \arrow[ddr, "e"'] & H \arrow[r, "\delta"] \arrow[d, "He"] & HH \arrow[r, "HHe"] & HHH \arrow[d, "H\overline{\omega}"] \\
& HH \arrow[urr, "\delta H"] \arrow[d, "m"] & HH \arrow[d, "H\varepsilon"] & HHH \arrow[l, "mH"] \arrow[d, "HH\varepsilon"] \\
& H \arrow[r, equal] \arrow[ur, "\delta"] & H & HH \arrow[l, "m"]
\end{tikzcd}
$$

commutes by naturality and since $\overline{\omega}$ is an entwining (Proposition 5.5). Now equations from 5.3 and $\overline{\xi} \cdot m \cdot \overline{\xi} H = \overline{\xi} \cdot m$ (Proposition 5.4(2)) yield commutativity of the diagram

$$
\begin{tikzcd}
H \arrow[r, "\delta"] & HH \arrow[r, "H\overline{\xi}"] & HH \arrow[r, "\overline{\xi}H"] \arrow[dl, "m"] & HH \arrow[d, "m"] \\
1 \arrow[u, "e"] \arrow[r, "e"] & H \arrow[r, "\overline{\xi}"'] & H & H \arrow[l, "\overline{\xi}"]
\end{tikzcd}
$$

and the splitting $\overline{\xi} = \iota^{\overline{\xi}} \cdot q^{\overline{\xi}}$ implies $m^{\overline{\xi}} \cdot \upsilon = e^{\overline{\xi}}$.

In the diagram

$$
\begin{tikzcd}
H^{\overline{\xi}} \arrow[r, "eH^{\overline{\xi}}"] \arrow[d, "\iota^{\overline{\xi}}"'] & HH^{\overline{\xi}} \arrow[r, "\delta H^{\overline{\xi}}"] \arrow[d, "H\iota^{\overline{\xi}}"'] & HHH^{\overline{\xi}} \arrow[r, "q^{\overline{\xi}}HH^{\overline{\xi}}"] \arrow[d, "H\overline{\xi}H^{\overline{\xi}}"'] & H^{\overline{\xi}}HH^{\overline{\xi}} \arrow[r, "H^{\overline{\xi}}q^{\overline{\xi}}H^{\overline{\xi}}"] \arrow[dr, "H^{\overline{\xi}}\,\overline{\xi}H^{\overline{\xi}}"'] & H^{\overline{\xi}}H^{\overline{\xi}}H^{\overline{\xi}} \arrow[d, "H^{\overline{\xi}}\,\iota^{\overline{\xi}}H^{\overline{\xi}}"] \\
H \arrow[r, "eH"'] \arrow[dddd, "\delta"'] & HH \arrow[d, "\delta H"] & HHH^{\overline{\xi}} \arrow[rr, "q^{\overline{\xi}}HH^{\overline{\xi}}"'] \arrow[d, "HH\iota^{\overline{\xi}}"] & & H^{\overline{\xi}}HH^{\overline{\xi}} \arrow[d, "H^{\overline{\xi}}H\iota^{\overline{\xi}}"] \\
& HHH \arrow[r, "H\overline{\xi}H"'] \arrow[ddd, "Hm"] & HHH \arrow[rr, "q^{\overline{\xi}}HH"'] \arrow[d, "Hm"] & & H^{\overline{\xi}}HH \arrow[d, "H^{\overline{\xi}}m"] \\
\arrow[r, phantom, "(a)"] & \arrow[r, phantom, "(b)"] & HH \arrow[rr, "q^{\overline{\xi}}H"'] \arrow[d, "Hq^{\overline{\xi}}"] & & H^{\overline{\xi}}H \arrow[d, "H^{\overline{\xi}}q^{\overline{\xi}}"] \\
HH \arrow[r, "\chi H"'] & HH \arrow[r, "Hq^{\overline{\xi}}"'] & HH^{\overline{\xi}} \arrow[rr, "q^{\overline{\xi}}H^{\overline{\xi}}"'] & & H^{\overline{\xi}}H^{\overline{\xi}}
\end{tikzcd}
$$

- the triangle commutes by the splitting of $\overline{\xi}$;
- region (a) commutes by Proposition 5.4 (4);
- region (b) commutes because Proposition 5.4 (2) induces $q^{\overline{\xi}} \cdot m \cdot \overline{\xi} H = q^{\overline{\xi}} \cdot m$;
- the other regions commute by naturality.

Since $\chi H \cdot \delta \cdot \iota^{\overline{\xi}} = \delta \cdot \iota^{\overline{\xi}}$ by Proposition 5.7, we obtain from this commutativity of the left hand diagram.

Next, in the diagram

$$\begin{tikzcd}
H^{\overline{\xi}} \arrow[r, "H^{\overline{\xi}}e"] \arrow[d, "\iota^{\overline{\xi}}"'] & H^{\overline{\xi}}H \arrow[r, "H^{\overline{\xi}}\delta"] \arrow[d, "\iota^{\overline{\xi}}H"'] & H^{\overline{\xi}}HH \arrow[r, "H^{\overline{\xi}}q^{\overline{\xi}}H"] \arrow[d, "\iota^{\overline{\xi}}HH"'] \arrow[dr, "H^{\overline{\xi}}\overline{\xi}H"'] & H^{\overline{\xi}}H^{\overline{\xi}}H \arrow[r, "H^{\overline{\xi}}H^{\overline{\xi}}q^{\overline{\xi}}"] \arrow[d, "H^{\overline{\xi}}\iota^{\overline{\xi}}H"] & H^{\overline{\xi}}H^{\overline{\xi}}H^{\overline{\xi}} \arrow[d, "H^{\overline{\xi}}\iota^{\overline{\xi}}H^{\overline{\xi}}"] \\
H \arrow[r, "He"] \arrow[dr, "He"'] \arrow[ddd, "\delta"'] & HH \arrow[r, "H\delta"'] & HHH \arrow[dr, "H\overline{\xi}H"] \arrow[d, phantom, "(a)"] & H^{\overline{\xi}}HH \arrow[d, "\iota^{\overline{\xi}}HH"] & H^{\overline{\xi}}HH^{\overline{\xi}} \arrow[d, "\iota^{\overline{\xi}}HH^{\overline{\xi}}"] \\
& HH \arrow[rr, "H\delta"'] \arrow[dd, phantom, "(b)"] & {} & HHH \arrow[d, "mH"] & HHH^{\overline{\xi}} \arrow[d, "mH^{\overline{\xi}}"] \\
& & & & \\
HH \arrow[rrr, "H\overline{\chi}"] \arrow[d, "q^{\overline{\xi}}H"'] & & & HH \arrow[r, "Hq^{\overline{\xi}}"'] \arrow[d, "q^{\overline{\xi}}H"] & HH^{\overline{\xi}} \arrow[d, "q^{\overline{\xi}}H^{\overline{\xi}}"] \\
H^{\overline{\xi}}H \arrow[rrr, "H\overline{\chi}"'] & & & H^{\overline{\xi}}H \arrow[r, "H^{\overline{\xi}}q^{\overline{\xi}}"'] & H^{\overline{\xi}}H^{\overline{\xi}}
\end{tikzcd}$$

- the top triangle commutes by the splitting of $\overline{\xi}$;
- region (a) commutes since from Proposition 5.4(1) and (3) we get

$$\overline{\xi}H \cdot \delta \cdot e = \overline{\xi}H \cdot \delta \cdot \overline{\xi} \cdot e = \delta \cdot \overline{\xi} \cdot e = \delta \cdot e;$$

- region (b) commutes by Proposition 5.4(4);
- the other regions commute by naturality.

