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On comonadicity of the extension-of-scalars functors

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Abstract

Building on a categorical approach of G. Janelidze and W. Tholen to descent theory for modules, we show how this theory can be presented at the level of enriched categories. Specializing to the monoidal category of bimodules over a separable algebra this gives a criterion for comonadicity of the extension-of-scalars functor associated to an extension of (not necessarily commutative) rings. As an application of this criterion, some known results on the comonadicity of such functors are obtained.

Keywords: Noncommutative descent; Extension-of-scalars functor; Comonadicity

1. Introduction

Grothendieck's faithfully flat descent theorem gives us, for any faithfully flat extension $i : A \to B$ of commutative rings, a very pleasant description of A-modules in terms of B-modules together with some additional structure, called *descent data*. To be more specific, recall that a descent datum on a B-module M is a $B \otimes_A B$-module morphism $\theta_M : M \otimes_A B \to B \otimes_A M$ for which

$$\begin{array}{ccc} M \otimes_A B & \xrightarrow{\theta_M} & B \otimes_A M \\ {\scriptstyle \tau_{M,B}}\downarrow & & \downarrow{\scriptstyle \alpha_M} \\ B \otimes_A M & \xrightarrow[\alpha_M]{} & M \end{array}$$

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and

$$\begin{array}{ccc} & M\otimes_A B\otimes_A B & \\ {\scriptstyle M\otimes_A\sigma_{B,B}}\swarrow & & \searrow{\scriptstyle \theta_M\otimes_A B}\\ M\otimes_A B\otimes_A B & & B\otimes_A M\otimes_A B\\ {\scriptstyle \theta_M\otimes_A B}\downarrow & & \downarrow{\scriptstyle B\otimes_A\theta_M}\\ B\otimes_A M\otimes_A B & \xrightarrow[B\otimes_A\sigma_{M,B}]{} & B\otimes_A B\otimes_A M\end{array}$$

commute, where σ with two subscripts denotes the symmetry isomorphism on the tensor product of the subscripts. Write **Des**(i) for the category whose objects are pairs (M, θ_M), where M is a B-module and θ_M is a descent datum on M, and write $K : {}_A\text{Mod} \to \mathbf{Des}(\mathrm{i})$ for the functor that takes $N \in {}_A\text{Mod}$ to the pair $(B \otimes_A N, B \otimes_A \sigma_{N,B})$. $i : A \to B$ is called an *effective descent morphism for modules* if the functor K is an equivalence of categories. Grothendieck's theorem asserts that faithfully flat extensions of commutative rings are effective.

Grothendieck's theorem has various extensions and generalizations. For instance, a complete characterization of effective descent morphisms of commutative rings was given by A. Joyal and M. Tierney (unpublished, but see [13]). M. Cipolla [8] made a first step in extending Grothendieck's result to noncommutative rings. This was further investigated in [15]. In [3] T. Brzeziński pointed out that the category of noncommutative descent data in the sense of Cipolla is (isomorphic) to the category of comodules over the Sweedler canonical coring associated to the ring extension $i : A \to B$. This idea was further developed in [6,7]. A more general situation, where the ring extension $i : A \to B$ is replaced by an A–B-bimodule M with M_B finitely generated and projective, was extensively studied in [9], wherein a generalized faithfully flat descent theorem for modules is proved. This theory was generalized in [7].

