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Entwining structures in monoidal categories ☆

Bachuki Mesablishvili

Razmadze Mathematical Institute, Tbilisi 0193, Georgia

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Abstract

Interpreting entwining structures as special instances of J. Beck's distributive law, the concept of entwining module can be generalized for the setting of arbitrary monoidal category. In this paper, we use the distributive law formalism to extend in this setting basic properties of entwining modules.

Keywords: Entwining module; (Braided) monoidal category; Hopf algebra

1. Introduction

The important notion of entwining structures has been introduced by T. Brzeziński and S. Majid in [5]. An entwining structure (over a commutative ring K) consists of a K-algebra A, a K-coalgebra C and a certain K-homomorphism $\lambda : C \otimes_K A \to A \otimes_K C$ satisfying some axioms. Associated to λ there is the category $\mathcal{M}_A^C(\lambda)$ of entwining modules whose objects are at the same time A-modules and C-comodules, with compatibility relation given by λ.

The algebra A can be identified with the monad $T = - \otimes_K A : \mathrm{Mod}_K \to \mathrm{Mod}_K$ whose Eilenberg–Moore category of algebras, $(\mathrm{Mod}_K)^T$, is (isomorphic to) the category of right A-modules. Similarly, C can be identified with the comonad $G = - \otimes_K C : \mathrm{Mod}_K \to \mathrm{Mod}_K$, and the corresponding Eilenberg–Moore category of coalgebras with the category of C-comodules. It turns out that to give an entwining structure $C \otimes_K A \to A \otimes_K C$ is to give a mixed distributive law $TG \to GT$ from the monad T to the comonad G in the sense of J. Beck [2], which

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E-mail address: bachi@rmi.acnet.ge.

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are in bijective correspondence with liftings (or extensions) $\bar{G}$ of the comonad G to the category $(\mathrm{Mod}_K)^T$; or, equivalently, liftings $\bar{T}$ of the monad T to the category $(\mathrm{Mod}_K)_G$. Moreover, the categories $\mathcal{M}_A^C(\lambda)$, $((\mathrm{Mod}_K)^T)_G$ and $((\mathrm{Mod}_K)_G)^T$ are isomorphic. Thus, the (mixed) distributive law formalism can be used to study entwining structures and the corresponding category of modules. In this article—based on this formalism—we extend in the context of monoidal categories some of basic results on entwining structures that appear in the literature (see, for example, [6,7,13]).

The paper is organized as follows. After recalling the notion of Beck's mixed distributive law and the basic facts about it, we define in Section 3 an entwining structure in any monoidal category. In Section 4, we prove some categorical results that are needed in the next section, but may also be of independent interest. Finally, in the last section we present our main results.

We refer to M. Barr and C. Wells [1], S. MacLane [10] and F. Borceux [3] for terminology and general results on (co)monads, and to T. Brzeziński and R. Wisbauer [6] for coring and comodule theory.

2. Mixed distributive laws

Let $\mathbf{T} = (T, \eta, \mu)$ be a monad and $\mathbf{G} = (G, \varepsilon, \delta)$ a comonad on a category $\mathcal{A}$. A *mixed distributive law* from $\mathbf{T} = (T, \eta, \mu)$ to $\mathbf{G} = (G, \varepsilon, \delta)$ is a natural transformation

$$\lambda : \mathbf{TG} \to \mathbf{GT}$$

for which the diagrams

$$\begin{array}{ccc} & G & \\ {\scriptstyle \eta G}\swarrow & & \searrow{\scriptstyle G\eta} \\ TG & \xrightarrow[\lambda]{} & GT, \end{array} \qquad \begin{array}{ccc} & TG & \\ {\scriptstyle \lambda}\swarrow & & \searrow{\scriptstyle T\varepsilon} \\ GT & \xrightarrow[\varepsilon T]{} & T, \end{array}$$

$$\begin{array}{ccccc} TG & \xrightarrow{T\delta} & TG^2 & \xrightarrow{\lambda G} & GTG \\ {\scriptstyle \lambda}\downarrow & & & & \downarrow{\scriptstyle G\lambda} \\ GT & & \xrightarrow[\delta T]{} & & GGT \end{array} \quad \text{and} \quad \begin{array}{ccccc} T^2G & \xrightarrow{T\lambda} & TGT & \xrightarrow{\lambda T} & GTT \\ {\scriptstyle \mu G}\downarrow & & & & \downarrow{\scriptstyle G\mu} \\ TG & & \xrightarrow[\lambda]{} & & GT \end{array}$$

commute.

Given a monad $\mathbf{T} = (T, \eta, \mu)$ on $\mathcal{A}$, write $\mathcal{A}^{\mathbf{T}}$ for the Eilenberg–Moore category of $\mathbf{T}$-algebras, and write $F^{\mathbf{T}} \dashv U^{\mathbf{T}} : \mathcal{A}^{\mathbf{T}} \to \mathcal{A}$ for the corresponding forgetful-free adjunction. Dually, if $\mathbf{G} = (G, \varepsilon, \delta)$ is a comonad on $\mathcal{A}$, then write $\mathcal{A}_{\mathbf{G}}$ for the category of $\mathbf{G}$-coalgebras, and write $F_{\mathbf{G}} \dashv U_{\mathbf{G}} : \mathcal{A}_{\mathbf{G}} \to \mathcal{A}$ for the corresponding forgetful-cofree adjunction.

2.1. Theorem. *(See [14].) Let $\mathbf{T} = (T, \eta, \mu)$ be a monad and $\mathbf{G} = (G, \varepsilon, \delta)$ a comonad on a category $\mathcal{A}$. Then the following structures are in bijective correspondences*:

- *mixed distributive laws $\lambda : \mathbf{TG} \to \mathbf{GT}$*;
- *comonads $\bar{\mathbf{G}} = (\bar{G}, \bar{\varepsilon}, \bar{\delta})$ on $\mathcal{A}^T$ that extend $\mathbf{G}$ in the sense that $U^T \bar{G} = G U^T$, $U^T \bar{\varepsilon} = \varepsilon U^T$ and $U^T \bar{\delta} = \delta U^T$*;

- *monads* $\bar{\mathbf{T}} = (\bar{T}, \bar{\eta}, \bar{\mu})$ *on* $\mathcal{A}_G$ *that extend* $\mathbf{T}$ *in the sense that* $U_G \bar{T} = T U_G$, $U_G \bar{\eta} = \eta U_G$ *and* $U_G \bar{\mu} = \mu U_G$.

