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MORE ON DESCENT THEORY FOR SCHEMES

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Abstract. In this paper we continue the investigation of some aspects of descent theory for schemes that was begun in [11]. Let **SCH** be a category of schemes. We show that quasi-compact pure morphisms of schemes are effective descent morphisms with respect to **SCH**-indexed categories given by (i) quasi-coherent modules of finite type, (ii) flat quasi-coherent modules, (iii) flat quasi-coherent modules of finite type, (iv) locally projective quasi-coherent modules of finite type. Moreover, we prove that a quasi-compact morphism of schemes is pure precisely when it is a stable regular epimorphism in **SCH**. Finally, we present an alternative characterization of pure morphisms of schemes.

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1. Introduction

The notion of *purity* for a morphism of schemes was introduced by the author in [11] with the aim to give a full characterization of those morphisms of schemes that are stable effective descent morphisms for quasi-coherent modules. Let us recall this notion here. Let $\mathbb{M}$ denote the class of closed immersions in the category of schemes, **SCH**. It was shown in [11] that $\mathbb{M}$ is the second factor of a factorization system $(\mathbb{E}, \mathbb{M})$ on **SCH**. Let $\mathbb{E}'$ be the stabilization of $\mathbb{E}$ [3]. A morphism of schemes is called *pure* if it lies in $\mathbb{E}'$. Various necessary and sufficient conditions for a morphism in **SCH** to be pure are given in [11]. In particular, it is shown that a quasi-coherent morphism of schemes is pure precisely when it is a stable effective descent morphism with respect to the **SCH**-indexed category given by quasi-coherent modules.

In the present paper, we discuss some general properties of pure morphisms. We give several examples of **SCH**-indexed categories for which every quasi-compact pure morphism of schemes is an effective descent morphism. Moreover, we show that a quasi-compact morphism is pure if and only if it is a stable regular epimorphism in **SCH**. Finally, we present an alternative description of pure morphisms in terms of schematically dominant morphisms.

Our reference for the general theory of schemes is [6], and we freely use the terminology and results from it. Notation is as in *loc.cit.* with some minor differences: For any scheme $\mathbf{X} = (\mathbf{X}, \mathcal{O}_{\mathbf{X}})$, the category of all $\mathcal{O}_{\mathbf{X}}$-modules is denoted by $\mathcal{M}(\mathbf{X})$, while $\mathcal{QCM}(\mathbf{X})$ (resp. $\mathcal{QCA}(\mathbf{X})$) is the notation for the category of all quasi-coherent $\mathcal{O}_{\mathbf{X}}$-modules (resp. $\mathcal{O}_{\mathbf{X}}$-algebras). When we consider the underlying topological space of $\mathbf{X}$, we denote this space by $\mathrm{sp}(\mathbf{X})$.

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And if U is an open subset of the underlying topological space of $\mathbf{X}$, we write $\mathbf{U}$ for the corresponding induced open subscheme structure on U.

2. Preliminaries on Descent Theory

Let $\mathbf{C}$ be a category admitting pullbacks, and let $p\colon c' \to c$ be an arbitrary morphism in $\mathbf{C}$. Consider the canonical projections

$$\pi_1, \pi_2\colon c' \times_c c' \to c'$$

and

$$\pi_{12}, \pi_{13}, \pi_{23}\colon c' \times_c c' \times_c c' \to c' \times_c c'$$

arising from taking pullbacks. Then one has the following equations:

$$\pi_1 \cdot \pi_{12} = \pi_1 \cdot \pi_{13}, \quad \pi_2 \cdot \pi_{13} = \pi_2 \cdot \pi_{23}, \quad \text{and} \quad \pi_1 \cdot \pi_{23} = \pi_2 \cdot \pi_{12}. \tag{1}$$

For a $\mathbf{C}$-indexed category $\mathcal{F} : \mathbf{C}^{op} \to \mathbf{CAT}$, a $\mathcal{F}$-*descent datum* on an object $x \in \mathcal{F}(c')$ *relative to the morphism* p consists of an isomorphism

$$\theta\colon \mathcal{F}(\pi_1)(x) \to \mathcal{F}(\pi_2)(x)$$

in $\mathcal{F}(c' \times_c c')$ such that the diagram

$$\begin{array}{ccc}
\mathcal{F}(\pi_{12})\mathcal{F}(\pi_1)(x) & \xrightarrow{\mathcal{F}(\pi_{12})(\theta)} & \mathcal{F}(\pi_{12})\mathcal{F}(\pi_2)(x) \\
\simeq \downarrow & & \downarrow \simeq \\
\mathcal{F}(\pi_{13})\mathcal{F}(\pi_1)(x) & & \mathcal{F}(\pi_{23})\mathcal{F}(\pi_1)(x) \\
{\scriptstyle \mathcal{F}(\pi_{13})(\theta)} \downarrow & & \downarrow {\scriptstyle \mathcal{F}(\pi_{23})(\theta)} \\
\mathcal{F}(\pi_{13})\mathcal{F}(\pi_2)(x) & \xrightarrow[\simeq]{} & \mathcal{F}(\pi_{23})\mathcal{F}(\pi_2)(x)
\end{array}$$

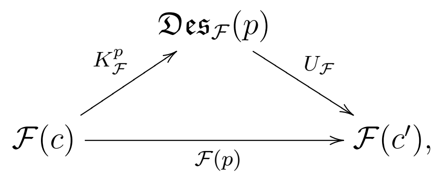
commutes. Here the unnamed morphisms are the canonical isomorphisms of the indexed category $\mathcal{F}$ arising from equations (1). A morphism of $\mathcal{F}$-descent data $h\colon (x, \theta) \to (x', \theta')$ is a morphism $h\colon x \to x'$ in $\mathcal{F}(c')$ such that $\theta' \cdot \mathcal{F}(\pi_1)(h) = \mathcal{F}(\pi_2)(h) \cdot \theta$.

The category whose objects are pairs (x, θ), where x is an object of $\mathcal{F}(c')$ and θ is a $\mathcal{F}$-descent datum on x is called the *category of* $\mathcal{F}$*-descent data relative to* p and denoted by $\mathfrak{Des}_{\mathcal{F}}(p)$.

If $y \in \mathcal{F}(c)$, then $\mathcal{F}(p)(y)$ comes equipped with canonical $\mathcal{F}$-descent datum given by the composite

$$\mathcal{F}(\pi_1)(\mathcal{F}(p)(y)) \simeq \mathcal{F}(p\pi_1)(y) = \mathcal{F}(p\pi_2)(y) \simeq \mathcal{F}(\pi_2)(\mathcal{F}(p)(y)).$$

Thus the functor $\mathcal{F}(p)$ factors as



where $U_{\mathcal{F}}$ is the evident forgetful functor, and $K^p_{\mathcal{F}}$ sends y to $\mathcal{F}(p)(y)$ equipped with canonical $\mathcal{F}$-descent datum.

p is called an *(effective) $\mathcal{F}$-descent morphism* if the functor $K^p_{\mathcal{F}}$ is full and faithful (an equivalence of categories). Moreover, p is a stable (effective) $\mathcal{F}$-descent morphism if every pullback of p is an (effective) $\mathcal{F}$-descent morphism.

More details on Descent Theory can be found in [7] and [8].