In [10], G. Janelidze and W. Tholen showed how the theory of comonads provides a purely categorical approach to descent theory for modules. It should be pointed out that their approach is quite different from the one taken by all the authors above and that [10] is the only paper that explains exactly how comonadicity is used in descent theory for modules. An important contribution of [10] is also the fact that it separates purely categorical arguments from module-theoretical ones. The approach of G. Janelidze and W. Tholen is based on the translation of descent data into coalgebras w.r.t. the comonad associated to a ring extension (note that the comonadic approach to descent is also used in [12]). According to this translation (which is a special case of a much more general result due to J. Bénabou and J. Roubaud [2], and Beck (unpublished), that asserts that when the so-called Beck–Chevalley condition is satisfied, descent reduces to monadicity), $i : A \to B$ is an effective descent morphism for modules iff the corresponding extension-of-scalars functor $B \otimes_A - : {}_A\text{Mod} \to {}_B\text{Mod}$ is comonadic. In particular, Grothendieck's theorem is a very special case of (the dual of) Beck's theorem on monadic functors [1]. Thus, comonadicity of extension-of-scalars functors plays an important role in descent theory for modules. In view of this fact, it becomes even more sensible to have manageable tests for comonadicity of the extension-of-scalars functors. Although there are several results obtained along these lines (see [4–10,12,15]) the question of comonadicity of such functors is not fully answered yet. The purpose of this note is to give such a test. Our contribution to this matter is to give a necessary and sufficient condition—based on our attempt to repeat Janelidze–Tholen's comonadic approach to descent theory for modules, but now at the level of enriched categories—for comonadicity of the extension-of-scalars functors associated to certain extensions of noncommutative rings $i : A \to B$ with A a separable algebra over a commutative ring.

For the basic definitions of category theory, see [11].

2. Preliminaries

A monoidal category $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ is called *biclosed* if for all $X \in \mathrm{Ob}(\mathcal{V}_0)$, the functors

$$- \otimes X, X \otimes - : \mathcal{V}_0 \to \mathcal{V}_0$$

have (chosen) right adjoints, denoted $[X, -]$ and $\{X, -\}$, respectively. In other words, a biclosed monoidal category consists of a monoidal category $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$, equipped with two functors

$$[-,-], \{-,-\} : \mathcal{V}_0^{\mathrm{op}} \times \mathcal{V}_0 \to \mathcal{V}_0,$$

for which there are natural isomorphisms

$$\mathcal{V}_0\big(X, [Y, Z]\big) \simeq \mathcal{V}_0(X \otimes Y, Z) \simeq \mathcal{V}_0\big(Y, \{X, Z\}\big). \tag{2.1}$$

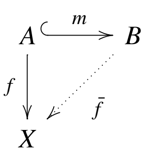
Recall that the adjunctions $X \otimes - \dashv \{X, -\}$ and $- \otimes X \dashv [X, -]$ are internal, in the sense that one has natural isomorphisms

$$\{X \otimes Y, Z\} \simeq \big\{Y, \{X, Z\}\big\} \tag{2.2}$$

and

$$[X \otimes Y, Z] \simeq \big[X, [Y, Z]\big]. \tag{2.3}$$

Let us recall that a morphism in a category $\mathcal{A}$ is a *regular monomorphism* if it is an equalizer of some pair of morphisms. Recall also that a *regular injective* object in $\mathcal{A}$ is an object $X \in \mathcal{A}$ which has the extension property with respect to regular monomorphisms; that is, if every extension problem



with m a regular monomorphism has a solution $\bar{f} : B \to X$ extending f along m, i.e., satisfying $\bar{f}m = f$. (The dual notions are the *regular epimorphism* and the *projective object*.)

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ be a monoidal category and let $f : X \to Y$ be a morphism in $\mathcal{V}_0$. We say that f is *right* (respectively *left*) *pure* if, for any $Z \in \mathrm{Ob}(\mathcal{V}_0)$, the morphism

$$f \otimes Z : X \otimes Z \to Y \otimes Z$$

$$(\text{respectively } Z \otimes f : Z \otimes X \to Z \otimes Y)$$

is a regular monomorphism.

Henceforth, we suppose without explicit mention that $\mathcal{V}$ is a finitely complete and finitely cocomplete monoidal biclosed category whose unit I for the tensor product is projective.