These correspondences are constructed as follows:

- *Given a mixed distributive law*

$$\lambda : \mathbf{TG} \to \mathbf{GT},$$

then $\bar{G}(a, \xi_a) = (G(a), G(\xi_a) \cdot \lambda_a)$, $\bar{\varepsilon}_{(a,\xi_a)} = \varepsilon_a$, $\bar{\delta}_{(a,\xi_a)} = \delta_a$, *for any* $(a, \xi_a) \in \mathcal{A}^{\mathbf{T}}$; *and* $\bar{T}(a, \nu_a) = (T(a), \lambda_a \cdot T(\nu_a))$, $\bar{\eta}_{(a,\nu_a)} = \eta_a$, $\bar{\mu}_{(a,\nu_a)} = \mu_a$ *for any* $(a, \nu_a) \in \mathcal{A}_{\mathbf{G}}$.
- *If* $\bar{\mathbf{G}} = (\bar{G}, \bar{\varepsilon}, \bar{\delta})$ *is a comonad on* $\mathcal{A}^{\mathbf{T}}$ *extending the comonad* $\mathbf{G} = (G, \varepsilon, \delta)$, *then the corresponding distributive law*

$$\lambda : \mathbf{TG} \to \mathbf{GT}$$

is given by

$$TG \xrightarrow{TG\eta} TGT = U^T F^T G U^T F^T = U^T F^T U^T \bar{G} F^T \xrightarrow{U^T \varepsilon^T \bar{G} F^T} U^T \bar{G} F^T = G U^T F^T = GT,$$

where $\varepsilon^{\mathbf{T}} : F^T U^T \to 1$ *is the counit of the adjunction* $F^T \dashv U^T$.
- *If* $\bar{\mathbf{T}} = (\bar{T}, \bar{\eta}, \bar{\mu})$ *is a monad on* $\mathcal{A}_{\mathbf{G}}$ *extending* $\mathbf{T} = (T, \eta, \mu)$, *then the corresponding mixed distributive law is given by*

$$TG = T U_G F_G = U_G \bar{T} F_G \xrightarrow{U_G \eta_G \bar{T} F_G} U_G F_G U_G \bar{T} F_G = U_G F_G T U_G F_G = GTG \xrightarrow{GT\varepsilon} GT,$$

where $\eta_G : 1 \to F_G U_G$ *is the unit of the adjunction* $U_G \dashv F_G$.

It follows from this theorem that if

$$\lambda : \mathbf{TG} \to \mathbf{GT}$$

is a mixed distributive law, then $(\mathcal{A}_{\mathbf{G}})^{\bar{\mathbf{T}}} = (\mathcal{A}^{\mathbf{T}})_{\bar{\mathbf{G}}}$. We write $(\mathcal{A}^{\mathbf{T}}_{\mathbf{G}})(\lambda)$ for this category. An object of this category is a three-tuple (a, ξ_a, ν_a), where $(a, \xi_a) \in \mathcal{A}^{\mathbf{T}}$, $(a, \nu_a) \in \mathcal{A}_{\mathbf{G}}$, for which $G(\xi_a) \cdot \lambda_a \cdot T(\nu_a) = \nu_a \cdot \xi_a$. A morphism $f : (a, \xi_a, \nu_a) \to (a', \xi'_a, \nu'_a)$ in $(\mathcal{A}^{\mathbf{T}}_{\mathbf{G}})(\lambda)$ is a morphism $f : a \to a'$ in $\mathcal{A}$ such that $\xi'_a \cdot T(f) = f \cdot \xi_a$ and $\nu'_a \cdot f = G(f) \cdot \nu_a$.

3. Entwining structures in monoidal categories

Let $\mathcal{V} = (V, \otimes, I)$ be a monoidal category with coequalizers such that the tensor product preserves the coequalizer in both variables. Then for all algebras $\mathbb{A} = (A, e_A, m_A)$ and $\mathbb{B} = (B, e_B, m_B)$ and all $M \in \mathcal{V}_A$, $N \in {}_A\mathcal{V}_B$ and $P \in {}_B\mathcal{V}$, the tensor product $M \otimes_A N$ exists and the canonical morphism $(M \otimes_A N) \otimes_B P \to M \otimes_A (M \otimes_B P)$ is an isomorphism. Using MacLane's coherence theorem (see, [10, XI.5]), we may assume without loss of generality that $\mathcal{V}$ is strict.

It is well known that every algebra $\mathbb{A} = (A, e_A, m_A)$ in $\mathcal{V}$ defines a monad $\mathbf{T}_{\mathbb{A}}$ on $\mathcal{V}$ by

- $T_{\mathbb{A}}(X) = X \otimes A$,
- $(\eta_{T_{\mathbb{A}}})_X = X \otimes e_A : X \to X \otimes A$,
- $(\mu_{T_{\mathbb{A}}})_X = X \otimes m_A : X \otimes A \otimes A \to X \otimes A$,

and that $\mathcal{V}^{\mathbf{T}_{\mathbb{A}}}$ is (isomorphic to) the category $\mathcal{V}_{\mathbb{A}}$ of right A-modules.

Dually, if $\mathbb{C} = (C, \varepsilon_C, \delta_C,)$ is a coalgebra (= comonoid) in $\mathcal{V}$, then one defines a comonad $\mathbf{G}_{\mathbb{C}}$ on $\mathcal{V}$ by

- $G_{\mathbb{C}}(X) = X \otimes C$,
- $(\varepsilon_{G_{\mathbb{C}}})_X = X \otimes \varepsilon_C : X \otimes C \to X$,
- $(\delta_{G_{\mathbb{C}}})_X = X \otimes \delta_C : X \otimes C \to X \otimes C \otimes C$,

and $\mathcal{V}_{\mathbf{G}_{\mathbb{C}}}$ is (isomorphic to) the category $\mathcal{V}^{\mathbb{C}}$ of right C-comodules.

Quite obviously, if λ is a mixed distributive law from $\mathbf{T}_{\mathbb{A}}$ to $\mathbf{G}_{\mathbb{C}}$, then the morphism

$$\lambda' = \lambda_I : C \otimes A \to A \otimes C$$

makes the following diagrams commutative:

$$\begin{array}{ccc} C & & \\ {\scriptstyle C\otimes e_A}\downarrow & \searrow^{e_A\otimes C} & \\ C\otimes A & \xrightarrow[\lambda']{} & A\otimes C, \end{array} \qquad \begin{array}{ccc} C\otimes A & & \\ {\scriptstyle \lambda'}\downarrow & \searrow^{\varepsilon_{\mathbb{C}}\otimes A} & \\ A\otimes C & \xrightarrow[A\otimes\varepsilon_{\mathbb{C}}]{} & A, \end{array}$$

$$\begin{array}{ccccc} C\otimes A & \xrightarrow{\delta_C\otimes A} & C\otimes C\otimes A & \xrightarrow{C\otimes\lambda'} & C\otimes A\otimes C \\ {\scriptstyle \lambda'}\downarrow & & & & \downarrow{\scriptstyle \lambda'\otimes C} \\ A\otimes C & & \xrightarrow[A\otimes\delta_C]{} & & A\otimes C\otimes C, \end{array}$$

$$\begin{array}{ccccc} C\otimes A\otimes A & \xrightarrow{\lambda'\otimes A} & A\otimes C\otimes A & \xrightarrow{A\otimes\lambda'} & A\otimes A\otimes C \\ {\scriptstyle C\otimes m_A}\downarrow & & & & \downarrow{\scriptstyle m_A\otimes C} \\ C\otimes A & & \xrightarrow[\lambda']{} & & A\otimes C. \end{array}$$

Conversely, if $\lambda' : C \otimes A \to A \otimes C$ is a morphism for which the above diagrams commute, then the natural transformation

$$- \otimes \lambda' : T_{\mathbb{A}} G_{\mathbb{C}}(-) = - \otimes C \otimes A \to - \otimes A \otimes C = G_{\mathbb{C}} T_{\mathbb{A}}(-)$$

is a mixed distributive law from the monad $\mathbf{T}_{\mathbb{A}}$ to the comonad $\mathbf{G}_{\mathbb{C}}$. It is easy to see that $\lambda' = (-\otimes\lambda')_I$. When I is a regular generator in $\mathcal{V}$ and the tensor product preserves all colimits in both variables, it is not hard to show that $\lambda \simeq -\otimes\lambda_I$. When this is the case, then the correspondences $\lambda \to \lambda_I$ and $\lambda' \to -\otimes\lambda'$ are inverses of each other.