Thus the whole diagram is commutative, and in the light of the equality $q^{\overline{\xi}} \cdot \overline{\chi} = q^{\overline{\xi}}$, derived from $\overline{\xi} \cdot \overline{\chi} = \overline{\xi}$ in Proposition 5.4(6), commutativity of the right hand diagram follows. $\square$

The following generalises [26, Proposition 4.2] and [29, Proposition 1.6] (see also [8, Proposition 4.4]).

5.11. Proposition. *Let $\mathsf{H} = (\underline{\mathsf{H}}, \overline{\mathsf{H}}, \tau)$ be a weak braided bimonad on $\mathbb{A}$. Then the quintuple $(H^{\overline{\xi}}, m^{\overline{\xi}}, e^{\overline{\xi}}; \delta_{\overline{\xi}}, \varepsilon_{\overline{\xi}})$ is a separable Frobenius monad.*

Proof. Using the results of Lemma 5.10, it is easy to verify that the diagram corresponding to (2.1) is commutative and that $m^{\overline{\xi}} \cdot \delta_{\overline{\xi}} = 1_{H^{\overline{\xi}}}$. $\square$

5.12. Proposition. *Let $\mathsf{H} = (\underline{\mathsf{H}}, \overline{\mathsf{H}}, \tau)$ be a weak braided bimonad on $\mathbb{A}$ and suppose the functor $K_\omega : \mathbb{A} \to \mathbb{A}^{\overline{\mathsf{H}}}_{\underline{\mathsf{H}}}(\omega)$ admits a right adjoint $\overline{K} : \mathbb{A}^{\overline{\mathsf{H}}}_{\underline{\mathsf{H}}}(\omega) \to \mathbb{A}$. Then*

$\underline{H}^{\overline{\xi}} = (H^{\overline{\xi}}, m^{\overline{\xi}}, e^{\overline{\xi}})$ is the monad generated by the adjunction $K_\omega \dashv \overline{K}$ and $\iota^{\overline{\xi}}$ is the corresponding morphism of monads $\underline{H}^{\overline{\xi}} \to \underline{H}$ (as in Proposition 3.5).

Proof. Suppose there is an adjunction

$$\overline{\eta}, \overline{\varepsilon} : K_\omega \dashv \overline{K} : \mathbb{A}^{\underline{\mathsf{H}}}_{\underline{\mathsf{H}}}(\omega) \to \mathbb{A};$$

consider the monad $(\overline{K}K_\omega, \overline{K}\overline{\varepsilon}K_\omega, \overline{\eta})$ it generates on $\mathbb{A}$. Write $\iota : \overline{K}K_\omega \to \underline{\mathsf{H}}$ for the corresponding morphism of monads (as in Proposition 3.5). Since, by Proposition 4.10, the functor $K_\omega : \mathbb{A} \to \mathbb{A}^{\underline{\mathsf{H}}}_{\underline{\mathsf{H}}}(\omega)$ admits a right adjoint if and only if for every $(a, h, \theta) \in \mathbb{A}^{\underline{\mathsf{H}}}_{\underline{\mathsf{H}}}(\omega)$, the pair of morphisms (4.7) has an equaliser in $\mathbb{A}$, and since $K_\omega(a) = (H(a), m_a, \delta_a)$ for all $a \in \mathbb{A}$, it follows from Proposition 5.7 that $\overline{K}K_\omega = H^{\overline{\xi}}$ and that $\iota^{\overline{\xi}} = \iota$.

According to Theorem 1.15, the unit $\overline{\eta} : 1 \to \overline{K}K_\omega = H^{\overline{\xi}}$ of the adjunction $K_\omega \dashv \overline{K}$ is the unique such natural transformation yielding, for all $a \in \mathbb{A}$, commutativity of the diagram

$$\begin{tikzcd} H^{\overline{\xi}}(a) \arrow[r, "\iota^{\overline{\xi}}_a"] & H(a) \\ & a. \arrow[ul, "\overline{\eta}_a"] \arrow[u, "e_a"'] \end{tikzcd}$$

It then follows that $\overline{\eta}_a = q^{\overline{\xi}}_a \cdot \iota^{\overline{\xi}}_a \cdot \overline{\eta}_a = q^{\overline{\xi}}_a \cdot e_a$.

Next, since for any $a \in \mathbb{A}$, $\overline{\varepsilon}_{K_\omega(a)} = m_a \cdot H(\iota^{\overline{\xi}}_a)$ by (3.4), it follows that

$$\overline{K}(\overline{\varepsilon}_{K_\omega(a)}) : \overline{K}K_\omega\overline{K}K_\omega(a) = H^{\overline{\xi}}H^{\overline{\xi}}(a) \to \overline{K}K_\omega(a) = H^{\overline{\xi}}(a)$$

is the unique morphism making the diagram

$$\begin{tikzcd} H^{\overline{\xi}}H^{\overline{\xi}}(a) \arrow[r, "(\iota^{\overline{\xi}})_{H^{\overline{\xi}}(a)}"] \arrow[dd, "\overline{K}(\overline{\varepsilon}_{K_\omega(a)})"'] & HH^{\overline{\xi}}(a) \arrow[d, "H(\iota^{\overline{\xi}}_a)"] \\ & HH(a) \arrow[d, "m_a"] \\ H^{\overline{\xi}}(a) \arrow[r, "\iota^{\overline{\xi}}_a"'] & H(a) \end{tikzcd}$$

commute and we calculate (recall (5.4))

$$\overline{K}(\overline{\varepsilon}_{K_\omega(a)}) = q^{\overline{\xi}}_a \cdot \iota^{\overline{\xi}}_a \cdot \overline{K}(\overline{\varepsilon}_{K_\omega(a)}) = q^{\overline{\xi}}_a \cdot m_a \cdot H(\iota^{\overline{\xi}}_a) \cdot \iota^{\overline{\xi}}_{H^{\overline{\xi}}(a)} = q^{\overline{\xi}}_a \cdot m_a \cdot (\iota^{\overline{\xi}}\iota^{\overline{\xi}})_a.$$

This completes the proof. □

$$
\begin{tikzcd}
HH^{\overline{\xi}}H \arrow[rr, "H\iota^{\overline{\xi}}H"] \arrow[ddd, "H\iota^{\overline{\xi}}H"'] & & H^3 \arrow[r, "mH"] \arrow[dl, "HeH^2"'] \arrow[d, "H\delta H"] & H^2 \arrow[r, "\delta H"] \arrow[d, phantom, "(2)"] & H^3 \arrow[r, "Hm"] \arrow[d, phantom, "(3)"] & H^2 \\
\arrow[r, phantom, "(1)"] & H^4 \arrow[ddl, "mH^2"'] \arrow[dd, "H^2m"'] \arrow[dr, "H\delta H^2"] & H^4 \arrow[r, "\omega H^2"'] \arrow[dr, "H^2m"] & H^4 \arrow[r, "H^2m"'] \arrow[ur, "HmH"'] \arrow[d, phantom, "(4)"] & H^3 \arrow[ur, "Hm"'] & \\
 & \arrow[dr, phantom, "(6)"] & H^5 \arrow[u, "H^2mH"] \arrow[r, phantom, "(7)"] \arrow[dr, "H^3m"'] & H^3 \arrow[ur, "\omega H"'] & \arrow[r, phantom, "(9)"] & \\
H^3 \arrow[r, phantom, "(5)"] \arrow[dr, "Hm"'] & H^3 \arrow[rr, "H\delta H"'] \arrow[d, "mH"] & \arrow[d, phantom, "(10)"] & H^4 \arrow[r, "\omega H^2"'] \arrow[u, "H^2m"] \arrow[ur, phantom, "(8)"] & H^4 \arrow[r, "HmH"] \arrow[uu, "H^2m"] & H^3 \arrow[uuu, "Hm"'] \\
 & H^2 \arrow[urrrr, "\delta H"'] & & & &
\end{tikzcd}
$$

Fig. 1.