Actually, we are interested in indexed categories that are induced by pullback-stable classes of morphisms. Recall (for instance from [8]) that any pullback-stable class $\mathbb{H}$ of morphisms in $\mathbf{C}$ can be considered as a $\mathbf{C}$-indexed category as follows: given an object $c \in \mathbf{C}$, the category $\mathbb{H}(c)$ is a full subcategory of the slice category $\mathbf{C}/c$ whose objects are morphisms in $\mathbb{H}$, and if $p : c' \to c$ is an arbitrary morphism in $\mathbf{C}$, then one has the pullback functor

$$p^* : \mathbf{C}/c \to \mathbf{C}/c'$$

which, because $\mathbb{H}$ is pullback-stable, restricts to a functor

$$\mathbb{H}(p) : \mathbb{H}(c) \to \mathbb{H}(c')$$

and so makes the assignment $c \to \mathbb{H}(c)$ into a $\mathbf{C}$-indexed category $\mathbb{H}$.

The following was mentioned in [14] but only in the case $\mathbb{H} =$ all morphisms.

Proposition 2.1. *Let $p : c' \to c$ be a morphism in $\boldsymbol{C}$, and let $\gamma : x \to c'$ be an object of $\mathbb{H}(c')$ with a monomorphism γ in $\boldsymbol{C}$. Then γ is equipped with $\mathbb{H}$-descent datum relative to p if and only if there is an isomorphism $\pi_1^*(x,\gamma) \simeq \pi_2^*(x,\gamma)$ in $\mathbb{H}(c' \times_c c')$. (Here $\pi_i, i = 1, 2,$ denote the projections of $c' \times_c c'$ onto the i-*th *factor.)*

We shall need the following result which is in fact just an adaptation of Corollary 2.7 of [7] to general indexed categories.

Theorem 2.2. *Let $\boldsymbol{C}$ be a category with pullbacks, and let*

$$\alpha : \mathcal{F} \Rightarrow \mathcal{F}' : \boldsymbol{C}^{op} \to \mathbf{CAT}$$

be a morphism of $\boldsymbol{C}$-indexed categories whose components are fully faithful (in which case we say that $\mathcal{F}$ is a $\boldsymbol{C}$-indexed full subcategory of $\mathcal{F}'$). A morphism $p : c' \to c$ in $\boldsymbol{C}$ which is an effective $\mathcal{F}'$-descent morphism is also an effective

$\mathcal{F}$-descent morphism if and only if the diagram in **CAT**

$$\begin{array}{ccc} \mathcal{F}(c) & \xrightarrow{\alpha_c} & \mathcal{F}'(c) \\ {\scriptstyle \mathcal{F}(p)}\downarrow & & \downarrow{\scriptstyle \mathcal{F}'(p)} \\ \mathcal{F}(c') & \xrightarrow[\alpha_{c'}]{} & \mathcal{F}'(c') \end{array}$$

is a (pseudo-)pullback.

3. Pure Stacks

Let $(\mathbf{C}, J)$ be a site (see, for example, [9]) such that $\mathbf{C}$ has (small) coproducts. For our purposes, a *stack* on $(\mathbf{C}, J)$ (or just a $\mathbf{C}$-*stack*) is a $\mathbf{C}$-indexed category $\mathcal{F} : \mathbf{C}^{op} \to \mathbf{CAT}$ such that for any cover $(c_i \to c)_{i \in I}$ in J, the induced morphism $\coprod_i c_i \to c$ is a stable effective $\mathcal{F}$-descent morphism.

Consider the category **SCH** of schemes and recall [10] that the *Zariski* topology (*Zar*) on **SCH** is defined by open immersions of schemes, and henceforth by a stack on **SCH** we will mean a stack on (**SCH**, *Zar*). Recall also that, for a given scheme $\mathbf{X} = (\mathbf{X}, \mathcal{O}_{\mathbf{X}})$, an $\mathcal{O}_{\mathbf{X}}$-module $\mathcal{M}$ is quasi-coherent if there is, for all $x \in \mathrm{sp}(\mathbf{X})$, an open neighborhood U of x such that $\mathcal{M}|_U$ is the cokernel of a morphism ${\mathcal{O}_{\mathbf{X}}}^{(I)}|_U \to {\mathcal{O}_{\mathbf{X}}}^{(J)}|_U$ of $\mathcal{O}_{\mathbf{X}}$-module for some collections of indices I and J. An $\mathcal{O}_{\mathbf{X}}$-algebra is quasi-coherent if it is quasi-coherent as $\mathcal{O}_{\mathbf{X}}$-module.

Write $\mathcal{QCM}$ (resp. $\mathcal{QCA}$) for the **SCH**-indexed category given by quasi-coherent modules (resp. algebras). It is well known (see, for example, [13]) that, if $(\mathbf{U}_i \to \mathbf{X})_{i \in I}$ is a cover in (*Zar*), then the canonical morphism $\coprod_i \mathbf{U}_i \to \mathbf{X}$ is a stable effective $\mathcal{QCM}$-descent morphism; the same is true for the **SCH**-indexed category $\mathcal{QCA}$ so that each of $\mathcal{QCM}$ and $\mathcal{QCA}$ is a stack on **SCH**.

A stack $\mathcal{F}$ on **SCH** is called *pure* if every pure morphism of affine schemes is an effective $\mathcal{F}$-descent morphism. According to the results in [11] , both $\mathcal{QCM}$ and $\mathcal{QCA}$ are pure **SCH**-stacks.

Theorem 3.1. *Let $\mathcal{F}$ and $\mathcal{F}'$ be pure stacks on* **SCH**, *and let $\alpha : \mathcal{F} \to \mathcal{F}'$ be a morphism of indexed categories whose components are fully faithful. Suppose that the following condition is satisfied:*

> *For any scheme $\mathbf{X}$, and for any open cover $(V_k)_{k \in K}$ of $\mathrm{sp}(\mathbf{X})$, the diagram in* **CAT**
>
> $$\begin{array}{ccc} \mathcal{F}(\mathbf{X}) & \xrightarrow{\alpha_{\mathbf{X}}} & \mathcal{F}'(\mathbf{X}) \\ \downarrow & & \downarrow \\ \prod_k \mathcal{F}(\boldsymbol{V}_k) & \xrightarrow{\prod_k \alpha_{\boldsymbol{V}_k}} & \prod_k \mathcal{F}'(\boldsymbol{U}_k) \end{array}$$
>
> *is a pullback (the vertical functors here arise from the canonical morphisms $\boldsymbol{V}_k \to \mathbf{X}$).*

Then a morphism $p : \mathbf{Y} \to \mathbf{X}$ of schemes which is a stable effective $\mathcal{F}'$-descent morphism is also a (stable) effective $\mathcal{F}$-descent morphism if and only if there *exists an open cover* $(U_i)_{i\in I}$ *of* $\mathrm{sp}(\mathbf{X})$ *such that the pullback projections* $\mathbf{Y}_i = \mathbf{Y}\times_{\mathbf{X}} \boldsymbol{U}_i \to \boldsymbol{U}_i$ *are (stable) effective* $\mathcal{F}$*-descent morphisms.*

Proof. For the non-trivial part, suppose first that there exists an open cover $(U_i)_{i\in I}$ of $\mathrm{sp}(\mathbf{X})$ such that in each pullback diagram

$$\begin{array}{ccc} \mathbf{Y}_i & \xrightarrow{p_i} & \mathbf{U}_i \\ {\scriptstyle g_i'}\downarrow & & \downarrow{\scriptstyle g_i} \\ \mathbf{Y} & \xrightarrow{p} & \mathbf{X}, \end{array}$$

where $g_i : \mathbf{U}_i \to \mathbf{X}$ is the canonical embedding, the morphism p_i is an effective $\mathcal{F}$-descent morphism. Since p is assumed to be a stable effective $\mathcal{F}'$-descent morphism, it follows that each p_i is an effective $\mathcal{F}'$-descent morphism. Then, by Theorem 2.2, each diagram