3.1. Definition. An entwining structure $(\mathbb{C}, \mathbb{A}, \lambda)$ consists of an algebra $\mathbb{A} = (A, e_A, m_A)$ and a coalgebra $\mathbb{C} = (C, \varepsilon_C, \delta_C)$ in $\mathcal{V}$ and a morphism $\lambda : C \otimes A \to A \otimes C$ such that the natural transformation

$$-\otimes\lambda : T_{\mathbb{A}}G_{\mathbb{C}}(-) = -\otimes C\otimes A \to -\otimes A\otimes C = G_{\mathbb{C}}T_{\mathbb{A}}(-)$$

is a mixed distributive law from the monad $\mathbf{T}_{\mathbb{A}}$ to the comonad $\mathbf{G}_{\mathbb{C}}$.

Let be $(\mathbb{C}, \mathbb{A}, \lambda)$ be an entwining structure and let $\bar{\mathbf{G}} = (\bar{G}, \bar{\varepsilon}, \bar{\delta})$ be the comonad on $\mathcal{V}_{\mathbb{A}}$ that extends $\mathbf{G} = \mathbf{G}_{\mathbb{C}}$. Then we know that, for any $(V, \xi_V) \in \mathcal{V}_{\mathbb{A}}$,

$$\bar{G}(V, \xi_V) = \big(V\otimes C, V\otimes C\otimes A \xrightarrow{V\otimes\lambda} V\otimes A\otimes C \xrightarrow{\xi_V\otimes C} V\otimes C\big).$$

In particular, since $(A, m_A) \in \mathcal{V}_{\mathbb{A}}$, $A\otimes C$ is a right A-module with right action

$$\xi_{A\otimes C} : A\otimes C\otimes A \xrightarrow{A\otimes\lambda} A\otimes A\otimes C \xrightarrow{m_a\otimes C} A\otimes C.$$

3.2. Lemma. *View $A\otimes C$ as a left A-module through $\bar{\xi}_{A\otimes C} = m_A\otimes C$. Then $(A\otimes C, \bar{\xi}_{A\otimes C}, \xi_{A\otimes C})$ is an A–A-bimodule.*

Proof. Clearly $(A\otimes C, \bar{\xi}_{A\otimes C}) \in {}_{\mathbb{A}}\mathcal{V}$. Moreover, since $(A\otimes\lambda)\cdot(m_A\otimes C\otimes A) = (m_A\otimes A\otimes C)\cdot(A\otimes A\otimes\lambda)$, it follows from the associativity of m_A that the diagram

$$\begin{array}{ccc}
A\otimes A\otimes C\otimes A & \xrightarrow{A\otimes A\otimes\lambda} & A\otimes A\otimes A\otimes C \\
\downarrow{\scriptstyle m_A\otimes C\otimes A} & & \downarrow{\scriptstyle A\otimes m_A\otimes C} \\
A\otimes C\otimes A & & A\otimes A\otimes C \\
\downarrow{\scriptstyle A\otimes\lambda} & & \downarrow{\scriptstyle m_A\otimes C} \\
A\otimes A\otimes C & \xrightarrow[m_A\otimes C]{} & A\otimes C
\end{array}$$

is commutative, which just means that $(A\otimes C, \bar{\xi}_{A\otimes C}, \xi_{A\otimes C})$ is an A–A-bimodule. $\square$

Since $\bar{\varepsilon}_{(A,m_A)} : \bar{G}(A, m_A) \to (A, m_A)$ and $\bar{\delta}_{(A,m_A)} : \bar{G}(A, m_A) \to \bar{G}^2(A, m_A)$ are morphisms of right A-modules, and since $U_{\mathbb{A}}(\bar{\varepsilon}_{(A,m_A)}) = (\varepsilon_{\mathbf{G}_{\mathbb{C}}})_A = (A\otimes C \xrightarrow{A\otimes\varepsilon_{\mathbb{C}}} A)$ and $U_A(\bar{\delta}_{(A,m_A)}) = (\delta_{\mathbf{G}_{\mathbb{C}}})_A = (A\otimes C \xrightarrow{A\otimes\delta_{\mathbb{C}}} A\otimes C\otimes C)$, it follows that $A\otimes C \xrightarrow{A\otimes\varepsilon_{\mathbb{C}}} A$ and $A\otimes C \xrightarrow{A\otimes\delta_{\mathbb{C}}} A\otimes C\otimes C$ are both morphisms of right A-modules. Clearly they are also morphisms of left A-modules with the obvious left A-module structures arising from the multiplication $m_A : A\otimes A \to A$, and hence morphisms of A–A-bimodules. Since $\mathbb{C} = (C, \varepsilon_C, \delta_C)$ is a coalgebra in $\mathcal{V}$, it follows that the triple $(\underline{A\otimes C})_\lambda = (A\otimes C, \varepsilon_{(\underline{A\otimes C})_\lambda}, \delta_{(\underline{A\otimes C})_\lambda})$, where $\varepsilon_{(\underline{A\otimes C})_\lambda} = A\otimes C \xrightarrow{A\otimes\varepsilon_{\mathbb{C}}} A$

and $\delta_{\underline{(A\otimes C)}_\lambda} = A\otimes C \xrightarrow{A\otimes\delta_{\mathbb{C}}} A\otimes C\otimes C$, is an A-coring. Since, for any $V\in\mathcal{V}_{\mathbb{A}}$, $V\otimes_A(A\otimes C)\simeq V\otimes C$, the comonad $\bar{\mathbf{G}}$ is isomorphic to the comonad $\mathbf{G}_{\underline{(A\otimes C)}_\lambda}$. Thus, any entwining structure $(\mathbb{C},\mathbb{A},\lambda)$ defines a right A-module structure $\xi_{A\otimes C}$ on $A\otimes C$ such that $(A\otimes C, \bar{\xi}_{A\otimes C} = m_A\otimes C, \xi_{A\otimes C})$ is an A–A-bimodule and the triple $\underline{(A\otimes C)}_\lambda = (A\otimes C, \varepsilon_{\underline{(A\otimes C)}_\lambda}, \delta_{\underline{(A\otimes C)}_\lambda})$ is an A-coring. Moreover, when this is the case, the comonad $\mathbf{G}_{\underline{(A\otimes C)}_\lambda}$ on $\mathcal{V}_{\mathbb{A}}$ extends the comonad $\mathbf{G}_{\mathbb{C}}$. It follows that $\mathcal{V}_{\mathbb{A}}^{\underline{(A\otimes C)}_\lambda} = \mathcal{V}_{\mathbb{A}}^{\mathbb{C}}(\lambda)$.