Theorem 2.1. *Let Q be an object of $\mathcal{V}_0$ for which the functor*

$$\{-, Q\} : \mathcal{V}_0^{\mathrm{op}} \to \mathcal{V}_0$$

is conservative (*that is, isomorphism-reflecting*) *and preserves regular epimorphisms. Then the following properties of a morphism $f : X \to Y$ of $\mathcal{V}_0$ are equivalent*:

(i) *The morphism f is right pure.*
(ii) *The morphism $f \otimes \{X, Q\} : X \otimes \{X, Q\} \to Y \otimes \{X, Q\}$ is a regular monomorphism.*
(iii) *The morphism $\{f, Q\} : \{Y, Q\} \to \{X, Q\}$ is a split epimorphism.*

Proof. (i) implies (ii) trivially. To see that (ii) implies (iii), let us assume that the morphism

$$f \otimes \{X, Q\} : X \otimes \{X, Q\} \to Y \otimes \{X, Q\}$$

is a regular monomorphism. Since the functor $\{-, Q\} : \mathcal{V}_0^{\mathrm{op}} \to \mathcal{V}_0$ preserves regular epimorphisms by hypothesis, the morphism

$$\big\{\{X, Q\}, \{f, Q\}\big\} : \big\{\{X, Q\}, \{Y, Q\}\big\} \to \big\{\{X, Q\}, \{X, Q\}\big\},$$

which is isomorphic by (2.2) to the morphism

$$\big\{f \otimes \{X, Q\}, Q\big\} : \big\{Y \otimes \{X, Q\}, Q\big\} \to \big\{X \otimes \{X, Q\}, Q\big\},$$

is a regular epimorphism in $\mathcal{V}_0$. Since I is assumed to be projective in $\mathcal{V}_0$, the functor

$$\mathcal{V}_0(I, -) : \mathcal{V}_0 \to \mathbf{Set}$$

takes regular epimorphisms to surjections. It follows that the map

$$\mathcal{V}_0\big(\{X, Q\}, \{f, Q\}\big)$$

of sets, which (using (2.1)) is isomorphic to the map

$$\mathcal{V}_0\big(I, \big\{\{X, Q\}, \{f, Q\}\big\}\big)$$

is surjective. But this means that every morphism

$$\{X, Q\} \to \{X, Q\}$$

factors through $\{f, Q\}$, that is to say, that $\{f, Q\}$ is a split epimorphism, as is seen from the special case of the identity morphism $1_{\{X,Q\}}$.

It remains to show that (iii) implies (i). If the morphism

$$\{f, Q\} : \{Y, Q\} \to \{X, Q\}$$

is a split epimorphism, then so is

$$\{Z, \{f, Q\}\} : \{Z, \{Y, Q\}\} \to \{Z, \{X, Q\}\}$$

too, for all $Z \in \mathrm{Ob}(\mathcal{V}_0)$. Identifying the morphism $\{Z, \{f, Q\}\}$ (via the isomorphism (2.2)) with $\{f \otimes Z, Q\}$, we see that the morphism

$$\{f \otimes Z, Q\} : \{Y \otimes Z, Q\} \to \{X \otimes Z, Q\}$$

is also a split epimorphism. We now observe that, since the functor $\{-, Q\} : \mathcal{V}_0^{\mathrm{op}} \to \mathcal{V}_0$ admits as a left adjoint the functor $[-, Q] : \mathcal{V}_0 \to \mathcal{V}_0^{\mathrm{op}}$, as can be seen from the following sequence of natural isomorphisms:

$$\mathcal{V}_0\big(X, \{Y, Q\}\big) \simeq \mathcal{V}_0(Y \otimes X, Q) \simeq \mathcal{V}_0\big(Y, [X, Q]\big) \simeq \mathcal{V}_0^{\mathrm{op}}\big([X, Q], Y\big),$$

to say that $\{-, Q\}$ is conservative and preserves regular epimorphisms is to say that it preserves and reflects regular epimorphisms. And since any split epimorphism is regular, it follows that the morphism $f \otimes Z : X \otimes Z \to Y \otimes Z$ is a regular monomorphism for all $Z \in \mathrm{Ob}(\mathcal{V}_0)$. Thus (iii) implies (i). The proof of the theorem is now complete. □