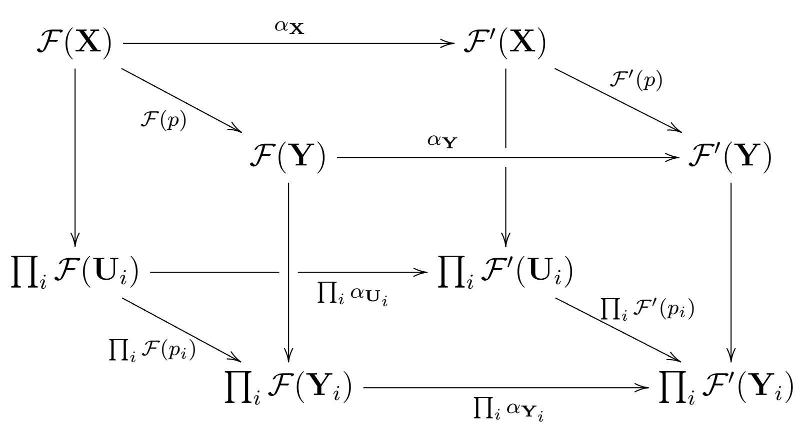
$$\begin{array}{ccc} \mathcal{F}(\mathbf{U}_i) & \xrightarrow{\alpha_{\mathbf{U}_i}} & \mathcal{F}'(\mathbf{U}_i) \\ {\scriptstyle \mathcal{F}(p_i)}\downarrow & & \downarrow{\scriptstyle \mathcal{F}'(p_i)} \\ \mathcal{F}(\mathbf{Y}_i) & \xrightarrow{\alpha_{\mathbf{Y}_i}} & \mathcal{F}'(\mathbf{Y}_i), \end{array}$$

(and hence also the diagram

$$\begin{array}{ccc} \prod_i \mathcal{F}(\mathbf{U}_i) & \xrightarrow{\prod_i \alpha_{\mathbf{U}_i}} & \prod_i \mathcal{F}'(\mathbf{U}_i) \\ {\scriptstyle \prod_i \mathcal{F}(p_i)}\downarrow & & \downarrow{\scriptstyle \prod_i \mathcal{F}'(p_i)} \\ \prod_i \mathcal{F}(\mathbf{Y}_i) & \xrightarrow{\prod_i \alpha_{\mathbf{Y}_i}} & \prod_i \mathcal{F}'(\mathbf{Y}_i) \quad) \end{array}$$

is a pullback.

We now consider the following commutative (up to an isomorphism) diagram



in which:

- the vertical morphisms are induced by the morphisms $g_i : \mathbf{U}_i \to \mathbf{X}$ and $g'_i : \mathbf{Y}_i \to \mathbf{Y}$;
- the bottom square is a pullback, as we have seen above;
- the back left and the front right squares are pullbacks, since $(U_i)_{i\in I}$ (resp. $(Y_i)_{i\in I}$) is an open cover of $\mathrm{sp}(\mathbf{X})$ (resp. of $\mathrm{sp}(\mathbf{Y})$).

It follows that the exterior and the bottom squares in the diagram

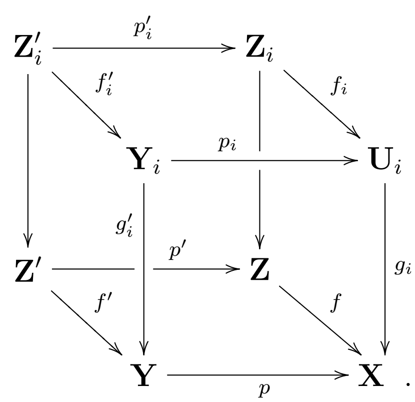
$$\begin{array}{ccc}
\mathcal{F}(\mathbf{X}) & \xrightarrow{\alpha_{\mathbf{X}}} & \mathcal{F}'(\mathbf{X}) \\
{\scriptstyle \mathcal{F}(p)}\downarrow & & \downarrow{\scriptstyle \mathcal{F}'(p)} \\
\mathcal{F}(\mathbf{Y} & \xrightarrow{\alpha_{\mathbf{Y}}} & \mathcal{F}'(\mathbf{Y}) \\
\downarrow & & \downarrow \\
\prod_i \mathcal{F}(\mathbf{Y}_i) & \xrightarrow{\prod_i \alpha_{\mathbf{Y}_i}} & \prod_i \mathcal{F}'(\mathbf{Y}_i)
\end{array}$$

are pullbacks. Hence the top square is also a pullback, and Theorem 2.2 tells us that p is an effective $\mathcal{F}$-descent morphism.

Suppose now that each p_i is a stable effective $\mathcal{F}$-descent morphism. To see that the morphism p is also a stable effective $\mathcal{F}$-descent morphism, consider an arbitrary morphism $f : \mathbf{Z} \to \mathbf{X}$ in **SCH** and form the pullback

$$\begin{array}{ccc}
\mathbf{Z}' & \xrightarrow{p'} & \mathbf{Z} \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
\mathbf{Y} & \xrightarrow{p} & \mathbf{X}
\end{array} \quad . \tag{2}$$

We are to show that p' is an effective $\mathcal{F}$-descent morphism. Let us first note that since p is a stable effective $\mathcal{F}'$-descent morphism and since the diagram (2) is a pullback, p' is also a stable effective $\mathcal{F}'$-descent morphism. Next, for each $i \in I$, consider the diagram all of whose faces are pullbacks



Since the top square is a pullback and since p_i is a stable effective $\mathcal{F}$-descent morphism by assumption, p'_i is an effective $\mathcal{F}$-descent morphism. Then as above we get that p' is an effective $\mathcal{F}$-descent morphism. This completes the proof. □

Proposition 3.2. *Let R be a commutative ring, and let $p : \mathbf{X} \to \mathrm{Spec}(R)$ be a quasi-compact pure morphism of schemes. If p is a (stable) effective $\mathcal{F}'$-descent morphism, it is also a (stable) effective $\mathcal{F}$-descent morphism.*

Proof. Since the morphism $p : \mathbf{X} \to \mathrm{Spec}(R)$ is quasi-compact, the topological space $\mathrm{sp}(\mathbf{X})$ is quasi-compact; so we can cover it with a finite number of open affine subsets $(\mathrm{sp}(\mathrm{Spec}(A_i)))_{i=1,n}$. If we set $A = \prod_i A_i$ and $e : \mathrm{Spec}(A) = \coprod_i \mathrm{Spec}(A_i) \to \mathbf{X}$ is the canonical morphism, then the composite

$$pe\colon\ \mathrm{Spec}(A) \to \mathrm{Spec}(R)$$

is a pure morphism of affine schemes (see [11]). In that case, by Theorem 2.2, the exterior rectangle of the diagram

$$\begin{array}{ccc}
\mathcal{F}(\mathrm{Spec}(R)) & \xrightarrow{\alpha_{\mathrm{Spec}(R)}} & \mathcal{F}'(\mathrm{Spec}(R)) \\
{\scriptstyle \mathcal{F}(p)}\downarrow & & \downarrow{\scriptstyle \mathcal{F}'(p)} \\
\mathcal{F}(\mathbf{X}) & \xrightarrow{\alpha_{\mathbf{X}}} & \mathcal{F}'(\mathbf{X}) \\
{\scriptstyle \mathcal{F}(e)}\downarrow & & \downarrow{\scriptstyle \mathcal{F}'(e)} \\
\mathcal{F}(\mathrm{Spec}(A)) & \xrightarrow{\alpha_{\mathrm{Spec}(A)}} & \mathcal{F}'(\mathrm{Spec}(A)).
\end{array}$$

is a pullback, since both $\mathcal{F}$ and $\mathcal{F}'$ are pure stacks on **SCH**. Moreover, the bottom square is also a pullback, since both $\mathcal{F}$ and $\mathcal{F}'$ are stacks on **SCH**, thus the top square is a pullback and Theorem 2.2 shows immediately that p is an effective $\mathcal{F}$-descent morphism.