Conversely, let $\mathbb{A}=(A,e_A,m_A)$ be an algebra and $\mathbb{C}=(C,\varepsilon_C,\delta_C)$ a coalgebra in $\mathcal{V}$, and suppose that $A\otimes C$ has the structure $\xi_{A\otimes C}$ of a right A-module such that the triple

$$\underline{A\otimes C} = \big((A\otimes C, m_A\otimes C, \xi_{A\otimes C}), A\otimes C \xrightarrow{A\otimes\varepsilon_{\mathbb{C}}} A, A\otimes C \xrightarrow{A\otimes\delta_{\mathbb{C}}} A\otimes C\otimes C\big) \tag{1}$$

is an A-coring. Then it is easy to see that the comonad $\mathbf{G}_{\underline{A\otimes C}}$ on $\mathcal{V}_{\mathbb{A}}$ extends the comonad $\mathbf{G}_{\mathbb{C}}$ on $\mathcal{V}$, and thus defines an entwining structure $\lambda_{\underline{A\otimes C}} : C\otimes A\to A\otimes C$.

Summarizing, we have

3.3. Theorem. *Let $\mathbb{A}=(A,e_A,m_A)$ be an algebra and $\mathbb{C}=(C,\varepsilon_C,\delta_C)$ a coalgebra in $\mathcal{V}$. Then there exists a bijection between right A-module structures $\xi_{A\otimes C}$ making $(A\otimes C, m_A\otimes C, \xi_{A\otimes C})$ an A-bimodule for which the triple* (1) *is an A-coring and entwining structures* $(\mathbb{C},\mathbb{A},\lambda)$*, given by*:

$$\xi_{A\otimes C} \longrightarrow \big(\lambda_{\underline{A\otimes C}} : C\otimes A \xrightarrow{e_A\otimes C\otimes A} A\otimes C\otimes A \xrightarrow{\xi_{A\otimes C}} A\otimes C\big)$$

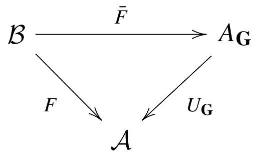
with inverse given by

$$\lambda \longrightarrow \big(\xi_{A\otimes C} : A\otimes C\otimes A \xrightarrow{A\otimes\lambda} A\otimes A\otimes C \xrightarrow{m_A\otimes C} A\otimes C\big).$$

Under this equivalence $\mathcal{V}_{\mathbb{A}}^{\underline{(A\otimes C)}_\lambda} = \mathcal{V}_{\mathbb{A}}^{\mathbb{C}}(\lambda)$.

4. Some categorical results

Let $\mathbf{G}=(G,\varepsilon,\delta)$ be a comonad on a category $\mathcal{A}$, and let $U_{\mathbf{G}} : A_{\mathbf{G}}\to\mathcal{A}$ be the forgetful functor. Fix a functor $F:\mathcal{B}\to\mathcal{A}$, and consider a functor $\bar{F}:\mathcal{B}\to A_{\mathbf{G}}$ making the diagram



(2)

commutative. Then $\bar{F}(b)=(F(b),\alpha_{F(b)})$ for some $\alpha_{F(b)} : F(b)\to GF(b)$. Consider the natural transformation

$$\bar{\alpha}_F : F\to GF, \tag{3}$$

whose b-component is $\alpha_{F(b)}$.

We shall need the following result, which is an immediate consequence of Propositions II.1.1 and II.1.4 in [8]:

4.1. Theorem. *Suppose that F has a right adjoint $U : \mathcal{A} \to \mathcal{B}$ with unit $\eta : 1 \to FU$ and counit $\varepsilon : FU \to 1$. Then the composite*

$$t_{\bar{F}} : FU \xrightarrow{\bar{\alpha}_F U} GFU \xrightarrow{G\varepsilon} G$$

is a morphism from the comonad $\mathbf{G}' = (FU, \varepsilon, F\eta U)$ generated by the adjunction $\eta, \varepsilon : F \dashv U : \mathcal{B} \to \mathcal{A}$ to the comonad $\mathbf{G}$. Moreover, the assignment

$$\bar{F} \to t_{\bar{F}}$$

yields a one to one correspondence between functors $\bar{F} : \mathcal{B} \to \mathcal{A}_G$ making the diagram (2) commutative and morphisms of comonads $t_{\bar{F}} : \mathbf{G}' \to \mathbf{G}$.

Write β_U for the composite $U \xrightarrow{\eta U} UFU \xrightarrow{Ut_{\bar{F}}} UG$.

4.2. Proposition. *Consider the following diagram*

$$UU_G \underset{\beta_U U_G}{\overset{UU_G\eta_G}{\rightrightarrows}} UGU_G = UU_G F_G U_G,$$

where $\eta_G : 1 \to F_G U_G$ is the unit of the adjunction $U_G \dashv F_G$. If the equalizer $\bar{U}$ of this pair of parallel natural transformations exists, then it is right adjoint to $\bar{F}$.

Proof. See the proof of Theorem A.1 in [8]. □

Let $\bar{F} : \mathcal{B} \to \mathcal{A}_{\mathbf{G}}$ be a functor making (2) commutative and let $t_{\bar{F}} : \mathbf{G}' \to \mathbf{G}$ be the corresponding morphism of comonads. Consider the following composition

$$\mathcal{B} \xrightarrow{K_{\mathbf{G}'}} \mathcal{A}_{\mathbf{G}'} \xrightarrow{\mathcal{A}_{t_{\bar{F}}}} \mathcal{A}_{\mathbf{G}},$$

where

- $K_{\mathbf{G}'} : \mathcal{B} \to \mathcal{A}_{\mathbf{G}'}$, $K_{\mathbf{G}'}(b) = (F(b), F(\eta_b))$ is the Eilenberg–Moore comparison functor for the comonad $\mathbf{G}'$.
- $A_{t_{\bar{F}}}$ is the functor

$$\big((a, \theta_a) \in \mathcal{A}'_{\mathbf{G}}\big) \to \big(\big(a, (t_{\bar{F}})_a \cdot \theta_a\big) \in \mathcal{A}_{\mathbf{G}}\big)$$

  induced by the morphism of comonads $t_{\bar{F}} : \mathbf{G}' \to \mathbf{G}$.

4.3. Lemma. *The diagram*

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{K_{G'}} & \mathcal{A}_{G'} \\ & \searrow^{\bar{F}} & \downarrow{\scriptstyle \mathcal{A}_{t_{\bar{F}}}} \\ & & \mathcal{A}_G \end{array} \tag{4}$$

is commutative.