There is of course a dual result:

Theorem 2.2. *Let Q be an object of $\mathcal{V}_0$ such that the functor*

$$[-, Q] : \mathcal{V}_0^{\mathrm{op}} \to \mathcal{V}_0$$

is conservative and preserves regular epimorphisms. Then the following properties of a morphism $f : X \to Y$ of $\mathcal{V}_0$ are equivalent:

(i) *The morphism f is left pure.*
(ii) *The morphism $[X, Q] \otimes f : [X, Q] \otimes X \to [X, Q] \otimes Y$ is a regular monomorphism.*
(iii) *The morphism $[f, Q] : [Y, Q] \to [X, Q]$ is a split epimorphism.*

An object Q of a monoidal biclosed category

$$\mathcal{V} = \big(\mathcal{V}_0, \otimes, I, [-,-], \{-,-\}\big)$$

is said to be *cyclic* if the functors $\{-, Q\}$ and $[-, Q]$ are naturally isomorphic. If Q is such an object, we shall denote by $[\![-, Q]\!]$ the functor $[-, Q] \simeq \{-, Q\}$.

Combining Theorems 2.1 and 2.2, we get:

Theorem 2.3. *Let Q be a cyclic object of $\mathcal{V}_0$ for which the functor*

$$[\![-, Q]\!] : \mathcal{V}_0^{\mathrm{op}} \to \mathcal{V}_0$$

is conservative and preserves regular epimorphisms (*equivalently, preserves and reflects regular epimorphisms*). *Then the following properties of a morphism $f : X \to Y$ of $\mathcal{V}_0$ are equivalent*:

(i) *The morphism f is left pure.*
(ii) *The morphism f is right pure.*
(iii) *The morphism $[\![X, Q]\!] \otimes f : [\![X, Q]\!] \otimes X \to [\![X, Q]\!] \otimes Y$ is a regular monomorphism.*
(iv) *The morphism $f \otimes [\![X, Q]\!] : X \otimes [\![X, Q]\!] \to Y \otimes [\![X, Q]\!]$ is a regular monomorphism.*
(v) *The morphism $[\![f, Q]\!] : [\![Y, Q]\!] \to [\![X, Q]\!]$ is a split epimorphism.*

3. A criterion for comonadicity of extension-of-scalars functors

In this section we present our main result.

Let us fix a commutative ring K with unit ($K = \mathbb{Z}$, the ring of integers, inclusive). All rings under consideration are associative unital K-algebras. A right or left module means a unital module. All bimodules are assumed to be K-symmetric. The K-categories of left and right modules over a ring A are denoted by ${}_A\mathrm{Mod}$ and Mod_A, respectively; while the category of (A, B)-bimodules is ${}_A\mathrm{Mod}_B$. We will use the notation ${}_BM_A$ to indicate that M is a left B, right A-module.

It is a well-known fact that, for a fixed ring A, the category ${}_A\mathrm{Mod}_A$ is a monoidal category with tensor product of two (A, A)-bimodules being their usual tensor product over A and the unit for this tensor product being the (A, A)-bimodule A. Moreover, this monoidal category is biclosed: If M and N are two (A, A)-bimodules, then $[M, N] = \mathrm{Mod}_A(M, N)$ and $\{M, N\} = {}_A\mathrm{Mod}(M, N)$.

For any (A, A)-bimodule M, the *character* (A, A)*-bimodule* of M is defined to be $M^+ = \mathbf{Ab}(M, \mathbb{Q}/\mathbb{Z})$ (where $\mathbf{Ab}$ is the category of abelian groups and $\mathbb{Q}/\mathbb{Z}$ is the rational circle abelian group). This is an (A, A)-bimodule via the actions $(afa')(m) = f(a'ma)$.