Now suppose that p is a stable effective $\mathcal{F}'$-descent morphism. Let $f : \mathbf{Y} \to \mathrm{Spec}(R)$ be an arbitrary morphism of schemes and let the pullback of f along p be

$$\begin{array}{ccc}
\mathbf{Z} & \xrightarrow{p'} & \mathbf{Y} \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
\mathbf{X} & \xrightarrow{p} & \mathrm{Spec}(R).
\end{array}$$

Take any open affine cover $(\mathrm{sp}(e_i) : \mathrm{sp}(\mathrm{Spec}(R_i)) \to \mathrm{sp}(\mathbf{Y}))$ of the topological space $Sp(\mathbf{Y})$, and consider for each $i \in I$ the pullback square

$$\begin{array}{ccc}
\mathbf{Z}_i & \xrightarrow{p'_i} & \mathrm{Spec}(R_i) \\
{\scriptstyle f'_i}\downarrow & & \downarrow{\scriptstyle e_i} \\
\mathbf{Z} & \xrightarrow{p'} & \mathbf{Y}.
\end{array}$$

Then, since p is a stable effective $\mathcal{F}'$-descent morphism by assumption, and since the combined diagram

$$\begin{array}{ccc} \mathbf{Z}_i & \xrightarrow{p'_i} & \mathrm{Spec}(R_i) \\ {\scriptstyle f'f'_i}\downarrow & & \downarrow{\scriptstyle fe_i} \\ \mathbf{X} & \xrightarrow{p} & \mathrm{Spec}(R) \end{array}$$

is a pullback, p'_i is an effective $\mathcal{F}'$-descent morphism. Moreover, the class of quasi-compact pure morphisms is stable under pullback, so that p'_i is a quasi-compact pure morphism of schemes. Then by the previous part of the proof applied to the morphism p', p'_i is an effective $\mathcal{F}$-descent morphism; whence by Theorem 3.1, p' is an effective $\mathcal{F}$-descent morphism and one concludes that p is a stable effective $\mathcal{F}$-descent morphism. □

In the light of the proposition, we get from Theorem 3.1 the following result.

Theorem 3.3. *In the situation of Theorem* 3.1, *any quasi-compact pure morphism of schemes that is a stable effective $\mathcal{F}'$-descent morphism is also a stable effective $\mathcal{F}$-descent morphism.*

We shall call an **SCH**-indexed category $\mathcal{F} : \mathbf{SCH}^{op} \to \mathbf{CAT}$ product-preserving if for any scheme $\mathbf{X}$ and any open cover $(U_k)_{k\in K}$ of $\mathrm{sp}(\mathbf{X})$, the canonical functor

$$\mathcal{F}\Big(\coprod_k \mathbf{U}_k\Big) \to \prod_k \mathcal{F}(\mathbf{U}_k)$$

induced by the functors

$$\mathcal{F}(i_k) : \mathcal{F}\Big(\coprod_k \mathbf{U}_k\Big) \to \mathcal{F}(\mathbf{U}_k)$$

is an equivalence of categories (here $i_k : \mathbf{U}_k \to \coprod_k \mathbf{U}_k$ is the coproduct inclusion).

Theorem 3.4. *Let $\alpha : \mathcal{F} \to \mathcal{F}'$ be a morphism of pure stacks on* **SCH** *whose components are full and faithful, and suppose that both $\mathcal{F}$ and $\mathcal{F}'$ are product-preserving. Then any quasi-compact pure morphism of schemes that is a stable effective $\mathcal{F}'$-descent morphism is also a stable effective $\mathcal{F}$-descent morphism.*

Proof. Let $\mathbf{X}$ be a scheme, and $(U_i)_{i\in I}$ an open cover of $\mathrm{sp}(\mathbf{X})$. Since, by hypothesis, both $\mathcal{F}$ and $\mathcal{F}'$ are **SCH**-stacks, the diagram

$$\begin{array}{ccc} \mathcal{F}(\mathbf{X}) & \xrightarrow{\alpha_{\mathbf{X}}} & \mathcal{F}'(\mathbf{X}) \\ {\scriptstyle \mathcal{F}(e)}\downarrow & & \downarrow{\scriptstyle \mathcal{F}'(e)} \\ \mathcal{F}(\coprod_i \mathbf{U}_i) & \xrightarrow{\alpha_{(\coprod_i \mathbf{U}_i)}} & \mathcal{F}'(\coprod_i \mathbf{U}_i), \end{array}$$

where $e : \coprod_i \mathbf{U}_i \to \mathbf{X}$ is the canonical morphism, is a pullback by Theorem 2.2.

Since $\mathcal{F}$ and $\mathcal{F}'$ are product-preserving **SCH**-indexed categories by assumption, the diagram

$$\begin{array}{ccc} \mathcal{F}(\mathbf{X}) & \xrightarrow{\alpha_{\mathbf{X}}} & \mathcal{F}'(\mathbf{X}) \\ \downarrow & & \downarrow \\ \prod_i \mathcal{F}(\mathbf{U}_i) & \xrightarrow{\prod_i \alpha_{\mathbf{U}_i}} & \prod_i \mathcal{F}'(\mathbf{U}_i) \end{array}$$

too is a pullback.

We can now deduce the result from Theorem 3.3. □

4. Some Properties of Pure Morphisms

We begin by recalling some definitions. Let $\mathbf{X} = (\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ be a scheme, and $\mathcal{M}$ an $\mathcal{O}_{\mathbf{X}}$-module. Then

- $\mathcal{M}$ is of finite type if for each point $x \in \mathrm{sp}(\mathbf{X})$, there is an open neighborhood U of x and a natural number n (n might depend on x) such that $\mathcal{M}|_U$ can be written as a quotient of $(\mathcal{O}_{\mathbf{X}}|_U)^n$ or, equivalently, if for each point $x \in \mathrm{sp}(\mathbf{X})$ there is an open neighborhood U of x and a family of sections $(m_i)_{i=\overline{1,n}}$, $m_i \in \mathcal{M}(U)$ such that $\mathcal{M}|_U$ is generated by this family, by which is meant that for each point $y \in U$, the images of m_i in the stalk $\mathcal{M}_y$ generate $\mathcal{M}_y$ as an $\mathcal{O}_{\mathbf{X},y}$-module;
- $\mathcal{M}$ is flat if $\mathcal{M}_x$ is a flat $\mathcal{O}_{\mathbf{X},x}$-module for every $x \in \mathrm{sp}(\mathbf{X})$;
- $\mathcal{M}$ is a free $\mathcal{O}_{\mathbf{X}}$-module (of finite rank) if it is isomorphic to a direct sum of (finite) copies of $\mathcal{O}_{\mathbf{X}}$, and $\mathcal{M}$ is locally free (of finite rank) if there is an open cover $(U_i)_{i\in I}$ of $\mathrm{sp}(\mathbf{X})$ for which each $\mathcal{M}|_{U_i}$ is a free $\mathcal{O}_{\mathbf{X}}|_{U_i}$-module (of finite rank);
- $\mathcal{M}$ is locally projective of finite type if $\mathcal{M}$ is locally a direct summand of a free $\mathcal{O}_{\mathbf{X}}$-module of finite rank.