Proof. Let $b \in \mathcal{B}$. Then $K_{\mathbf{G}'}(b) = (F(b), F(\eta_b))$ and $\mathcal{A}_{t_{\bar{F}}}(F(b), F(\eta_b)) = (F(b), (t_{\bar{F}})_{F(b)} \cdot F(\eta_b))$. Since $(t_{\bar{F}})_{F(b)}$ is the composite

$$FUF(b) \xrightarrow{(\bar{\alpha}_F)_{UF(b)}} GFUF(b) \xrightarrow{G\varepsilon_{F(b)}} GF(b),$$

and since by naturality of $\bar{\alpha}_F$, the diagram

$$\begin{array}{ccc} F(b) & \xrightarrow{(\bar{\alpha})_b} & GF(b) \\ {\scriptstyle F(\eta_b)}\downarrow & & \downarrow{\scriptstyle GF(\eta_b)} \\ FUF(b) & \xrightarrow[(\bar{\alpha})_{UF(b)}]{} & GFUF(b) \end{array}$$

commutes, we have

$$(t_{\bar{F}})_{F(b)} \cdot F(\eta_b) = G(\varepsilon_{F(b)}) \cdot (\bar{\alpha}_F)_{UF(b)} \cdot F(\eta_b) = G(\varepsilon_{F(b)}) \cdot GF(\eta_b) \cdot (\bar{\alpha}_F)_b = (\bar{\alpha}_F)_b = \alpha_{F(b)}.$$

Thus

$$\begin{aligned} (\mathcal{A}_{t_{\bar{F}}} \cdot K_{\mathbf{G}'})(b) &= \mathcal{A}_{t_{\bar{F}}}\big(K_{\mathbf{G}'}(b)\big) = \mathcal{A}_{t_{\bar{F}}}\big(F(b), F(\eta_b)\big) \\ &= \big(F(b), (t_{\bar{F}})_{F(b)} \cdot F(\eta_b)\big) = \big(F(b), \alpha_{F(b)}\big), \end{aligned}$$

which just means that $\mathcal{A}_{t_{\bar{F}}} \cdot K_{\mathbf{G}'} = \bar{F}$. □

We are now ready to prove the following

4.4. Theorem. *Let* $\mathbf{G}$ *be a comonad on a category* $\mathcal{A}$, $\eta, \varepsilon : F \dashv U : \mathcal{B} \to \mathcal{A}$ *an adjunction and* $\bar{F} : \mathcal{B} \to \mathcal{A}_G$ *a functor with* $U_G \cdot \bar{F} = F$. *Then the following are equivalent*:

(i) *The functor* $\bar{F}$ *is an equivalence.*
(ii) *The functor* F *is comonadic* (*i.e. the functor* $K_{\mathbf{G}'}$ *is an equivalence of categories*) *and the morphism of comonads*

$$t_{\bar{F}} : \mathbf{G}' = (FU, \varepsilon, F\eta U) \to \mathbf{G}$$

is an isomorphism.

(iii) *The morphism of comonads*

$$t_{\bar{F}} : \mathbf{G}' = (FU, \varepsilon, F\eta U) \to \mathbf{G}$$

is an isomorphism, the functor F is conservative and for any $(X, x) \in \mathcal{A}_{\mathbf{G}}$, it preserves the equalizer of the pair of parallel morphisms

$$U(X) \underset{\eta_{U(X)}}{\longrightarrow} UG'(X) \underset{U((t_{\bar{F}})_X)}{\longrightarrow} UG(X), \qquad U(X) \xrightarrow{U(x)} UG(X). \tag{5}$$

Proof. Suppose that $\bar{F}$ is an equivalence of categories. Then F is isomorphic to the comonadic functor $U_{\mathbf{G}}$ and thus is comonadic. Hence the comparison functor $K_{\mathbf{G}'} : \mathcal{B} \to \mathcal{A}_{\mathbf{G}'}$ is an equivalence and it follows from the commutative diagram (4) that $\mathcal{A}_{t_{\bar{F}}}$ is also an equivalence, and since the diagram

$$\begin{array}{ccc} \mathcal{A}_{G'} & \xrightarrow{\mathcal{A}_{t_{\bar{F}}}} & \mathcal{A}_G \\ & {\scriptstyle U_{G'}} \searrow \quad \swarrow {\scriptstyle U_G} & \\ & \mathcal{A} & \end{array}$$

is commutative, $t_{\bar{F}}$ is an isomorphism of comonads. So (i) ⇒ (ii).

Suppose now that $t_{\bar{F}} : \mathbf{G}' \to \mathbf{G}$ is an isomorphism of comonads and F is comonadic. Then

- $K_{\mathbf{G}'}$ is an equivalence, since F is comonadic.
- $\mathcal{A}_{t_{\bar{F}}}$ is an equivalence, since $t_{\bar{F}}$ is an isomorphism.

And it now follows from the commutative diagram (4) that $\bar{F}$ is also an equivalence. Thus (ii) ⇒ (i).

When $t_{\bar{F}}$ is an isomorphism of comonads, to say that F preserves the equalizer of the pair of morphisms (5) is to say that F preserves the equalizer of the pair of morphisms

$$U(X) \underset{U((t_{\bar{F}}^{-1})_X) \cdot U(x)}{\overset{\eta_{U(X)}}{\rightrightarrows}} UG'(X),$$

which we can rewrite as

$$U(X) \underset{U((t_{\bar{F}}^{-1})_X \cdot x)}{\overset{\eta_{U(X)}}{\rightrightarrows}} UG'(X) = UFU(X). \tag{6}$$

Since $t_{\bar{F}}$ is an isomorphism of comonads, $\mathcal{A}_{t_{\bar{F}}}$ is an equivalence of categories, and thus each object $(X, x') \in \mathcal{A}_{\mathbf{G}'}$ is isomorphic to the $\mathbf{G}'$-coalgebra $(X, (t_{\bar{F}}^{-1})_X \cdot x)$, where $(X, x) \in \mathcal{A}_{\mathbf{G}}$. It follows that when $t_{\bar{F}}$ is an isomorphism of comonads, to say that F preserves the equalizer of (5) for each $(X,x) \in \mathcal{A}_{\mathbf{G}}$ is to say that F preserves the equalizer of (6) for each $(X,x') \in \mathcal{A}_{\mathbf{G}'}$. Thus, when $t_{\bar{F}}$ is an isomorphism of comonads, $\bar{F}$ is an equivalence of categories iff F is conservative and preserves the equalizer of (6) for each $(X,x') \in \mathcal{A}_{\mathbf{G}'}$, which according to (the dual of) Beck's theorem (see [10, VII. 2. Theorem 1, p. 147]), is to say that the functor F is comonadic. Hence (ii) and (iii) are equivalent. This completes the proof of the theorem. □

4.5. Remark. A different proof of the fact that (ii) and (iii) are equivalent was already given by J. Gómez-Torrecillas (see Theorem 2.7 in [9]).

5. Some applications

Let $(\mathbb{C}, \mathbb{A}, \lambda)$ be an entwining structure in a monoidal category $\mathcal{V} = (V, \otimes, I)$, and let $g : I \to C$ be a group-like element of $\mathbb{C}$. (Recall that a morphism $g : I \to C$ is said to be a group-like element of $\mathbb{C}$ if the following diagrams

$$\begin{array}{ccc} I & \xrightarrow{g} & C \\ & \searrow^{=} \;(1) & \downarrow{\scriptstyle \varepsilon_{\mathbb{C}}} \\ & & I, \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{g} & C \\ & \searrow^{g\otimes g} \;(2) & \downarrow{\scriptstyle \delta_{\mathbb{C}}} \\ & & C \otimes C \end{array}$$

are commutative.)