Lemma 3.1. *The character bimodule A^+ of the (A, A)-bimodule A is a cyclic object of the monoidal biclosed category ${}_A\mathrm{Mod}_A$ of (A, A)-bimodules.*

Proof. The following string of natural isomorphisms

$$\begin{aligned}\{-, A^+\} &= \{-, \mathbf{Ab}(A, \mathbb{Q}/\mathbb{Z})\} = {}_A\mathrm{Mod}(-, \mathbf{Ab}(A, \mathbb{Q}/\mathbb{Z}))\\ &\simeq \mathbf{Ab}(A \otimes_A -, \mathbb{Q}/\mathbb{Z}) \simeq \mathbf{Ab}(-, \mathbb{Q}/\mathbb{Z}) \simeq \mathbf{Ab}(- \otimes_A A, \mathbb{Q}/\mathbb{Z})\\ &\simeq \mathrm{Mod}_A(-, \mathbf{Ab}(A, \mathbb{Q}/\mathbb{Z})) = [-, A^+]\end{aligned}$$

shows that the functors

$$\{-, A^+\}, [-, A^+] : ({}_A\mathrm{Mod}_A)^{\mathrm{op}} \to {}_A\mathrm{Mod}_A$$

are naturally equivalent. □

Since the functor $[\![-, A^+]\!]$ is naturally equivalent to $\mathbf{Ab}(-, \mathbb{Q}/\mathbb{Z})$ and since $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator in $\mathbf{Ab}$, we have that

Lemma 3.2. *The functor*

$$[\![-, A^+]\!] : ({}_A\mathrm{Mod}_A)^{\mathrm{op}} \to {}_A\mathrm{Mod}_A$$

is exact and conservative.

Before we prove our main result we state the following slight improvement of Lemma 5.2 in [10]:

Theorem 3.3. *Let $i : A \to B$ be a homomorphism of rings. If the induced morphism $i^+ : B^+ \to A^+$ is a split epimorphism of (A, A)-bimodules, then the functors*

$$- \otimes_A B : \mathrm{Mod}_A \to \mathrm{Mod}_B$$

and

$$B \otimes_A - : {}_A\mathrm{Mod} \to {}_B\mathrm{Mod}$$

are both comonadic.[1]

Recall (for example, from [14]) that a morphism $f : M \to N$ of right A-modules is called *pure* if $f \otimes_A 1_L : M \otimes_A L \to N \otimes_A L$ is injective for every left A-module L. Pure morphisms in the category of left A-modules are defined analogously.

The main result of this note is contained in the following:

Theorem 3.4. *Let $i : A \to B$ be a homomorphism of rings. If A is a separable K-algebra, then the following are equivalent*:

(i) *i is a pure morphism of left A-modules.*
(ii) *i is a pure morphism of right A-modules.*
(iii) *$i^+ : B^+ \to A^+$ is a split epimorphism of (A, A)-bimodules.*
(iv) *The functor $- \otimes_A B : \mathrm{Mod}_A \to \mathrm{Mod}_B$ is comonadic.*
(v) *The functor $B \otimes_A - : {}_A\mathrm{Mod} \to {}_B\mathrm{Mod}$ is comonadic.*

Proof. We remark first that, by left–right symmetry, it suffices to prove the equivalence of (i), (iii) and (v).

Suppose that $i : A \to B$ is a pure morphism of left A-modules. Then, for any $X \in {}_A\mathrm{Mod}_A$, the morphism $X \otimes_A i : X \otimes_A A \to X \otimes_A B$ is a monomorphism in **Ab**; and since the forgetful functor ${}_A\mathrm{Mod}_A \to \mathbf{Ab}$ reflects monomorphisms, the morphism $X \otimes_A i$, seen as a morphism in ${}_A\mathrm{Mod}_A$, is a monomorphism. Thus, if i is a pure morphism of left A-modules, then it is left pure in ${}_A\mathrm{Mod}_A$. And since assuming A be K-separable is, just by definition, the same as assuming A be projective in ${}_A\mathrm{Mod}_A$, it follows from Theorem 2.3 that the morphism $[\![i, A^+]\!] \simeq i^+$ is a split epimorphism of (A, A)-bimodules, provided that i is a pure morphism of left A-modules. Thus (i) implies (iii).