It follows immediately from Theorem II.11 of [1] that

Proposition 4.1. *Let $\mathbf{X} = (\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ be a scheme, and let $\mathcal{M}$ be an $\mathcal{O}_{\mathbf{X}}$-module. Then $\mathcal{M}$ is locally projective of finite type if and only if it is locally free of finite rank.*

Proposition 4.2. *Let $\mathbf{X} = (\mathbf{X}, \mathcal{O}_{\mathbf{X}})$ be a scheme. For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$, the following statements are equivalent:*

(i) *for any open subset $U \subseteq \mathrm{sp}(\mathbf{X})$, the functor*

$$(\mathcal{M}|_U) \otimes_{(\mathcal{O}_{\mathbf{X}}|_U)} - : \mathcal{M}(\boldsymbol{U}) \to \mathcal{M}(\boldsymbol{U})$$

is exact; that is, $\mathcal{M}|_U$ is flat in $\mathcal{M}(\boldsymbol{U})$;

(ii) *for any open subset $U \subseteq \mathrm{sp}(\mathbf{X})$, the functor*

$$(\mathcal{M}|_U) \otimes_{(\mathcal{O}_{\mathbf{X}}|_U)} - : \mathcal{QCM}(\boldsymbol{U}) \to \mathcal{QCM}(\boldsymbol{U})$$

is exact; that is, $\mathcal{M}|_U$ is flat in $\mathcal{QCM}(\boldsymbol{U})$;

(iii) *$\mathcal{M}$ is a flat $\mathcal{O}_{\mathbf{X}}$-module; that is, $\mathcal{M}_x$ is a flat $\mathcal{O}_{\mathbf{X},x}$-module for any $x \in \mathrm{sp}(\mathbf{X})$.*

Proof. Clearly, (i) always implies (ii).

Since $((\mathcal{M}|_U)\otimes_{(\mathcal{O}_{\mathbf{X}}|_U)}\mathcal{N})_x=\mathcal{M}_x\otimes_{\mathcal{O}_{\mathbf{X},x}}\mathcal{N}_x$ for every $\mathcal{N}\in\mathcal{M}(\mathbf{U})$ and every $x\in U$, (iii) implies both (i) and (ii). It remains to show that (ii) implies (iii).

Fix $x\in\mathrm{sp}(\mathbf{X})$. To show that $\mathcal{M}_x$ is a flat $\mathcal{O}_{\mathbf{X},x}$-module, it is sufficient (see [2]) to show that for any finitely generated ideal $\mathcal{I}$ of $\mathcal{O}_{\mathbf{X},x}$, the canonical morphism

$$\mathcal{I}\otimes_{\mathcal{O}_{\mathbf{X},x}}\mathcal{M}_x\to\mathcal{M}_x$$

is a monomorphism. So consider a finitely generated ideal $\mathcal{I}$ of $\mathcal{O}_{\mathbf{X},x}$, and let $a_1,\dots,a_n$ be elements of $\mathcal{O}_{\mathbf{X},x}$ such that $\mathcal{I}$ is generated as a $\mathcal{O}_{\mathbf{X},x}$-module by them. Since

$$\mathrm{colim}_{U\ni x}\mathcal{O}_{\mathbf{X}}(U)=\mathcal{O}_{\mathbf{X},x}$$

we can find a neighborhood V of x and sections $f_1,\dots,f_n\in\mathcal{O}_{\mathbf{X}}(V)$ with $(f_i)_x=a_i$ for every i. The family $(f_i)_{i=\overline{1,n}}$ induces a morphism

$$(\mathcal{O}_{\mathbf{X}}|_V)^n\to\mathcal{O}_{\mathbf{X}}|_V,$$

and if $\mathcal{J}\subseteq\mathcal{O}_{\mathbf{X}}|_V$ is the image of this morphism, then $\mathcal{J}$ is a quasi-coherent $\mathcal{O}_{\mathbf{X}}|_V$-module, since the image of any morphism of quasi-coherent modules is quasi-coherent [6]. Moreover, it is clear that $\mathcal{J}_x\simeq\mathcal{I}$.

Now, since $\mathcal{M}|_V$ is flat in $\mathcal{QCM}(\mathbf{V})$ by hypothesis, the morphism

$$(\mathcal{M}|_V)\otimes_{(\mathcal{O}_{\mathbf{X}}|_V)}\mathcal{J}\to(\mathcal{M}|_V)\otimes_{(\mathcal{O}_{\mathbf{X}}|_V)}(\mathcal{O}|_V)\simeq\mathcal{M}|_V$$

is a monomorphism. It follows that the stalk of this morphism at x $-$ which is (isomorphic to) the morphism $\mathcal{M}_x\otimes_{\mathcal{O}_{\mathbf{X},x}}\mathcal{I}\to\mathcal{M}_x$ $-$ is a monomorphism as well. Hence $\mathcal{M}_x$ is a flat $\mathcal{O}_{\mathbf{X},x}$-module. □

Proposition 4.3. *Let $\mathbf{X}=(\mathbf{X},\mathcal{O}_{\mathbf{X}})$ be a scheme. An $\mathcal{O}_{\mathbf{X}}$-module $\mathcal{M}$ is (locally projective) of finite type if and only if for any open subset $U\subseteq\mathrm{sp}(\mathbf{X})$, $\mathcal{O}_{\mathbf{X}}|_U$-module $\mathcal{M}|_U$ is (locally projective) of finite type.*

Proof. The first part of the Proposition follows easily from the definition, while the second one is Theorem II.1 in [1]. □

Let $\mathcal{P}$ be a property of quasi-coherent modules of being (i) of finite type, or (ii) flat, or (iii) flat of finite type, or (iv) locally projective of finite type. Combining the two proposition above with the definition of quasi-coherent modules of finite type, we get:

Proposition 4.4. *Let $\mathbf{X}=(\mathbf{X},\mathcal{O}_{\mathbf{X}})$ be scheme, and let $(U_i)_{i\in I}$ be arbitrary open cover of $\mathrm{sp}(\mathbf{X})$. Then the canonical morphism*

$$e:\coprod_i\boldsymbol{U}_i\to\mathbf{X}$$

descends the property $\mathcal{P}$. In other words, an $\mathcal{O}_{\mathbf{X}}$-module $\mathcal{M}$ has $\mathcal{P}$ if and only if each $\mathcal{M}|_{U_i}$ does so.

Let $\mathcal{QCM}_{\mathcal{P}}$ denote the **SCH**-indexed category given by quasi-coherent modules having $\mathcal{P}$. Then we have the evident morphism $i:\mathcal{QCM}_{\mathcal{P}}\to\mathcal{QCM}$, turning $\mathcal{QCM}_{\mathcal{P}}$ into an **SCH**-indexed full subcategory of $\mathcal{QCM}$.

We can now give an alternative form of Proposition 4.4 as follows:

Proposition 4.5. *For any scheme* $\mathbf{X}$ *and any open cover* $(U_i)_{i\in I}$ *of* $\mathrm{sp}(\mathbf{X})$*, the diagram*

$$\begin{array}{ccc} \mathcal{QCM}_{\mathcal{P}}(\mathbf{X}) & \xrightarrow{\mathcal{QCM}_{\mathcal{P}}(e)} & \mathcal{QCM}_{\mathcal{P}}(\coprod_i \boldsymbol{U}_i) \\ {\scriptstyle i_{\mathbf{X}}}\downarrow & & \downarrow{\scriptstyle i_{(\coprod_i \boldsymbol{U}_i)}} \\ \mathcal{QCM}(\mathbf{X}) & \xrightarrow{\mathcal{QCM}(e)} & \mathcal{QCM}(\coprod_i \boldsymbol{U}_i) \end{array}$$

is a pullback in **CAT**.