By symmetry, we also have:

5.13. Proposition. *Let* $\mathsf{H} = (\underline{\mathsf{H}}, \overline{\mathsf{H}}, \tau)$ *be a weak braided bimonad on* $\mathbb{A}$ *and suppose the functor* $K_\omega : \mathbb{A} \to \mathbb{A}^{\overline{H}}_{\underline{H}}(\omega)$ *admits a left adjoint* $\underline{K} : \mathbb{A}^{\overline{H}}_{\underline{H}}(\omega) \to \mathbb{A}$. *Then* $\overline{\mathsf{H}}_{\overline{\xi}} = (H^{\overline{\xi}}, \delta_{\overline{\xi}}, \varepsilon_{\overline{\xi}})$ *is the comonad generated by the adjunction* $\underline{K} \dashv K_\omega$ *and* $q^{\overline{\xi}} : \overline{\mathsf{H}} \to \overline{\mathsf{H}}_{\overline{\xi}}$ *is a comonad morphism.*

According to Section 1.7, the monad morphism $\iota^{\overline{\xi}} : \underline{\mathsf{H}}^{\overline{\xi}} \to \underline{\mathsf{H}}$ equips H with an $\underline{\mathsf{H}}^{\overline{\xi}}$-bimodule structure, where the left and right actions are the composites, respectively,

$$\rho_l : H^{\overline{\xi}}H \xrightarrow{\iota^{\overline{\xi}}H} HH \xrightarrow{m} H, \qquad \rho_r : HH^{\overline{\xi}} \xrightarrow{H\iota^{\overline{\xi}}} HH \xrightarrow{m} H.$$

5.14. Proposition. *For a weak braided bimonad* $\mathsf{H} = (\underline{\mathsf{H}}, \overline{\mathsf{H}}, \tau)$ *on* $\mathbb{A}$, $\sigma = Hm \cdot \delta H$ *coequalises the pair* $(\rho_r H, H\rho_l)$, *i.e.,*

$$\sigma \cdot \rho_r H = \sigma \cdot H\rho_l.$$

Proof. Since (5.5) is an equaliser diagram and since in Fig. 1

- diagram (1) commutes since e is the unit of the monad T;
- diagrams (2) and (10) commute by Proposition 5.5;

- diagrams (3), (7) and (9) commute by associativity of m;
- diagrams (4), (5), (6) and (8) commute by naturality,

one sees that $\sigma \cdot \rho_r H = Hm \cdot \delta H \cdot Hm \cdot H\iota^{\overline{\xi}} H = \sigma \cdot H\rho_l$. □

Suppose now that the tensor product $H \otimes_{\underline{\mathsf{H}}^{\overline{\xi}}} H$ exists, i.e., there is a coequaliser diagram

$$\begin{tikzcd} HH^{\overline{\xi}}H \arrow[r, shift left, "\rho_r H"] \arrow[r, shift right, "H\rho_l"'] & HH \arrow[r, "\ell"] & H \otimes_{\underline{\mathsf{H}}^{\overline{\xi}}} H , \end{tikzcd} \tag{5.7}$$

where $\ell = \mathrm{can}^{H,H}_{\underline{\mathsf{H}}^{\overline{\xi}}}$. Note that, by Proposition 1.9, $H \otimes_{\underline{\mathsf{H}}^{\overline{\xi}}} H$ has a right $\underline{\mathsf{H}}$-module structure such that ℓ is a morphism of right $\underline{\mathsf{H}}$-modules. Moreover, since σ coequalises $\rho_r H$ and $H\rho_l$, the composite $\overline{p} \cdot \sigma : HH \to \overline{G}$ (see (4.6)) also coequalises them, and since diagram (5.7) is a coequaliser, there exists a unique natural transformation $\gamma : H \otimes_{\underline{\mathsf{H}}^{\overline{\xi}}} H \to \overline{G}$ making the diagram

$$\begin{tikzcd} HH^{\overline{\xi}}H \arrow[r, shift left, "\rho_r H"] \arrow[r, shift right, "H\rho_l"'] & HH \arrow[r, "\ell"] \arrow[dr, "\overline{p} \cdot \sigma"'] & H \otimes_{\underline{\mathsf{H}}^{\overline{\xi}}} H \arrow[d, dashed, "\exists ! \gamma"] \\ & & \overline{G} \end{tikzcd} \tag{5.8}$$

commute. It follows – since $U_{\underline{\mathsf{H}}} t\phi_{\underline{\mathsf{H}}} = \overline{p} \cdot \sigma$ by Proposition 4.8 – that the diagram

$$\begin{tikzcd} HH \arrow[r, "\sigma"] \arrow[d, "\ell"'] \arrow[dr, "U_{\underline{\mathsf{H}}} t\phi_{\underline{\mathsf{H}}}"] & HH \arrow[d, "\overline{p}"] \\ H \otimes_{\underline{\mathsf{H}}^{\overline{\xi}}} H \arrow[r, "\gamma"'] & \overline{G} \end{tikzcd} \tag{5.9}$$

commutes. Precomposing this square with He and using $\sigma \cdot He = \delta$ (e.g. [23, 5.2]), we get

$$\overline{p} \cdot \delta = \gamma \cdot \ell \cdot He. \tag{5.10}$$

5.15. *H* as $\boldsymbol{H_{\overline{\xi}}}$-bicomodule. The comonad morphism $q^{\overline{\xi}} : \overline{\mathsf{H}} \to \overline{\mathsf{H}}_{\overline{\xi}}$ equips H with an $\overline{\mathsf{H}}_{\overline{\xi}}$-bicomodule structure, where the left and right coactions are the composites, respectively,

$$\theta_l : H \xrightarrow{\delta} HH \xrightarrow{q^{\overline{\xi}} H} H_{\overline{\xi}} H, \qquad \theta_l : H \xrightarrow{\delta} HH \xrightarrow{Hq^{\overline{\xi}}} HH_{\overline{\xi}}.$$

Moreover, there is a unique natural transformation $\gamma' : \overline{T} \to H \otimes^{\overline{H}_{\overline{\xi}}} H$ making the triangle in the diagram

$$\begin{tikzcd}
H \otimes_{\overline{H}^{\overline{\xi}}} H \arrow[r, "\mathrm{can}"] & HH \arrow[r, shift left, "\theta_r H"] \arrow[r, shift right, "H\theta_l"'] & HH_{\overline{\xi}}H \\
\overline{T} \arrow[u, dashed, "\exists!\gamma'"] \arrow[ur, "\sigma\cdot\bar{i}\,'"'] & &
\end{tikzcd} \tag{5.11}$$

commute. Here $\bar{i}\,'$ and $\overline{T}$ are as defined in the diagrams (4.6).