5.1. Proposition. *If $\mathbb{C}$ has a group-like element $g : I \to C$, then A is a right C-comodule through the morphism*

$$g_A : A \xrightarrow{g\otimes A} C \otimes A \xrightarrow{\lambda} A \otimes C.$$

Proof. Consider the diagram

$$\begin{array}{ccccc} A & \xrightarrow{g\otimes A} & C \otimes A & \xrightarrow{\lambda} & A \otimes C \\ & \searrow^{=} & \downarrow{\scriptstyle \varepsilon_{\mathbb{C}}\otimes A} & & \downarrow{\scriptstyle A\otimes\varepsilon_{\mathbb{C}}} \\ & & A & = & A. \end{array}$$

The triangle is commutative by (1) of the definition of g and the square is commutative by the definition of λ (see the second commutative diagram in the definition of entwining structures).

Now, we have to show that the following diagram

$$\begin{array}{ccccc} A & \xrightarrow{g\otimes A} & C \otimes A & \xrightarrow{\lambda} & A \otimes C \\ \downarrow{\scriptstyle g\otimes A} & & & & \\ C \otimes A & & & & \downarrow{\scriptstyle A\otimes\delta_{\mathbb{C}}} \\ \downarrow{\scriptstyle \lambda} & & & & \\ A \otimes C & \xrightarrow[g\otimes A\otimes C]{} & C \otimes A \otimes C & \xrightarrow[\lambda\otimes C]{} & A \otimes C \otimes C \end{array}$$

is also commutative, which it is since

$$(A \otimes \delta_{\mathbb{C}})\lambda = (\lambda \otimes C)(C \otimes \lambda)(\delta_{\mathbb{C}} \otimes A)$$

by the definition of λ and since the diagram (2) of definition of group-like elements is commutative. $\square$

Suppose now that $\mathcal{V}$ admits equalizers. For any $(M, \alpha_M) \in \mathcal{V}^{\mathbb{C}}$, write $((M, \alpha_M)^{\mathbb{C}}, i_M)$ for the equalizer of the morphisms

$$(M, \alpha_M)^{\mathbb{C}} \xrightarrow{\ i_M\ } M \underset{Mg}{\overset{\alpha_M}{\rightrightarrows}} M \otimes C.$$

5.2. Proposition. *$A^{\mathbb{C}} = (A, g_A)^{\mathbb{C}}$ is an algebra in $\mathcal{V}$ and $i_A : A^{\mathbb{C}} \to A$ is an algebra morphism.*

Proof. Consider the diagram

$$\begin{array}{ccccccc} & & & \multicolumn{3}{c}{\overset{A\otimes g}{\curvearrowright}} & \\ A^{\mathbb{C}} & \xrightarrow{i_A} & A & \xrightarrow[g\otimes A]{} & C \otimes A & \xrightarrow[\lambda]{} & A \otimes C. \\ {\scriptstyle e_{A^{\mathbb{C}}}}\uparrow & \nearrow{\scriptstyle e_A} & & & & & \\ I & & & & & & \end{array} \tag{7}$$

Since

$$g \otimes - : 1_{\mathcal{V}} = I \otimes - \to C \otimes -$$

is a natural transformation, the diagram

$$\begin{array}{ccc} I & \xrightarrow{\ g\ } & C \\ {\scriptstyle e_A}\downarrow & & \downarrow{\scriptstyle C\otimes e_A} \\ A & \xrightarrow[g\otimes A]{} & C \otimes A \end{array}$$

is commutative. Similarly, since $e_A \otimes - : 1_{\mathcal{V}} = I \otimes - \to C \otimes -$ is a natural transformation, the following diagram is also commutative:

$$\begin{array}{ccc} I & \xrightarrow{\ e_A\ } & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle A\otimes g} \\ C & \xrightarrow[e_A\otimes C]{} & A \otimes C. \end{array}$$

Now we have:

$$\lambda(g \otimes A)e_A = \lambda(C \otimes e_A)g = \text{by the definition of } \lambda$$
$$= (e_A \otimes C)g = (A \otimes g)e_A.$$

Thus there exists a unique morphism $e_A : I \to A^{\mathbb{C}}$ for which $i_A \cdot e_{A^{\mathbb{C}}} = e_A$.

Since

- the diagram

$$\begin{array}{ccc} A \otimes A & \xrightarrow{g \otimes A \otimes A} & C \otimes A \otimes A \\ {\scriptstyle m_A}\downarrow & & \downarrow{\scriptstyle C \otimes m_A} \\ A & \xrightarrow[g \otimes A]{} & C \otimes A \end{array}$$

  is commutative by naturality of $g \otimes -$;
- $\lambda(C \otimes m_A) = (m_A \otimes C)(A \otimes \lambda)(\lambda \otimes A)$ by the definition of λ;
- $\lambda(g \otimes A)i_A = (A \otimes g)i_A$, since i_A is an equalizer of $\lambda(g \otimes A)$ and $A \otimes g$;
- the diagram

$$\begin{array}{ccc} A \otimes A & \xrightarrow{A \otimes A \otimes g} & A \otimes A \otimes C \\ {\scriptstyle m_A}\downarrow & & \downarrow{\scriptstyle m_A \otimes C} \\ A & \xrightarrow[A \otimes g]{} & A \otimes C \end{array}$$

  is commutative by naturality of $m_A \otimes -$,

we have

$$\begin{aligned} \lambda(g \otimes A)m_A(i_A \otimes i_A) &= \lambda(C \otimes m_A)(g \otimes A \otimes A)(i_A \otimes i_A) \\ &= (m_A \otimes C)(A \otimes \lambda)(\lambda \otimes A)(g \otimes A \otimes A)(i_A \otimes i_A) \\ &= (m_A \otimes C)(A \otimes \lambda)(A \otimes g \otimes A)(i_A \otimes i_A) \\ &= (m_A \otimes C)(A \otimes A \otimes g)(i_A \otimes i_A) \\ &= (A \otimes g)m_A(i_A \otimes i_A). \end{aligned}$$

Thus the morphism $m_A \cdot (i_A \otimes i_A)$ equalizes the morphisms $\lambda \cdot (g \otimes A)$ and $A \otimes g$, and hence there is a unique morphism

$$m_{A^{\mathbb{C}}} : A^{\mathbb{C}} \otimes A^{\mathbb{C}} \to A^{\mathbb{C}}$$

such that the diagram

$$\begin{array}{ccc} A^{\mathbb{C}} \otimes A^{\mathbb{C}} & \xrightarrow{i_A \otimes i_A} & A \otimes A \\ {\scriptstyle m_{A^{\mathbb{C}}}} \downarrow & & \downarrow {\scriptstyle m_A} \\ A^{\mathbb{C}} & \xrightarrow[i_A]{} & A \end{array} \tag{8}$$

commutes. It is now straightforward to show that the triple $(A^{\mathbb{C}}, e_{A^{\mathbb{C}}}, m_{A^{\mathbb{C}}})$ is an algebra in $\mathcal{V}$; moreover, the triangle of the diagram (7) and the diagram (8) show that i_A is an algebra morphism. □