Applying Theorem 2.2 to the morphism $i : \mathcal{QCM}_{\mathcal{P}} \to \mathcal{QCM}$ of **SCH**-indexed categories, and recalling from [11] that $e : \coprod_i \mathbf{U}_i \to \mathbf{X}$ is a stable effective $\mathcal{QCM}$-descent morphism, we get

Proposition 4.6. $\mathcal{QCM}_{\mathcal{P}}$ *is an* **SCH**-*stack.*

Now Theorem 1.1 of [12] gives

Proposition 4.7. $\mathcal{QCM}_{\mathcal{P}}$ *is a pure* **SCH**-*stack.*

One easily checks that both $\mathcal{QCM}_{\mathcal{P}}$ and $\mathcal{QCM}$ preserve products; and taking into account Proposition 4.7, Theorem 3.4 and Theorem 5.15 of [11], we get

Theorem 4.8. *Every quasi-compact pure morphism of schemes is a stable effective* $\mathcal{QCM}_{\mathcal{P}}$*-descent morphism.*

5. Pure Morphisms via Stable Regular Epimorphisms

Let $p : \mathbf{Y} \to \mathbf{X}$ be a morphism in **SCH**. In the category of topological spaces, **Top**, we may factorize $p_0 : \mathrm{sp}(\mathbf{Y}) \to \mathrm{sp}(\mathbf{X})$ as

$$\mathrm{sp}(\mathbf{Y}) \xrightarrow{\overline{p}_0} p_0(\mathrm{sp}(\mathbf{Y})) \xrightarrow{i_0} \mathrm{sp}(\mathbf{X}),$$

where $\overline{p}_0$ is a surjection and i_0 is a subspace embedding.

Suppose now that $g : \mathbf{Y} \to \mathbf{Z}$ is a morphism of schemes with $g\pi_1 = g\pi_2$, where π_1 and π_2 are the projections

$$\mathbf{Y} \times_{\mathbf{X}} \mathbf{Y} \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \mathbf{Y} .$$

Recall that we write $\mathbb{M}$ for the class of closed immersions in **SCH**.

Proposition 5.1. *With the previous notations, suppose that* $p : \mathbf{Y} \to \mathbf{X}$ *is an effective* $\mathbb{M}$*-descent morphism, then there exists a unique map* $h : p_0(\mathrm{sp}(\mathbf{Y})) \to \mathrm{sp}(\boldsymbol{Z})$ *of topological spaces making the diagram*

$$\begin{array}{ccc} & p_0(\mathrm{sp}(\mathbf{Y})) & \\ {\scriptstyle \overline{p}_0}\nearrow & & \searrow{\scriptstyle h} \\ \mathrm{sp}(\mathbf{Y}) & \xrightarrow[g_0]{} & \mathrm{sp}(\boldsymbol{Z}) \end{array}$$

commute.

Proof. We first note that if such a continuous map of topological spaces exists, it is unique because $\overline{p}_0$ is surjective.

Now let $|\ |: \mathbf{Top} \to \mathbf{Sets}$ denote the forgetful functor from the category of topological spaces to the category of sets. Then the map $|\overline{p}_0| : |\operatorname{sp}(\mathbf{Y})| \to |p_0(\operatorname{sp}(\mathbf{Y}))|$ of sets is a surjection and since in **Sets** all surjections are regular epimorphisms, $|\overline{p}_0|$ is the coequalizer of its kernel-pair. Since $i_0 : p_0(\operatorname{sp}(\mathbf{Y})) \to \operatorname{sp}(\mathbf{X})$ is a subspace embedding, the map $|i_0|$ is monic, and so the kernel-pair of $|\overline{p}_0|$ is also the kernel-pair of $|p_0| = |i_0\overline{p}_0| = |i_0||\overline{p}_0|$. Thus

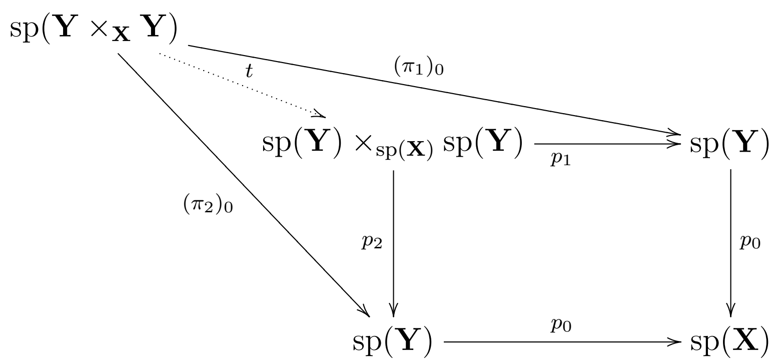
$$|\operatorname{sp}(\mathbf{Y})| \times_{|\operatorname{sp}(\mathbf{X})|} |\operatorname{sp}(\mathbf{Y})| \underset{p_2}{\overset{p_1}{\rightrightarrows}} |\operatorname{sp}(\mathbf{Y})| \xrightarrow{|\overline{p}_0|} |p_0(\operatorname{sp}(\mathbf{Y}))|\ ,$$

where p_1 and p_2 are the projections, is a coequalizer diagram in **Sets**.

Next, the unique morphism

$$t : \operatorname{sp}(\mathbf{Y} \times_{\mathbf{X}} \mathbf{Y}) \to \operatorname{sp}(\mathbf{Y}) \times_{\operatorname{sp}(\mathbf{X})} \operatorname{sp}(\mathbf{Y})$$

of topological spaces rendering the diagram



commutative, is surjective (see Lemma 2.3.1 of [13]); since the functor $|\ |: \mathbf{Top} \to \mathbf{Sets}$ preserves limits, the map

$$|\operatorname{sp}(\mathbf{Y} \times_{\mathbf{X}} \mathbf{Y})| \to |\operatorname{sp}(\mathbf{Y})| \times_{|\operatorname{sp}(\mathbf{X})|} |\operatorname{sp}(\mathbf{Y})|,$$

induced by the map $|t|$, is a surjection. It follows that the diagram

$$|\operatorname{sp}(\mathbf{Y} \times_{\mathbf{X}} \mathbf{Y})| \underset{|(\pi_2)_0|}{\overset{|(\pi_1)_0|}{\rightrightarrows}} |\operatorname{sp}(\mathbf{Y})| \xrightarrow{|\overline{p}_0|} |p_0(\operatorname{sp}(\mathbf{Y}))|$$

is a coequalizer diagram in **Sets**. But clearly $|g_0| \cdot |(\pi_1)_0| = |g_0| \cdot |(\pi_2)_0|$. Hence there exists a unique map $h : |p_0(\operatorname{sp}(\mathbf{Y}))| \to |\operatorname{sp}(\mathbf{Z})|$ of sets such that $h \cdot |\overline{p}_0| = |g_0|$.

Our next aim is to show that h is a continuous map of topological spaces.

Let F be a closed subset of $\operatorname{sp}(\mathbf{Z})$. We may consider F as a closed subscheme of $\mathbf{Z}$ with its induced closed structure [6]; we denote this scheme by $\mathbf{F}$. Then $\operatorname{sp}(\mathbf{F}) = F$ and the canonical embedding $j : \mathbf{F} \to \mathbf{Z}$ is a closed immersion so that j lies in $\mathbb{M}$.