5.16. Proposition. *Let $\mathsf{H} = (\underline{H}, \overline{H}, \tau)$ be a weak braided bimonad on $\mathbb{A}$. Viewing H as a right $\underline{H}^{\overline{\xi}}$-module by the structure map $HH^{\overline{\xi}} \xrightarrow{H\iota^{\overline{\xi}}} HH \xrightarrow{m} H$, then $q^{\overline{\xi}} : H \to H^{\overline{\xi}}$ is a morphism of right $\underline{H}^{\overline{\xi}}$-modules.*

Proof. For this, we have to show commutativity of the diagram

$$\begin{tikzcd}
HH^{\overline{\xi}} \arrow[rr, "q^{\overline{\xi}}H^{\overline{\xi}}"] \arrow[d, "H\iota^{\overline{\xi}}"'] & & H^{\overline{\xi}}H^{\overline{\xi}} \arrow[d, "m^{\overline{\xi}}"] \\
HH \arrow[r, "m"] & H \arrow[r, "q^{\overline{\xi}}"] & H^{\overline{\xi}},
\end{tikzcd}$$

and since $m^{\overline{\xi}} = q^{\overline{\xi}} \cdot m \cdot (\iota^{\overline{\xi}}\iota^{\overline{\xi}})$, this can be rewritten as

$$\begin{tikzcd}
HH^{\overline{\xi}} \arrow[r, "\overline{\xi}H^{\overline{\xi}}"] \arrow[d, "H\iota^{\overline{\xi}}"'] & HH^{\overline{\xi}} \arrow[r, "H\iota^{\overline{\xi}}"] & HH \arrow[d, "m"] \\
HH \arrow[urr, dashed, "\overline{\xi}H"'] \arrow[d, "m"'] & & H \arrow[d, "q^{\overline{\xi}}"] \\
H \arrow[rr, "q^{\overline{\xi}}"] & & H^{\overline{\xi}}.
\end{tikzcd}$$

In this diagram, the triangle commutes by naturality of composition, and the trapezoid commutes, since $\iota^{\overline{\xi}}$ is a (split) monomorphism and $\overline{\xi}\cdot m\cdot\overline{\xi}H = \overline{\xi}\cdot m$ by Proposition 5.4(2). This completes the proof. $\square$

Consider now the diagram

$$\begin{tikzcd}
HH^{\overline{\xi}}H \arrow[r, shift left, "\rho_r H"] \arrow[r, shift right, "H\rho_l"'] \arrow[d, "q^{\overline{\xi}}H^{\overline{\xi}}H"'] & HH \arrow[r, "\ell"] \arrow[d, "q^{\overline{\xi}}H"] & H\otimes_{\underline{H}^{\overline{\xi}}}H \arrow[rrd, dotted, "\widetilde{q}"] & & \\
H^{\overline{\xi}}H^{\overline{\xi}}H \arrow[r, shift left, "m^{\overline{\xi}}H"] \arrow[r, shift right, "H^{\overline{\xi}}\rho_l"'] & H^{\overline{\xi}}H \arrow[rr, "\iota^{\overline{\xi}}H"] \arrow[rrr, bend right, "\rho_l"'] & & HH \arrow[r, "m"] & H
\end{tikzcd} \tag{5.12}$$

in which $q^{\overline{\xi}}H \cdot \rho_r H = m^{\overline{\xi}}H \cdot q^{\overline{\xi}}H^{\overline{\xi}}H$, since $q^{\overline{\xi}}$ is a morphism of right $\underline{\mathsf{H}}^{\overline{\xi}}$-modules by Proposition 5.16, $q^{\overline{\xi}}H \cdot H\rho_l = H^{\overline{\xi}}\rho_l \cdot q^{\overline{\xi}}H^{\overline{\xi}}H$ because of naturality of composition, and the bottom row is a split coequaliser, since the pair (H, ρ_l) is a left $\underline{\mathsf{H}}^{\overline{\xi}}$-module (see Remark 1.17 (1)). It then follows

5.17. Proposition. *Let $\mathsf{H} = (\underline{\mathsf{H}}, \overline{\mathsf{H}}, \tau)$ be a weak braided bimonad on $\mathbb{A}$. In the situation described above, there is a unique morphism $\widetilde{q} : H \otimes_{\underline{H}^{\overline{\xi}}} H \to H$ making the trapezoid in the diagram (5.12) commutative and this is a morphism of right $\underline{\mathsf{H}}$-modules.*

Proof. According to Section 1.7, the morphisms $q^{\overline{\xi}}H, \iota^{\overline{\xi}}H$ and m are morphisms of right $\underline{\mathsf{H}}$-modules. Then the composite $m \cdot \iota^{\overline{\xi}}H \cdot q^{\overline{\xi}}H = \rho_l \cdot q^{\overline{\xi}}H$ (and hence also $\widetilde{q} \cdot \ell$) is a morphism of right $\underline{\mathsf{H}}$-modules, implying – since ℓ and ℓH are both epimorphisms of right $\underline{\mathsf{H}}$-modules – that $\widetilde{q}$ is also a morphism of right $\underline{\mathsf{H}}$-modules. □

5.18. Proposition. *Suppose $\gamma : H \otimes_{\underline{H}^{\overline{\xi}}} H \to \overline{G}$ in (5.8) is an epimorphism. If the morphisms $f, g : H \to H$ are such that $f * 1 = g * 1$, then $f * \xi = g * \xi$.*

Proof. If $f, g : H \to H$ are morphisms such that $f * 1 = g * 1$, then

$$m \cdot fH \cdot \delta = m \cdot gH \cdot \delta$$

and since $\sigma \cdot He = \delta$, we have

$$m \cdot fH \cdot \sigma \cdot He = m \cdot gH \cdot \sigma \cdot He.$$

According to Section 1.7, fH and gH can be seen as morphisms of the right $\underline{\mathsf{H}}$-module (HH, Hm) to itself, while m is a morphism from the right $\underline{\mathsf{H}}$-module (HH, Hm) to the $\underline{\mathsf{H}}$-module (H, m). Moreover, σ is also a morphism of right $\underline{\mathsf{H}}$-modules (e.g. [23, Section 5.1]). Thus the composites $m \cdot fH \cdot \sigma$ and $m \cdot gH \cdot \sigma$ both are morphisms of right $\underline{\mathsf{H}}$-modules. It then follows from the right hand version of [23, Lemma 3.2] that

$$m \cdot fH \cdot \sigma = m \cdot gH \cdot \sigma.$$

Next, since $\sigma = \kappa \cdot \sigma = \overline{i} \cdot \overline{p} \cdot \sigma$ and $\overline{p} \cdot \sigma = \gamma \cdot \ell$ (by (5.9)), we have

$$m \cdot fH \cdot \overline{i} \cdot \gamma \cdot \ell = m \cdot gH \cdot \overline{i} \cdot \gamma \cdot \ell,$$

and l and γ being epimorphisms we get $m \cdot fH \cdot \overline{i} = m \cdot gH \cdot \overline{i}$, thus

$$m \cdot fH \cdot \kappa = m \cdot gH \cdot \kappa.$$

Recalling that $\kappa \cdot He = \omega \cdot eH$ and $\omega \cdot eH = H\xi \cdot \delta$ (see (4.4), (4.2)(ii)), we get

$$m \cdot fH \cdot \kappa \cdot He = m \cdot fH \cdot \omega \cdot eH = m \cdot fH \cdot H\xi \cdot \delta = f * \xi,$$

and similarly, one derives $m \cdot gH \cdot \kappa \cdot He = g * \xi$. Thus, $f * \xi = g * \xi$. □

Since κ is (clearly) a morphism of right $\underline{\mathsf{H}}$-modules, Proposition 1.4 yields

5.19. Proposition. *The composite* $\overline{G}H \xrightarrow{\overline{i}H} HHH \xrightarrow{Hm} HH \xrightarrow{\overline{p}} \overline{G}$ *makes* $\overline{G}$ *into a right* $\underline{H}$*-module such that* $\overline{p} : HH \to \overline{G}$ *and* $\overline{i} : \overline{G} \to HH$ *both are morphisms of right* $\underline{H}$*-modules.*

5.20. Corollary. $\gamma : H \otimes_{\underline{H}^{\overline{\xi}}} H \to \overline{G}$ *is a morphism of right* $\underline{H}$*-modules.*

Proof. Since, in diagram (5.9), the morphisms σ, $\overline{p}$ and ℓ are all morphisms of right $\underline{\mathsf{H}}$-modules (see [23, 5.1] and Proposition 5.19), and since ℓ and ℓH are both epimorphisms, it follows that γ is also a morphism of right $\underline{\mathsf{H}}$-modules. □

6. Weak braided Hopf monads

In this section, we define an antipode for weak braided bimonads $\mathsf{H} = (\underline{H}, \overline{H}, \tau)$ and formulate various forms of the Fundamental Theorem. The definition corresponds to that in [10,2] and in other papers on (generalisations of) weak Hopf algebras. For the notations we refer to the preceding section.