5.3. Proposition. $(A, m_A, g_A) \in \mathcal{V}_{\mathbb{A}}^{\mathbb{C}}(\lambda)$.

Proof. Since $(A, m_A) \in \mathcal{V}_{\mathbb{A}}$ and $(A, g_A) \in \mathcal{V}^{\mathbb{C}}$, it only remains to show that the following diagram is commutative:

$$\begin{array}{ccccc} A \otimes A & \xrightarrow{g_A \otimes A} & A \otimes C \otimes A & \xrightarrow{A \otimes \lambda} & A \otimes A \otimes C \\ {\scriptstyle m_A} \downarrow & & & & \downarrow {\scriptstyle m_A \otimes C} \\ A & & \xrightarrow[g_A]{\hspace{4em}} & & A \otimes C. \end{array} \tag{9}$$

By the definition of g_A, we can rewrite it as

$$\begin{array}{ccccccc} A \otimes A & \xrightarrow{g \otimes A \otimes A} & C \otimes A \otimes A & \xrightarrow{\lambda \otimes A} & A \otimes C \otimes A & \xrightarrow{A \otimes \lambda} & A \otimes A \otimes C \\ {\scriptstyle m_A} \downarrow & & \vdots {\scriptstyle C \otimes m_A} & & & & \downarrow {\scriptstyle m_A \otimes C} \\ A & \xrightarrow[g \otimes A]{} & C \otimes A & & \xrightarrow[\lambda]{\hspace{4em}} & & A \otimes C. \end{array}$$

But this diagram is commutative, since

- the left square commutes because of naturality of $g \otimes -$;
- the right square commutes because of the definition of λ. □

The algebra morphism $i_A : A^{\mathbb{C}} \to A$ makes A an $A^{\mathbb{C}}$–$A^{\mathbb{C}}$-bimodule and thus induces the *extension-of-scalars* functor

$$F_{i_A} : \mathcal{V}_{A^{\mathbb{C}}} \to \mathcal{V}_A,$$
$$(X, \rho_X) \to (X \otimes_{A^{\mathbb{C}}} A, X \otimes_{A^{\mathbb{C}}} m_A),$$

and the forgetful functor

$$U_{i_A} : \mathcal{V}_A \to \mathcal{V}_{A^{\mathbb{C}}},$$
$$(Y, \varrho_Y) \to \big(Y, \varrho_Y \cdot (Y \otimes i_A)\big),$$

which is right adjoint to F_{i_A}. The corresponding comonad on $\mathcal{V}_A$ makes $A \otimes_{A^{\mathbb{C}}} A$ into an A-coring with the following counit and comultiplication:

$$\varepsilon : A \otimes_{A^{\mathbb{C}}} A \xrightarrow{q} A \otimes A \xrightarrow{m_A} A$$

(where q is the canonical morphism) and

$$\delta : A \otimes_{A^{\mathbb{C}}} A = A \otimes_{A^{\mathbb{C}}} A^{\mathbb{C}} \otimes_{A^{\mathbb{C}}} A \xrightarrow{A \otimes_{A^{\mathbb{C}}} i_A \otimes_{A^{\mathbb{C}}} A} A \otimes_{A^{\mathbb{C}}} A \otimes_{A^{\mathbb{C}}} A$$
$$= (A \otimes_{A^{\mathbb{C}}} A)_A \otimes (A \otimes_{A^{\mathbb{C}}} A).$$

We write $\underline{A \otimes_{A^{\mathbb{C}}} A}$ for this A-coring.

5.4. Lemma. *For any $X \in \mathcal{V}_{A^{\mathbb{C}}}$, the triple*

$$(X \otimes_{A^{\mathbb{C}}} A, X \otimes_{A^{\mathbb{C}}} m_A, X \otimes_{A^{\mathbb{C}}} g_A)$$

is an object of the category $\mathcal{V}_{\mathbb{A}}^{\mathbb{C}}(\lambda)$.

Proof. Clearly $(X \otimes_{A^{\mathbb{C}}} A, X \otimes_{A^{\mathbb{C}}} m_A) \in \mathcal{V}_{\mathbb{A}}$ and $(X \otimes_{A^{\mathbb{C}}} A, X \otimes_{A^{\mathbb{C}}} g_A) \in \mathcal{V}^{\mathbb{C}}$. Moreover, by (9), the following diagram

$$\begin{array}{ccccc}
X \otimes_{A^{\mathbb{C}}} A \otimes A & \xrightarrow{X \otimes_{A^{\mathbb{C}}} g_A \otimes A} & X \otimes_{A^{\mathbb{C}}} A \otimes C \otimes A & \xrightarrow{X \otimes_{A^{\mathbb{C}}} A \otimes \lambda} & X \otimes_{A^{\mathbb{C}}} A \otimes A \otimes C \\
{\scriptstyle X \otimes_{A^{\mathbb{C}}} m_A} \downarrow & & & & \downarrow {\scriptstyle X \otimes_{A^{\mathbb{C}}} m_A \otimes C} \\
X \otimes_{A^{\mathbb{C}}} A & & \xrightarrow[X \otimes_{A^{\mathbb{C}}} g_A]{} & & X \otimes_{A^{\mathbb{C}}} A \otimes C
\end{array}$$

is commutative. Thus, $(X \otimes_{A^{\mathbb{C}}} A, X \otimes_{A^{\mathbb{C}}} m_A, X \otimes_{A^{\mathbb{C}}} g_A) \in \mathcal{V}_{\mathbb{A}}^{\mathbb{C}}(\lambda)$. □

The lemma shows that the assignment

$$X \to (X \otimes_{A^{\mathbb{C}}} A, X \otimes_{A^{\mathbb{C}}} m_A, X \otimes_{A^{\mathbb{C}}} g_A)$$

yields a functor

$$\bar{F} : \mathcal{V}_{\mathbb{A}} \to \mathcal{V}_{\mathbb{A}}^{\mathbb{C}}(\lambda) = \mathcal{V}_{\mathbb{A}}^{\underline{(A \otimes C)}_\lambda}.$$

It is clear that $U_{\underline{(A \otimes C)}_\lambda} \cdot \bar{F} = F_{i_A}$, where $U_{\underline{(A \otimes C)}_\lambda} : \mathcal{V}_{\mathbb{A}}^{\underline{(A \otimes C)}_\lambda} \to \mathcal{V}_{\mathbb{A}}$ is the underlying functor. It now follows from Theorem 4.1 that the composite

$$A \otimes_{A^{\mathbb{C}}} A \xrightarrow{A \otimes g_A} A \otimes A \otimes C \xrightarrow{m_A \otimes C} A \otimes C$$

is a morphism of A-corings $\underline{A \otimes_{A^{\mathbb{C}}} A} \to (\underline{A \otimes C})_\lambda$. We write *can* for this morphism. We say that A is $(\mathbb{C}, g)$-Galois if *can* is an isomorphism of A-corings.