(iii) implies (v) by Theorem 3.3.

It is well known that the functor

$$- \otimes_A B : \mathrm{Mod}_A \to \mathrm{Mod}_B$$

admits as a right adjoint the functor

$$\mathrm{Mod}_B(B, -) : \mathrm{Mod}_B \to \mathrm{Mod}_A$$

[1] As is shown in [12], the conditions that the functors $- \otimes_A B$ and $B \otimes_A -$ are both conservative can be omitted.

and that the unit η of this adjunction has components

$$\eta_X : X \otimes_A i : X \simeq X \otimes_A A \to X \otimes_A B, \quad X \in \mathrm{Mod}_A.$$

Thus i is a pure morphism of left A-modules precisely when η is componentwise a monomorphism. According to Theorem 9 of Section 2.3 of [1], this is, in particular, the case when the functor $- \otimes_A B$ is comonadic. So (v) implies (i). This completes the proof of the theorem. □

Note that, since any algebra of matrices over K is separable, Theorem 3.4 immediately applies to any ring extension having for its domain an algebra of matrices over K.

4. Some particular cases

In this section we state some consequences of our main theorem. To state the first one, we need a definition. Let A, B be rings. Recall [7] that an (A, B)-bimodule M is said to be *totally faithful* as a left A-module if the morphism

$$X \to \mathrm{Mod}_B(M, X \otimes_A M), \qquad m \to x \otimes_A m,$$

is injective for every $X \in \mathrm{Mod}_A$, or equivalently, if the unit of the adjunction

$$- \otimes_A M \dashv \mathrm{Mod}_B(M, -) : \mathrm{Mod}_B \to \mathrm{Mod}_A$$

is pointwise a monomorphism.

Theorem 4.1. *Let A and B be rings, M an (A, B)-bimodule with M_B finitely generated and projective, $\mathcal{E}_M = \mathrm{Mod}_B(M, M)$ the right endomorphism ring of M_B and*

$$i_M : A \to \mathcal{E}_M, \qquad a \to [m \to am]$$

the corresponding ring homomorphism. If A is K-separable, then the following are equivalent:

(i) *The bimodule ${}_AM_B$ is totally faithful as a left A-module.*
(ii) *The bimodule ${}_BM^*_A$ is totally faithful as a right A-module. (Here we denote by M^* the dual $\mathrm{Mod}_B(M, B)$ of M_B which is a (B, A)-bimodule in a canonical way.)*
(iii) *The morphism $(i_M)^+ : (\mathcal{E}_M)^+ \to A^+$ is a split epimorphism of (A, A)-bimodules.*
(iv) *The functor $- \otimes_A M : \mathrm{Mod}_A \to \mathrm{Mod}_B$ is comonadic.*
(iv) *The functor $M^* \otimes_A - : {}_A\mathrm{Mod} \to {}_B\mathrm{Mod}$ is comonadic.*

Proof. Immediate from Theorem 3.4 using that:

- ${}_AM_B$ (respectively ${}_BM^*_A$) is totally faithful as a left (respectively a right) A-module if and only if $i_M : A \to \mathcal{E}_M$ is a pure morphism of left (respectively right) A-modules (see [7, Lemma 2.2] or [12, Proposition 7.3]);
- the functor $- \otimes_A M : \mathrm{Mod}_A \to \mathrm{Mod}_B$ (respectively $M^* \otimes_A - : {}_A\mathrm{Mod} \to {}_B\mathrm{Mod}$) is comonadic if and only if the functor $- \otimes_A \mathcal{E}_M : \mathrm{Mod}_A \to \mathrm{Mod}_{\mathcal{E}_M}$ (respectively $\mathcal{E}_M \otimes_A - : {}_A\mathrm{Mod} \to {}_{\mathcal{E}_M}\mathrm{Mod}$) is so (see [12, Theorem 7.5]). □

As a special case of Theorem 4.1 one can take $A = K$. Then, since obviously K is K-separable, we recover a result by Caenepeel, De Groot and Vercruysse [7].