6.1. Definition. Given a weak braided bimonad H, a natural transformation $S : H \to H$ is called an *antipode* if

$$1 * S = \xi, \quad S * 1 = \overline{\xi}, \quad S * 1 * S = S.$$

Since $\xi * 1 = 1 = 1 * \overline{\xi}$ (see (4.4), (4.11)), we also get $1 * S * 1 = 1$.

A weak braided bimonad H with an antipode is called a *weak braided Hopf monad* or a *weak τ-Hopf monad.*

6.2. Proposition. *Let* $\mathsf{H} = (\underline{\mathsf{H}}, \overline{\mathsf{H}}, \tau)$ *be a weak braided Hopf monad on a Cauchy complete category* $\mathbb{A}$*. Then the functor* $K_\omega : \mathbb{A} \to \mathbb{A}^{\overline{H}}_{\underline{H}}(\omega)$ *admits a right adjoint* $\overline{K} : \mathbb{A}^{\overline{H}}_{\underline{H}}(\omega) \to \mathbb{A}$*.*

Proof. Suppose that H has an antipode S. By Proposition 4.10, we have to show that for any $(a, h, \theta) \in \mathbb{A}^{\overline{\mathsf{H}}}_{\underline{\mathsf{H}}}(\omega)$, the pair (4.7) has an equaliser. We claim that the pair is cosplit by the composite $d : H(a) \xrightarrow{S_a} H(a) \xrightarrow{h} a$. Indeed, in the proof of [2, Proposition 3.5 (ii)] it is shown that

$$H(h) \cdot \delta_a \cdot e_a \cdot d \cdot \theta = \theta \cdot d \cdot \theta.$$

It remains to prove that

$$d \cdot H(h) \cdot \delta_a \cdot e_a = 1.$$

For this, consider the diagram

$$\begin{tikzcd}
a \arrow[r, "e_a"] \arrow[ddrr, "e_a"'] & H(a) \arrow[r, "\delta_a"] \arrow[ddr, "\overline{\xi}_a"] & HH(a) \arrow[r, "H(h)"] \arrow[d, "S_{H(a)}"] & H(a) \arrow[d, "S_a"] \\
 & & HH(a) \arrow[r, "H(h)"] \arrow[d, "m_a"] & H(a) \arrow[d, "h"] \\
 & & H(a) \arrow[r, "h"'] & a,
\end{tikzcd}$$

in which

- the left triangle commutes by Proposition 5.4 (1);
- the right triangle commutes because S is an antipode;
- the top square commutes by naturality of S;
- the bottom square commutes since $(a, h) \in \mathbb{A}_{\underline{\mathsf{H}}}$.

So the whole diagram commutes and since $h \cdot e_a = 1$, the outer paths show that the desired equality holds.

Thus the composite $d \cdot \theta : a \xrightarrow{\theta} H(a) \xrightarrow{S_a} H(a) \xrightarrow{h} a$ is an idempotent and if $a \xrightarrow{q_a} \overline{a} \xrightarrow{\iota_a} a$ is a splitting of this idempotent, then the diagram

$$\begin{tikzcd}
\overline{a} \arrow[r, "\iota_a"] & a \arrow[r, "e_a"'] \arrow[rrr, bend left, "\theta"] & H(a) \arrow[r, "\delta_a"'] & HH(a) \arrow[r, "H(h)"'] & H(a)
\end{tikzcd}$$

is a (split) equaliser in $\mathbb{A}$. Thus, $\overline{K}$ exists and for any $(a, h, \theta) \in \mathbb{A}_{\underline{\mathsf{H}}}^{\overline{\mathsf{H}}}(\omega)$, $\overline{K}(a, h, \theta) = \overline{a}$. $\square$

Dual to Proposition 6.2, we observe:

6.3. Proposition. *Let $\mathsf{H} = (\underline{H}, \overline{H}, \tau)$ be a weak braided Hopf monad on a Cauchy complete category $\mathbb{A}$. Then the functor $K_\omega : \mathbb{A} \to \mathbb{A}_{\underline{H}}^{\overline{H}}(\omega)$ admits a left adjoint $\underline{K} : \mathbb{A}_{\underline{H}}^{\overline{H}}(\omega) \to \mathbb{A}$ that takes $(a, h, \theta) \in \mathbb{A}_{\underline{H}}^{\overline{H}}(\omega)$ to the object $\overline{a}$ which splits the idempotent $a \xrightarrow{\theta} H(a) \xrightarrow{S_a} H(a) \xrightarrow{h} a$. Moreover, the diagram*

$$\begin{tikzcd}
H(a) \arrow[r, "H(\theta)"'] \arrow[rrr, bend left, "h"] & HH(a) \arrow[r, "m_a"'] & H(a) \arrow[r, "\varepsilon_a"'] & a \arrow[r, "q_a"'] & \overline{a}
\end{tikzcd}$$

is a (split) coequaliser.

6.4. Proposition. *Let $H = (\underline{H}, \overline{H}, \tau)$ be a weak braided Hopf monad on a Cauchy complete category $\mathbb{A}$. Then, for any $(a, h, \theta) \in \mathbb{A}_{\underline{H}}^{\overline{H}}(\omega)$, the diagram*

$$\begin{tikzcd} a \arrow[r, "\theta"] \arrow[d, equal] & H(a) \arrow[d, "H(\beta_a)"] \\ a & H(a) \arrow[l, "h"] \end{tikzcd},$$

with the idempotent $\beta_a : a \xrightarrow{\theta} H(a) \xrightarrow{S_a} H(a) \xrightarrow{h} a$ (see proof of Proposition 6.2) is commutative.

Proof. We compute

$$\begin{aligned} h \cdot H(\beta_a) \cdot \theta &= h \cdot H(h) \cdot H(S_a) \cdot H(\theta) \cdot \theta \\ &= h \cdot m_a \cdot H(S_a) \cdot \delta \cdot \theta \\ &= h \cdot \varepsilon_{H(a)} \cdot \omega_a \cdot e_{H(a)} \cdot \theta \\ &= \varepsilon_a \cdot H(h) \cdot \omega_a \cdot H(\theta) \cdot e_a \\ &= \varepsilon_a \cdot \theta \cdot h \cdot e_a = 1. \end{aligned}$$

The second and sixth equations hold since $(a, h) \in \mathbb{A}_{\underline{\mathsf{H}}}$ and $(a, \theta) \in \mathbb{A}^{\overline{\mathsf{H}}}$, the third one holds by the definition of an antipode, the forth one by naturality of e and ε, and the fifth one by the fact that $(a, h, \theta) \in \mathbb{A}_{\underline{\mathsf{H}}}^{\overline{\mathsf{H}}}(\omega)$. □

6.5. Proposition. *Let $H = (\underline{H}, \overline{H}, \tau)$ be a weak braided Hopf monad on a Cauchy complete category $\mathbb{A}$. Then*

(1) *$K_\omega : \mathbb{A} \to \mathbb{A}_{\underline{H}}^{\overline{H}}(\omega)$ admits both a left and a right adjoint $\overline{K}, \underline{K} : \mathbb{A}_{\underline{H}}^{\overline{H}}(\omega) \to \mathbb{A}$;*
(2) *the unit $\underline{\eta} : 1 \to K\underline{K}$ of $\underline{K} \dashv K$ is a split monomorphism, while the counit $\overline{\varepsilon} : K\overline{K} \to 1$ of $K \dashv \overline{K}$ is a split epimorphism.*