Since closed immersions are stable under pullback, we have the pulling-back functor

$$g^* : \mathbb{M}(\mathbf{Z}) \to \mathbb{M}(\mathbf{Y}).$$

Write $(\mathbf{F}', j')$ for the object $g^*(\mathbf{F}, j)$ of the category $\mathbb{M}(\mathbf{Y})$. Then there is a canonical isomorphism

$$\theta : \pi_1^*(\mathbf{F}', j') \to \pi_2^*(\mathbf{F}', j')$$

in $\mathbf{SCH} \downarrow (\mathbf{Y} \times_{\mathbf{X}} \mathbf{Y})$ (and hence in $\mathbb{M}(\mathbf{Y} \times_{\mathbf{X}} \mathbf{Y})$) arising from

$$\begin{aligned}\pi_1^*(\mathbf{F}', j') &= \pi_1^*(g^*(\mathbf{F}, j)) \simeq (g\pi_1)^*(\mathbf{F}, j) \\ &= (g\pi_2)^*(\mathbf{F}, j) \simeq \pi_2^*(g^*(\mathbf{F}, j)) = \pi_2^*(\mathbf{F}', j').\end{aligned}$$

Since every closed immersion is a monomorphism in $\mathbf{SCH}$ (see [11]), Proposition 2.1 gives that $((\mathbf{F}', j'), \theta) \in \mathfrak{Des}_{\mathbb{M}}(p)$. Then by our assumption on p, there exists a closed immersion $k : \mathbf{G} \to \mathbf{X}$ such that

$$j' \simeq \mathbf{Y} \times_{\mathbf{X}} k.$$

Since k is a closed immersion in $\mathbf{SCH}$, the map $k_0 : \mathrm{sp}(\mathbf{G}) \to \mathrm{sp}(\mathbf{X})$ of topological spaces is injective, and hence the inverse image $p_0^{-1}(k_0(\mathrm{sp}(\mathbf{G})))$ of the topological space $k_0(\mathrm{sp}(\mathbf{G}))$ is (isomorphic to) $\mathrm{sp}(\mathbf{F}')$. It is easy to see that $h(i_0^{-1}(k_0(\mathrm{sp}(\mathbf{G})))) \subseteq F$, and hence $i_0^{-1}(k_0(\mathrm{sp}(\mathbf{G}))) \subseteq h^{-1}(F)$. Moreover, we have

$$\begin{aligned}(\overline{p}_0)^{-1}(i_0^{-1}(k_0(\mathrm{sp}(\mathbf{G})))) &= p_0^{-1}(k_0(\mathrm{sp}(\mathbf{G})) \\ = \mathrm{sp}(\mathbf{F}') = g_0^{-1}(F) &= (h\overline{p}_0)^{-1}(F) = (\overline{p}_0)^{-1}(h^{-1}(F)).\end{aligned}$$

Since $\overline{p}_0 : \mathrm{sp}(\mathbf{Y}) \to p_0(\mathrm{sp}(\mathbf{Y}))$ is surjective in $\mathbf{Top}$, one has that $i_0^{-1}(k_0(\mathrm{sp}(\mathbf{G}))) = h^{-1}(F)$. But, since k is a closed immersion in $\mathbf{SCH}$, $k_0(\mathrm{sp}(\mathbf{G}))$ is a closed subset of $\mathrm{sp}(\mathbf{X})$. It follows that $i_0^{-1}(k_0(\mathrm{sp}(\mathbf{G})))$ (and hence $h^{-1}(F)$) is a closed subset of $p_0(\mathrm{sp}(\mathbf{Y}))$. Since F was an arbitrary closed subset of $\mathrm{sp}(\mathbf{Z})$, it proves that h is a continuous map of topological spaces. This completes the proof. □

Corollary 5.2. *In the situation of the previous proposition, suppose that p_0 is a surjective map of topological spaces, then for any morphism $g : \mathbf{Y} \to \boldsymbol{Z}$ of schemes, there is a unique map $h : \mathrm{sp}(\mathbf{X}) \to \mathrm{sp}(\boldsymbol{Z})$ in **Top** such that $g_0 = hp_0$.*

The following simple observation is well known:

Proposition 5.3. *Pure morphisms descend surjections; that is, if $\psi : R \to S$ is a pure morphism of commutative rings, and if in the pushout diagram*

$$\begin{array}{ccc} R & \xrightarrow{\psi} & S \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ A & \longrightarrow & S \otimes_R A \end{array}$$

f' is surjective, then so is f.

Since the opposite category of commutative rings is equivalent to the category of affine schemes, it follows from the above that our next statement is true.

Proposition 5.4. *Let $p : \mathbf{Y} \to \mathbf{X}$ be a pure morphism of affine schemes, and let*

$$\begin{array}{ccc} \mathbf{Y}' & \xrightarrow{p'} & \mathbf{X}' \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j} \\ \mathbf{Y} & \xrightarrow{p} & \mathbf{X} \end{array}$$

be a pullback in **SCH**. *If j' is a closed immersion, then so is j.*

Now Theorem 2.2 gives

Theorem 5.5. *Every pure morphism of affine schemes is an effective $\mathbb{M}$-descent morphism.*

Proposition 5.6. *Let $\psi : R \to S$ be a pure morphism of commutative rings, and $\psi^a : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ the corresponding morphism of affine schemes. Then $(\psi^a)_0$ is a surjection of topological spaces.*

Proof. For any given $\wp \in \mathrm{sp}(\mathrm{Spec}(R))$, let $R(\wp)$ denote the field of fractions of $R/\wp$. Since ψ is pure, the morphism $R(\wp) \to R(\wp) \otimes_R S$ is a monomorphism. Now $R(\wp) \neq 0$ follows that $R(\wp) \otimes_R S \neq 0$; that is, $\mathrm{sp}(\mathrm{Spec}(R(\wp) \otimes_R S)) \neq \varnothing$. But $((\psi^a)_0)^{-1}(\wp) = \mathrm{sp}(\mathrm{Spec}(R(\wp) \otimes_R S))$ so that $((\psi^a)_0)^{-1}(\wp) \neq \varnothing$ and this just means that ψ^a_0 is surjective. □

Theorem 5.7. *Let $\psi : R \to S$ be a pure morphism of commutative rings. Then the diagram*

$$\mathrm{Spec}(S \otimes_R S) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \mathrm{Spec}(S) \xrightarrow{\psi^a} \mathrm{Spec}(R)$$

is a coequalizer diagram in **SCH**.

Proof. We have to show that for any scheme $\mathbf{X}$ and any morphism $g : \mathrm{Spec}(S) \to \mathbf{X}$ with $g\pi_1 = g\pi_2$, there exists a unique morphism $h : \mathrm{Spec}(R) \to \mathbf{X}$ such that $g = h\psi^a$.

When $\mathbf{X}$ is an affine scheme, the result follows from the exactness of

$$R \xrightarrow{\psi} S \underset{i_2}{\overset{i_1}{\rightrightarrows}} S \otimes_R S$$

in the category of commutative rings (here $i_1(s) = 1 \otimes s$ and $i_2(s) = s \otimes 1$).

Consider now the case where $\mathbf{X}$ is an arbitrary scheme. Since $(\psi^a)_0$ is surjective (see Proposition 5.6) and since for any $r \in R$, the morphism $\psi_r : R_r \to S_{\psi(r)}$ of commutative rings is pure, it follows, as in the proof of Theorem 2.17 of [10], that such h is uniquely determined whenever it exists.

In view of the uniqueness of h, it is enough to define it locally.

By Corollary 5.2, there exists a map $h_0 : \mathrm{sp}(\mathrm{Spec}(R)) \to \mathrm{sp}(\mathbf{X})$ of topological spaces with $h_0 \cdot (\psi^a)_0 = g_0$.