Theorem 4.2. (Caenepeel, De Groot and Vercruysse [7]) *Let A be a ring and let M be a (K, A)-bimodule with M_A finitely generated and projective. Then the following are equivalent*:

(i) *The morphism $i_M : K \to \mathcal{E}_M = \mathrm{Mod}_A(M, M)$ is a pure morphism of left K-modules.*
(ii) *The morphism $i_M : K \to \mathcal{E}_M = \mathrm{Mod}_A(M, M)$ is a pure morphism of right K-modules.*
(iii) *The bimodule ${}_K M_A$ is totally faithful as a left K-module.*
(iv) *The bimodule ${}_A M^*_K$ is totally faithful as a right K-module.*
(v) *The functor $- \otimes_K M : \mathrm{Mod}_K \to \mathrm{Mod}_A$ is comonadic.*
(vi) *The functor $M^* \otimes_K - : {}_K\mathrm{Mod} \to {}_A\mathrm{Mod}$ is comonadic.*

For the special case in which $M = A$, we recapture easily the following result of Joyal and Tierney (unpublished, but see [13]). Recall (for example, from [10]) that a homomorphism $i : K \to A$ of commutative rings is said to be *effective for descent* if the extension-of-scalars functor

$$A \otimes_K - : \mathrm{Mod}_K \to \mathrm{Mod}_A$$

is comonadic.

Theorem 4.3. (Joyal and Tierney) *A homomorphism $i : K \to A$ of commutative rings is effective for descent if and only if it is a pure morphism of* (*say left*) *K-modules.*

References

[1] M. Barr, C. Wells, Toposes, Triples, and Theories, Grundlehren Math. Wiss., vol. 278, Springer-Verlag, 1985.
[2] J. Bénabou, J. Roubaud, Monades et descente, C. R. Acad. Sci. 270 (1970) 96–98.
[3] T. Brzeziński, The structure of corings. Induction functors, Maschke-type theorem, and Frobenius and Galois-type properties, Algebr. Represent. Theory 5 (2002) 389–410.
[4] T. Brzeziński, The bicategory of corings, math.RA/0408042v1.
[5] T. Brzeziński, A note on coring extensions, math.RA/0410020v1.
[6] S. Caenepeel, Galois corings from the descent theory point of view, in: Galois Theory, Hopf Algebras and Semiabelian Categories, in: Fields Inst. Commun., vol. 43, Amer. Math. Soc., Providence, RI, 2004, pp. 163–186.
[7] S. Caenepeel, E. De Groot, J. Vercruysse, Galois theory for comatrix corings: Descent theory, Morita theory, Frobenius and separability properties, Trans. Amer. Math. Soc., in press.
[8] M. Cipolla, Discesa fedelmente piatta dei moduli, Rend. Circ. Mat. Palermo 25 (1976) 43–46.
[9] L. El Kaoutit, J. Gómez-Torrecillas, Comatrix corings: Galois corings, descent theory, and a structure theorem for cosemisimple corings, Math. Z. 244 (2003) 887–906.
[10] G. Janelidze, W. Tholen, Facets of descent, III: Monadic descent for rings and algebras, Appl. Categ. Structures 12 (2004) 461–476.
[11] S. MacLane, Categories for the Working Mathematician, Grad. Texts in Math., vol. 5, Springer-Verlag, Berlin, 1971.
[12] B. Mesablishvili, Monads of effective descent type and comonadicity, Theory Appl. Categ. 16 (2006) 1–45.
[13] B. Mesablishvili, Pure morphisms of commutative rings are effective descent morphisms for modules—A new proof, Theory Appl. Categ. 7 (2000) 38–42.
[14] T.Y. Lam, Lectures on Modules and Rings, Grad. Texts in Math., Springer-Verlag, New York, 1999.
[15] P. Nuss, Noncommutative descent and non-abelian cohomology, K-Theory 12 (1997) 23–74.