Proof. According to Propositions 6.2 and 6.3, $K_\omega : \mathbb{A} \to \mathbb{A}_{\underline{\mathsf{H}}}^{\overline{\mathsf{H}}}(\omega)$ admits both left and right adjoints $\underline{K}, \overline{K}$. For any object $(a, h, \theta) \in \mathbb{A}_{\underline{\mathsf{H}}}^{\overline{\mathsf{H}}}(\omega)$, $\beta_a : a \xrightarrow{\theta} H(a) \xrightarrow{S_a} H(a) \xrightarrow{h} a$ is idempotent (see proof of Proposition 6.2) with a splitting $a \xrightarrow{q_a} \overline{a} \xrightarrow{\iota_a} a$ and

$$\underline{K}(a, h, \theta) = \overline{K}(a, h, \theta) = \overline{a}.$$

Moreover,

$$\begin{tikzcd} H(a) \arrow[r, "H(\theta)"'] \arrow[rrr, bend left, "h"] & HH(a) \arrow[r, "m_a"'] & H(a) \arrow[r, "\varepsilon_a"'] & a \arrow[r, "q_a"'] & \overline{a} \end{tikzcd}$$

is the defining coequaliser diagram for $\underline{K}(a, h, \theta)$, while

$$\begin{tikzcd} \overline{a} \arrow[r, "\iota_a"] & a \arrow[r, "e_a"'] \arrow[rrr, bend left, "\theta"] & H(a) \arrow[r, "\delta_a"'] & HH(a) \arrow[r, "H(h)"'] & H(a) \end{tikzcd}$$

is the defining equaliser diagram for $\overline{K}(a, h, \theta)$.

It is easy to verify directly, using (1.4) and (1.5), that

$$\underline{\eta}_{(a,\,h,\,\theta)} = H(q_a) \cdot \theta \quad \text{and} \quad \overline{\varepsilon}_{(a,\,h,\,\theta)} = h \cdot H(\iota_a).$$

Now, in view of Proposition 6.4, we compute

$$\overline{\varepsilon}_{(a,\,h,\,\theta)} \cdot \underline{\eta}_{(a,\,h,\,\theta)} = h \cdot H(\iota_a) \cdot H(q_a) \cdot \theta = h \cdot H(\beta_a) \cdot \theta = 1.$$

Thus $\overline{\varepsilon}_{((a,\,h),\,\theta)}$ is a split epimorphism, while $\underline{\eta}_{((a,\,h),\,\theta)}$ is a split monomorphism. $\square$

We are now ready to state and prove our main result. It subsumes the original version of the Fundamental Theorem proved for weak Hopf algebras over fields in [10, Theorem 3.9] as well as various generalisations, for example, for algebras over commutative rings in [32, Theorem 5.12], [13, Theorem 36.16], for Hopf algebroids in [6, Theorem 4.14], and for weak braided Hopf algebras on monoidal categories [2, Proposition 3.6].

6.6. Fundamental Theorem. *Let $H = (\underline{H}, \overline{H}, \tau)$ be a weak braided bimonad on a Cauchy complete category $\mathbb{A}$, and T and G the monad and comonad induced on $\mathbb{A}^{\overline{H}}$ and on $\mathbb{A}_{\underline{H}}$, respectively. Then the following are equivalent:*

(a) *H is a weak braided Hopf monad;*
(b) *the functor $K_\omega : \mathbb{A} \to \mathbb{A}^{\overline{H}}_{\underline{H}}(\omega)$ admits both left and right adjoints, and the right adjoint is monadic;*
(c) *the functor $K_\omega : \mathbb{A} \to \mathbb{A}^{\overline{H}}_{\underline{H}}(\omega)$ admits both left and right adjoints, and the left adjoint is comonadic;*
(d) *the induced natural transformation $\gamma : H \otimes_{\underline{H}^{\overline{\xi}}} H \to \overline{G}$ is an isomorphism;*
(e) *the induced natural transformation $\gamma' : \overline{T} \to H \otimes^{\overline{H}^{\overline{\xi}}} H$ is an isomorphism.*

Moreover, if the (equivalent) conditions above hold, then there is an equivalence of categories $\mathbb{A}_{\underline{H}^{\overline{\xi}}} \simeq \mathbb{A}^{\overline{H}}_{\underline{H}}(\omega)$.

Proof. (a) $\Rightarrow$ (b). If H is a weak braided Hopf monad, then, by Propositions 6.2 and 6.3, the functor $K_\omega : \mathbb{A} \to \mathbb{A}^{\overline{\mathsf{H}}}_{\underline{\mathsf{H}}}(\omega)$ admits a right adjoint $\overline{K}$ and a left adjoint $\underline{K}$. So it remains to prove that $\overline{K}$ is monadic. By Proposition 6.5, the counit of the adjunction $K \dashv \overline{K}$ is a split epimorphism. Moreover, since $\mathbb{A}$ is assumed to be Cauchy complete, so is $\mathbb{A}^{\overline{\mathsf{H}}}_{\underline{\mathsf{H}}}(\omega)$ by Proposition 1.4. Applying now the dual of [21, Proposition 3.16] gives that $\overline{K}$ is monadic.

(b) $\Rightarrow$ (d). $\underline{\mathsf{H}}^{\overline{\xi}}$ is separable Frobenius by Proposition 5.11, and hence, by Proposition 2.2, the change-of-base functor $(\iota^{\overline{\xi}})_! : \mathbb{A}_{\underline{\mathsf{H}}^{\overline{\xi}}} \to \mathbb{A}_{\mathsf{H}}$ exists. It then follows from Proposition 3.6 that the comparison functor $K_{\underline{\mathsf{H}}^{\overline{\xi}}} : (\mathbb{A}_{\underline{\mathsf{H}}})^{\mathsf{G}} \to \mathbb{A}_{\underline{\mathsf{H}}^{\overline{\xi}}}$ admits a left adjoint $L_{\underline{\mathsf{H}}^{\overline{\xi}}} : \mathbb{A}_{\underline{\mathsf{H}}^{\overline{\xi}}} \to (\mathbb{A}_{\underline{\mathsf{H}}})^{\mathsf{G}}$ such that $U^{\mathsf{G}} L_{\underline{\mathsf{H}}^{\overline{\xi}}} = (\iota^{\overline{\xi}})_!$. The situation is illustrated by the diagram

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, bend left, "\phi_{\underline{\mathsf{H}}^{\overline{\xi}}}"'] \arrow[rrrr, bend right, "K"'] \arrow[drr, "\phi_{\underline{\mathsf{H}}}"'] & & \mathbb{A}_{\underline{\mathsf{H}}^{\overline{\xi}}} \arrow[ll, "U_{\underline{\mathsf{H}}^{\overline{\xi}}}"'] \arrow[rr, bend right, "L_{\underline{\mathsf{H}}^{\overline{\xi}}}"'] \arrow[d, "(\iota^{\overline{\xi}})_!"'] & & (\mathbb{A}_{\underline{\mathsf{H}}})^{\mathsf{G}} \arrow[ll, "K_{\underline{\mathsf{H}}^{\overline{\xi}}}"'] \arrow[llll, bend right, "\overline{K}"'] \arrow[dll, "U^{\mathsf{G}}"] \\
& & \mathbb{A}_{\underline{\mathsf{H}}} & &
\end{tikzcd} . \tag{6.1}$$