Let $z \in \mathrm{sp}(\mathrm{Spec}(R))$ and $z' \in \mathrm{sp}(\mathrm{Spec}(S))$ with $(\psi^a)_0(z') = z$. Choose an affine open neighborhood U of $g_0(z') \in \mathrm{sp}(\mathbf{X})$. Then, since $h_0 \cdot (\psi^a)_0 = g_0$,

$h_0^{-1}(U) \subseteq \mathrm{sp}(\mathrm{Spec}(R))$ is an open neighborhood of z, we can find an element $r \in R$ such that $z \in \mathrm{sp}(\mathrm{Spec}(R_r)) \subseteq h_0^{-1}(U)$. Then

$$\mathrm{sp}(\mathrm{Spec}(S_{\psi(r)})) = (\psi^a)_0^{-1}(\mathrm{sp}(\mathrm{Spec}(R_r))) \subseteq (\psi_0^a)^{-1}(h_0^{-1}(U)) = g_0^{-1}(U).$$

We then have a diagram

$$\begin{array}{ccccc} \mathrm{Spec}(S_{\psi(r)} \otimes_{R_r} S_{\psi(r)}) & \overset{\pi_1'}{\underset{\pi_2'}{\rightrightarrows}} & \mathrm{Spec}(S_{\psi(r)}) & \xrightarrow{(\psi_r)^a} & \mathrm{Spec}(R) \\ & & & \searrow^{g'} & \\ & & & & \mathbf{U}, \end{array}$$

where $g' : \mathrm{Spec}(S_{\psi(r)}) \to \mathbf{U}$ is the restriction of $g : \mathrm{Spec}(S) \to \mathbf{X}$ and where $g'\pi_1' = g'\pi_2'$. We have seen that $(\psi_r)^a$ is a pure morphism of schemes, hence the problem is reduced to the case where $\mathbf{X}$ is an affine scheme. □

Corollary 5.8. *Let $p : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ be a pure morphism of affine schemes. Then p is a regular epimorphism in* **SCH**.

Theorem 5.9. *Let $p : \mathbf{Y} \to \mathbf{X}$ be a quasi-compact pure morphism of schemes. Then the diagram*

$$\mathbf{Y} \times_{\mathbf{X}} \mathbf{Y} \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} \mathbf{Y} \xrightarrow{p} \mathbf{X}$$

is a coequalizer diagram in **SCH**.

Proof. Without loss of generality, one may assume that $\mathbf{X}$ is affine. Since p is quasi-compact, there is an affine scheme $\mathbf{Y}'$ and a stable regular epimorphism $e : \mathbf{Y}' \to \mathbf{Y}$ such that the composite

$$\mathbf{Y}' \xrightarrow{e} \mathbf{Y} \xrightarrow{p} \mathbf{X}$$

is a pure morphism of affine schemes (see the proof of Proposition 3.2). Then pe is a regular epimorphism according to Corollary 5.8. It follows that p is a regular epimorphism [4]. □

Since the class of quasi-compact pure morphisms is stable under pullback, we have

Corollary 5.10. *Any quasi-compact pure morphism of schemes is a stable regular epimorphism.*

Since any stable regular epimorphism in **SCH** is pure [11], from the previous corollary we get

Theorem 5.11. *A quasi-compact morphism of schemes is pure if and only if it is a stable regular epimorphism.*

From Corollary 2.4 of [7] we obtain immediately

Theorem 5.12. *A quasi-compact pure morphism of schemes is a descent morphism with respect to the* **SCH***-indexed category given by*

$$\mathbf{X} \to \mathbf{SCH} \downarrow \mathbf{X}.$$

6. Stable Schematically Dominant Morphisms

Let us recall [6] that a morphism $p : \mathbf{Y} \to \mathbf{X}$ of schemes is a pair (p_0, p_1), where $p_0 : \mathrm{sp}(\mathbf{Y}) \to \mathrm{sp}(\mathbf{X})$ is a continuous map and $p_1 : \mathcal{O}_{\mathbf{X}} \to (p_0)_*(\mathcal{O}_{\mathbf{Y}})$ is a morphism of sheaves of commutative rings on $\mathrm{sp}(\mathbf{X})$.

According to Grothendieck and Dieudonné [5], a morphism $p : \mathbf{Y} \to \mathbf{X}$ of schemes is called *schematically dominant* if the canonical morphism

$$p_1 : \mathcal{O}_{\mathbf{X}} \to (p_0)_*(\mathcal{O}_{\mathbf{Y}})$$

is injective. A schematically dominant morphism is stable if it remains schematically dominant under pullback along any morphism.

Proposition 6.1. *If a composite qp of morphisms in* **SCH** *is schematically dominant, then so is q; in other words, the class of schematically dominant morphisms has the strong right cancellation property.*

Proof. Let $p : \mathbf{Y} \to \mathbf{X}$, $q : \mathbf{X} \to \mathbf{Z}$ be morphisms in **SCH** such that the composite pq is schematically dominant. Then the canonical morphism

$$(qp)_1 : \mathcal{O}_{\mathbf{Z}} \to ((qp)_0)_*(\mathcal{O}_{\mathbf{Y}})$$

is a monomorphism. But, by definition, the morphism $(qp)_1$ is, to within an isomorphism, the composite

$$\mathcal{O}_{\mathbf{Z}} \xrightarrow{q_1} (q_0)_*(\mathcal{O}_{\mathbf{X}}) \xrightarrow{(q_0)_*(p_1)} (q_0)_*(e_{0*}(\mathcal{O}_{\mathbf{Y}})),$$

and it follows that $q_1 : \mathcal{O}_{\mathbf{Z}} \to (q_0)_*(\mathcal{O}_{\mathbf{X}})$ is a monomorphism, i.e. q is schematically dominant. □

A corollary follows immediately:

Corollary 6.2. *The class of stable schematically dominant morphisms has the strong right cancellation property.*

Proposition 6.3. *A schematically dominant closed immersion is an isomorphism.*

Proof. Since the question is local, we may assume that the codomain of a given schematically dominant closed immersion of schemes is affine. Then, since any closed subscheme of an affine scheme is affine and arises from an ideal [6], the proposition follows from the purely algebraic fact that if $\mathcal{I}$ is an ideal of a commutative ring R such that the canonical surjection $R \to R/\mathcal{I}$ is an isomorphism, then $\mathcal{I} = 0$. □

Proposition 6.4. *A stable schematically dominant morphism of schemes is pure. If, in addition, the morphism is quasi-compact, the converse also holds.*

Proof. Let a stable schematically dominant morphism $p : \mathbf{Y} \to \mathbf{X}$ have the $(\mathbb{E}, \mathbb{M})$-factorization $p = me$. Then, since $me = p$ is schematically dominant and since m is a closed immersion, it follows from Proposition 6.3 that m is an isomorphism. Hence $p \in \mathbb{E}$ by Proposition 3.2 in [11].

It is now easy to see that any stable dominant morphism lies in $\mathbb{E}'$, which means that any stable dominant morphism is pure.

Suppose now that $p : \mathbf{Y} \to \mathbf{X}$ is a quasi-compact pure morphism of schemes. We are to show that p is a stable schematically dominant morphism; but this is a purely local problem and we may thus assume that $\mathbf{X}$ is affine, say, $\mathbf{X} = \mathrm{Spec}(R)$.

Since p is quasi-compact and pure, it follows, as in the proof of Proposition 3.2, that there exist an affine scheme $\mathrm{Spec}(S)$ and a morphism $e : \mathrm{Spec}(S) \to \mathbf{Y}$ of schemes such that the composite $pe : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a pure morphism of affine schemes. But by Proposition 3.9 in [11], a morphism of affine schemes is pure if and only if the corresponding morphism of commutative rings is pure. Therefore the morphism $R \to S$ of commutative rings corresponding to pe is pure; in particular, it is a monomorphism. Note that this morphism is nothing but the global section of the canonical morphism $(pe)_1 : \widetilde{R} \to ((pe)_0)_*(\widetilde{S})$; hence the morphism pe is schematically dominant, and then so is p by Proposition 6.1. □

Now Theorem 5.11 and Proposition 6.4 give

Theorem 6.5. *A quasi-compact morphism of schemes is stable schematically dominant if and only if it is pure if and only if it is a stable regular epimorphism.*

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