Next, from the proof of Proposition 2.2 we know that the defining coequaliser of $(\iota^{\overline{\xi}})_!$,

$$\phi_{\underline{\mathsf{H}}} H^{\overline{\xi}} U_{\underline{\mathsf{H}}^{\overline{\xi}}} = \phi_{\underline{\mathsf{H}}} U_{\underline{\mathsf{H}}^{\overline{\xi}}} \phi_{\underline{\mathsf{H}}^{\overline{\xi}}} U_{\underline{\mathsf{H}}^{\overline{\xi}}} \underset{\varrho U_{\underline{\mathsf{H}}^{\overline{\xi}}}}{\overset{\phi_{\underline{\mathsf{H}}} U_{\underline{\mathsf{H}}^{\overline{\xi}}} \varepsilon_{\underline{\mathsf{H}}^{\overline{\xi}}}}{\rightrightarrows}} \phi_{\underline{\mathsf{H}}} U_{\underline{\mathsf{H}}^{\overline{\xi}}} \xrightarrow{\ q\ } (\iota^{\overline{\xi}})_! \,, \tag{6.2}$$

where ρ is the composite $\phi_{\underline{\mathsf{H}}} H^{\overline{\xi}} \xrightarrow{\phi_{\underline{\mathsf{H}}} \iota^{\overline{\xi}}} \phi_{\underline{\mathsf{H}}} H = \phi_{\underline{\mathsf{H}}} U_{\underline{\mathsf{H}}} \phi_{\underline{\mathsf{H}}} \xrightarrow{\varepsilon_{\underline{\mathsf{H}}} \phi_{\underline{\mathsf{H}}}} \phi_{\underline{\mathsf{H}}}$, is absolute (i.e., is preserved by any functor). It then follows from Remark 1.17 (2) that the tensor product $H \otimes_{\underline{\mathsf{H}}^{\overline{\xi}}} H$ is just the functor $U_{\underline{\mathsf{H}}} (\iota^{\overline{\xi}})_! (\iota^{\overline{\xi}})^* \phi_{\underline{\mathsf{H}}}$. Moreover, $\ell = U_{\underline{\mathsf{H}}}\, q (\iota^{\overline{\xi}})^* \phi_{\underline{\mathsf{H}}}$.

Since $U_{\underline{\mathsf{H}}^{\overline{\xi}}} (\iota^{\overline{\xi}})^* = U_{\underline{\mathsf{H}}}$, it follows from Proposition 3.7 and Equation (3.8) that the diagram

$$\begin{tikzcd}
\phi_{\underline{\mathsf{H}}} U_{\underline{\mathsf{H}}^{\overline{\xi}}} (\iota^{\overline{\xi}})^* = \phi_{\underline{\mathsf{H}}} U_{\underline{\mathsf{H}}} \arrow[r, "q(\iota^{\overline{\xi}})^*"] \arrow[dr, "t_K"'] & (\iota^{\overline{\xi}})_! (\iota^{\overline{\xi}})^* \arrow[d, "t_{L_{\underline{\mathsf{H}}^{\overline{\xi}}}}"] \\
& \widetilde{G}
\end{tikzcd}$$

commutes, and since $U_{\underline{\mathsf{H}}} \widetilde{G} = G$, $G \phi_{\underline{\mathsf{H}}} = \overline{G}$, and (6.2) is absolute, the diagram

$$\begin{tikzcd}
HH \arrow[r, "\ell"] \arrow[dr, "U_{\underline{\mathsf{H}}}\, t_K \phi_{\underline{\mathsf{H}}}"'] & H \otimes_{\underline{\mathsf{H}}^{\overline{\xi}}} H \arrow[d, "U_{\underline{\mathsf{H}}}\, t_{L_{\underline{\mathsf{H}}^{\overline{\xi}}}}\, \phi_{\underline{\mathsf{H}}}"] \\
 & \overline{G}
\end{tikzcd}$$

commutes. Since ℓ is an epimorphism and (5.9) is also commutative, it follows that $\gamma = U_{\underline{\mathsf{H}}}\, t_{L_{\underline{\mathsf{H}}^{\overline{\xi}}}}\, \phi_{\underline{\mathsf{H}}}$.

Now, since $\overline{K}$ is assumed to be monadic, the comparison functor $K_{\underline{\mathsf{H}}^{\overline{\xi}}} : (\mathbb{A}_{\underline{\mathsf{H}}})^{\mathsf{G}} \to \mathbb{A}_{\underline{\mathsf{H}}^{\overline{\xi}}}$ is an equivalence, and hence its left adjoint $L_{\underline{\mathsf{H}}^{\overline{\xi}}} : \mathbb{A}_{\underline{\mathsf{H}}^{\overline{\xi}}} \to (\mathbb{A}_{\underline{\mathsf{H}}})^{\mathsf{G}}$ is also an equivalence. Applying Theorem 3.2 to the right commutative triangle in diagram (6.1) gives that $t_{L_{\underline{\mathsf{H}}^{\overline{\xi}}}}$ is an isomorphism. Quite obviously, $\gamma = U_{\underline{\mathsf{H}}} t_{L_{\underline{\mathsf{H}}^{\overline{\xi}}}} \phi_{\underline{\mathsf{H}}}$ is then also an isomorphism.

(d) $\Rightarrow$ (a). Suppose that the natural transformation $\gamma : H \otimes_{\underline{\mathsf{H}}^{\overline{\xi}}} H \to \overline{G}$ is an isomorphism. Then we claim that the composite

$$S : H \xrightarrow{He} HH \xrightarrow{\overline{p}} \overline{G} \xrightarrow{\gamma^{-1}} H \otimes_{\underline{\mathsf{H}}^{\overline{\xi}}} H \xrightarrow{\widetilde{q}} H$$

is an antipode for H. Note first that by Propositions 5.17, 5.19, and Corollary 5.20, the composite

$$\widetilde{q}_1 : HH \xrightarrow{\overline{p}} \overline{G} \xrightarrow{\gamma^{-1}} H \otimes_{\underline{\mathsf{H}}^{\overline{\xi}}} H \xrightarrow{\widetilde{q}} H$$

is a morphism of right $\underline{\mathsf{H}}$-modules. To show that $S * 1 = \overline{\xi}$, consider the diagram

$$\begin{tikzcd}
H \arrow[r, "\delta"] \arrow[d, "He"'] & HH \arrow[r, "HeH"] \arrow[dr, equal] & HHH \arrow[r, "\overline{p}H"] \arrow[d, "Hm"] & \overline{G}H \arrow[r, "\gamma^{-1}H"] & (H \otimes_{\underline{\mathsf{H}}^{\overline{\xi}}} H)H \arrow[r, "\widetilde{q}H"] & HH \arrow[d, "m"] \\
HH \arrow[rrrr, bend right, "\ell"'] & (1) & HH \arrow[r, "\overline{p}"'] & \overline{G} \arrow[r, "\gamma^{-1}"] & H \otimes_{\underline{\mathsf{H}}^{\overline{\xi}}} H \arrow[r, "\widetilde{q}"'] & H
\end{tikzcd}$$

in which

- the rectangle commutes since the $\widetilde{q}_1$ is a morphism of right $\underline{\mathsf{H}}$-modules;
- the triangle commutes since e is the unit of the monad $\underline{\mathsf{H}}$;
- part (1) commutes since $\overline{p} \cdot \delta = \gamma \cdot \ell \cdot He$ by (5.10), and therefore $\gamma^{-1} \cdot \overline{p} \cdot \delta = \ell \cdot He$.

So the whole diagram commutes and we have $S * 1 = \widetilde{q} \cdot \ell \cdot He$.

Commutativity of the trapezoid in (5.12) shows $\widetilde{q} \cdot \ell = m \cdot \overline{\xi} H$, and since $\overline{\xi} H \cdot He = He \cdot \overline{\xi}$ by naturality of composition, we get

$$S * 1 = \widetilde{q} \cdot \ell \cdot He = m \cdot \overline{\xi}H \cdot He = m \cdot He \cdot \overline{\xi} = \overline{\xi},$$

and from (4.11) we conclude $1 * S * 1 = 1 * \overline{\xi} = 1$.

In the diagram

$$\begin{tikzcd}
H \arrow[r, "\delta"] \arrow[dd, "He"'] & HH \arrow[r, "HeH"] \arrow[d, "H\xi"] & HHH \arrow[r, "\widetilde{q_1}H"] \arrow[d, "HH\xi"] & HH \arrow[d, "H\xi"] \\
\arrow[r, phantom, "(1)"] & HH \arrow[r, "HeH"] \arrow[dr, equal] & HHH \arrow[r, "\widetilde{q_1}H"] \arrow[d, "Hm"] & HH \arrow[d, "m"] \\
HH \arrow[rr, "\kappa"'] & & HH \arrow[r, "\widetilde{q_1}"'] & H
\end{tikzcd}$$

- part (1) commutes by (4.2)(ii) and (4.4),
- the top squares commute by naturality of composition,
- the bottom square commutes since $\widetilde{q_1}$ is a morphism of right $\underline{\mathsf{H}}$-modules, and
- the triangle commutes since e is the unit of the monad $\underline{\mathsf{H}}$.

Now, commutativity of the diagram and the equation $\overline{p} \cdot \kappa = \overline{p} \cdot \overline{i} \cdot \overline{p} = \overline{p}$ imply

$$\begin{aligned} S * \xi &= m \cdot H\xi \cdot \widetilde{q_1}H \cdot HeH \cdot \delta = \widetilde{q_1} \cdot \kappa \cdot He \\ &= \widetilde{q} \cdot \gamma^{-1} \cdot \overline{p} \cdot \kappa \cdot He = \widetilde{q} \cdot \gamma^{-1} \cdot \overline{p} \cdot He = S. \end{aligned}$$

Recalling $1 * \overline{\xi} = 1 = \xi * 1$ (from (4.11), (4.4)) yields

$$1 * S * 1 = 1 * \overline{\xi} = 1 = \xi * 1,$$

and applying Lemma 5.18 shows $1 * S = 1 * S * \xi = \xi * \xi = \xi$ and, eventually,

$$S * 1 * S = S * \xi = S, \quad \overline{\xi} * S = S * 1 * S = S.$$

Thus, S is an antipode and hence H is a weak braided Hopf monad, as asserted.

The implications (a)⇒(c)⇒(e)⇒(a) hold by symmetry and the final assertion follows from the proof of the implication (b)⇒(d). □

6.7. Remark. When a weak braided bimonad H has an antipode S, it can be shown that the composite

$$\overline{G} \xrightarrow{\overline{i}} HH \xrightarrow{\delta H} HHH \xrightarrow{HSH} HHH \xrightarrow{Hm} HH \xrightarrow{\ell} H \otimes_{\underline{\mathsf{H}}^{\overline{\xi}}} H$$

is the two-sided inverse of $\gamma : H \otimes_{\underline{\mathsf{H}}^{\overline{\xi}}} H \to \overline{G}$, while the composite

$$H \otimes^{\overline{\mathsf{H}}^{\overline{\xi}}} H \xrightarrow{\text{can}} HH \xrightarrow{H\delta} HHH \xrightarrow{HSH} HHH \xrightarrow{mH} HH \xrightarrow{\overline{p}\,'} \overline{T}$$

is the two-sided inverse of $\gamma' : \overline{T} \to H \otimes^{\overline{\mathsf{H}}^{\overline{\xi}}} H$.

6.8. Theorem. *Let $H = (\underline{H}, \overline{H}, \tau)$ be a weak braided bimonad on a Cauchy complete category $\mathbb{A}$, and T and G the monad and comonad it induces on $\mathbb{A}^{\overline{H}}$ and $\mathbb{A}_{\underline{H}}$, respectively.*

(1) *If H preserves existing coequalisers, the following are equivalent:*
  (a) *H is a weak braided Hopf monad;*
  (b) *the functor $K_\omega : \mathbb{A} \to \mathbb{A}_{\underline{H}}^{\overline{H}}(\omega)$ admits a monadic right adjoint;*
  (c) *the induced natural transformation $\gamma : H \otimes_{\underline{H}^{\overline{\xi}}} H \to \overline{G}$ is an isomorphism.*

(2) *If H preserves existing equalisers, the following are equivalent:*
  (a) *H is a weak braided Hopf monad;*
  (b) *the functor $K_\omega : \mathbb{A} \to \mathbb{A}_{\underline{H}}^{\overline{H}}(\omega)$ admits a comonadic left adjoint;*
  (c) *the induced natural transformation $\gamma' : \overline{T} \to H \otimes^{\overline{H}^{\overline{\xi}}} H$ is an isomorphism.*

Moreover, if the (equivalent) conditions above hold, then there is an equivalence of categories $\mathbb{A}_{\underline{H}^{\overline{\xi}}} \simeq \mathbb{A}_{\underline{H}}^{\overline{H}}(\omega)$.

Proof. By symmetry, it suffices to prove (1). Then, in the light of Proposition 6.6, we need only to show that (b) implies (c). So suppose the functor $K_\omega : \mathbb{A} \to \mathbb{A}_{\underline{\mathsf{H}}}^{\overline{\mathsf{H}}}(\omega)$ admits a monadic right adjoint $\overline{K}$. Then the comparison functor $K_{\underline{\mathsf{H}}^{\overline{\xi}}} : (\mathbb{A}_{\underline{\mathsf{H}}})^{\mathsf{G}} \to \mathbb{A}_{\underline{\mathsf{H}}^{\overline{\xi}}}$ is an equivalence and hence has a left adjoint (inverse) $L_{\underline{\mathsf{H}}^{\overline{\xi}}} : \mathbb{A}_{\underline{\mathsf{H}}^{\overline{\xi}}} \to (\mathbb{A}_{\underline{\mathsf{H}}})^{\mathsf{G}}$. It then follows from Proposition 3.6 that $U^{\mathsf{G}} L_{\underline{\mathsf{H}}^{\overline{\xi}}} = (\iota^{\overline{\xi}})_!$.

Since H is assumed to preserve existing coequalisers, the forgetful functor $U_{\mathsf{T}} : \mathbb{A}_{\mathsf{T}} \to \mathbb{A}$ also preserves existing coequalisers (e.g. [11, Proposition 4.3.2]). Since colimits in functor categories are calculated componentwise, this implies that the image of the coequaliser (6.2) under the functor $[\mathbb{A}_{\underline{\mathsf{H}}^{\overline{\xi}}}, U_{\mathsf{H}}] : [\mathbb{A}_{\underline{\mathsf{H}}^{\overline{\xi}}}, \mathbb{A}_{\mathsf{H}}] \to [\mathbb{A}_{\underline{\mathsf{H}}^{\overline{\xi}}}, \mathbb{A}]$ is again a coequaliser. Then $\gamma = U_{\underline{\mathsf{H}}}\, t_{L_{\underline{\mathsf{H}}^{\overline{\xi}}}}\, \phi_{\underline{\mathsf{H}}}$ by exactly the same argument used in the proof of the implication (b)⇒(d) of Proposition 6.6.

Now, since $L_{\underline{\mathsf{H}}^{\overline{\xi}}}$ is an equivalence of categories and $U^{\mathsf{G}} L_{\underline{\mathsf{H}}^{\overline{\xi}}} = (\iota^{\overline{\xi}})_!$, it follows by Theorem 3.2 that $t_{L_{\underline{\mathsf{H}}^{\overline{\xi}}}}$ (and hence also γ) is an isomorphism. Thus (b) implies (c), as required. □

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