Applying Theorem 4.4 to the commutative diagram

$$\begin{array}{ccc} \mathcal{V}_{A^{\mathbb{C}}} & \xrightarrow{\bar{F}} & \mathcal{V}_{\mathbb{A}}{}^{(\underline{A\otimes C})_\lambda} \\ & {\scriptstyle F_{i_A} = - \otimes_{A^{\mathbb{C}}} A} \searrow & \downarrow {\scriptstyle U_{(\underline{A\otimes C})_\lambda}} \\ & & \mathcal{V}_{\mathbb{A}} \end{array}$$

we get:

5.5. Theorem. *Let* $(\mathbb{C}, \mathbb{A}, \lambda)$ *be an entwining structure, and let* $g : I \to C$ *be a group-like element of* $\mathbb{C}$. *Then the functor*

$$\bar{F} : \mathcal{V}_{A^{\mathbb{C}}} \to \mathcal{V}_{\mathbb{A}}^{\mathbb{C}}(\lambda)$$

is an equivalence if and only if A *is* $(\mathbb{C}, g)$*-Galois and the functor* F *is comonadic.*

Let $\mathbb{A} = (A, e_A, m_A)$ and $\mathbb{B} = (B, e_B, m_B)$ be algebras in $\mathcal{V}$ and let $M \in {}_{\mathbb{A}}\mathcal{V}_{\mathbb{B}}$. We call ${}_AM$ (respectively M_B)

- *flat*, if the functor $- \otimes_A M : \mathcal{V}_A \to \mathcal{V}_B$ (respectively $M \otimes_B - : {}_B\mathcal{V} \to {}_A\mathcal{V}$) preserves equalizers;
- *faithfully flat*, if the functor $- \otimes_A M : \mathcal{V}_A \to \mathcal{V}_B$ (respectively $M \otimes_B - : {}_B\mathcal{V} \to {}_A\mathcal{V}$) is conservative and flat (equivalently, preserves and reflects equalizers);

5.6. Theorem. *Let* $(\mathbb{C}, \mathbb{A}, \lambda)$ *be an entwining structure, and let* $g : I \to C$ *be a group-like element of* $\mathbb{C}$. *If* C *is flat, then the following are equivalent*

(i) *The functor*

$$\bar{F} : \mathcal{V}_{A^{\mathbb{C}}} \to \mathcal{V}_{\mathbb{A}}^{\mathbb{C}}(\lambda) = \mathcal{V}_{\mathbb{A}}{}^{(\underline{A\otimes C})_\lambda}$$

is an equivalence of categories.

(ii) A *is* $(\mathbb{C}, g)$*-Galois and* ${}_{A^{\mathbb{C}}}A$ *is faithfully flat.*

Proof. Since any left adjoint functor that is conservative and preserves equalizers is comonadic by a simple and well-known application (of the dual of) Beck's theorem, one direction is clear from Theorem 5.5; so suppose that $\bar{F}$ is an equivalence of categories. Then, by Theorem 5.5, A is $(\mathbb{C}, g)$-Galois and the functor F_{i_A} is comonadic. Since any comonadic functor is conservative, F_{i_A} is also conservative. Thus, it only remains to show that ${}_{A^{\mathbb{C}}}A$ is flat.

Since C is flat by our assumption, ${}_A(A \otimes C)$ is also flat. It follows that the underlying functor of the comonad $\mathbf{G}_{(\underline{A\otimes C})_\lambda}$ on $\mathcal{V}_A$ preserves equalizers. It is well known (see, for example, Proposition 4.3.2 in [3]) that if $\mathbf{G} = (G, \varepsilon_G, \delta_G)$ is a comonad on a category $\mathcal{A}$, and if $\mathcal{A}$ has some type of limits preserved by G, then the category $\mathcal{A}_{\mathbf{G}}$ has the same type of limits and these are preserved by the underlying functor $U_{\mathbf{G}} : \mathcal{A}_{\mathbf{G}} \to \mathcal{A}$. Thus the functor $U_{(\underline{A\otimes C})_\lambda} : \mathcal{V}_{\mathbb{A}}{}^{(\underline{A\otimes C})_\lambda} \to \mathcal{V}_{\mathbb{A}}$

preserves equalizers, and since $\bar{F}$ is an equivalence of categories, the functor $F_{i_A} = - \otimes_{A^{\mathbb{C}}} A$ also preserves equalizers, which just means that ${}_{A^{\mathbb{C}}}A$ is flat. This completes the proof. □

Note that, for entwining structures between ordinary algebras and coalgebras, this result is proved by T. Brzezinski (see Theorem 5.6 in [4]).

6. The case of braided monoidal categories

Throughout of this paper, we shall assume that our $\mathcal{V}$ is a strict braided monoidal category with braiding $\sigma_{X,Y} : X \otimes Y \to Y \otimes X$. Then the tensor product of two (co)algebras in $\mathcal{V}$ is again a (co)algebra; the multiplication $m_{A\otimes B}$ and the unit $e_{A\otimes B}$ of the tensor product of two algebras $\mathbb{A} = (A, e_A, m_A)$ and $\mathbb{B} = (B, e_{\mathbb{B}}, m_{\mathbb{B}})$ are given through

$$m_{A\otimes B} = (m_A \otimes m_B)(A \otimes \sigma_{A,B} \otimes B)$$

and

$$e_{A\otimes B} = e_A \otimes e_B.$$

A bialgebra $\mathbb{H} = (\bar{H} = (H, e_H, m_H), \underline{H} = (H, \varepsilon_H, \delta_H))$ in $\mathcal{V}$ is an algebra $\bar{H} = (H, e_H, m_H)$ and a coalgebra $\underline{H} = (H, \varepsilon_H, \delta_H)$, where ε_H and δ_H are algebra morphisms, or, equivalently, e_H and m_H are coalgebra morphisms.

A Hopf algebra $\mathbb{H} = (\bar{H} = (H, e_H, m_H), \underline{H} = (H, \varepsilon_H, \delta_H), S)$ in $\mathcal{V}$ is a bialgebra $\mathbb{H}$ with a morphism $S : H \to H$, called the antipode of $\mathbb{H}$, such that

$$m_H(H \otimes S)\delta_H = m_H(S \otimes H)\delta_H = e_H \cdot \varepsilon_H.$$

Recall that for any bialgebra $\mathbb{H}$, the category $\mathcal{V}^{\underline{H}}$ is monoidal: The tensor product $(X, \delta_X) \otimes (Y, \delta_Y)$ of two right $\mathbb{H}$-comodules (X, δ_X) and (Y, δ_Y) is their tensor product $X \otimes Y$ in $\mathcal{V}$ with the coaction

$$\delta_{X\otimes Y} : X \otimes Y \xrightarrow{\delta_X \otimes \delta_Y} X \otimes H \otimes Y \otimes H \xrightarrow{X\otimes\sigma_{X,Y}\otimes Y} X \otimes Y \otimes H \otimes H \xrightarrow{X\otimes Y\otimes m_H} X \otimes Y \otimes H.$$

The unit object for this tensor product is I with trivial $\underline{H}$-comodule structure $e_H : I \to H$.

6.1. Proposition. *Let $\mathbb{H} = (\bar{H} = (H, e_H, m_H), \underline{H} = (H, \varepsilon_H, \delta_H))$ be a bialgebra in $\mathcal{V}$. For any algebra $\mathbb{A} = (A, e_A, m_A)$ in $\mathcal{V}$, the following conditions are equivalent*:

- *$\mathbb{A} = (A, e_A, m_A)$ is an algebra in the monoidal category $\mathcal{V}^{\underline{H}}$*;
- *$\mathbb{A} = (A, e_A, m_A)$ is an H-comodule algebra*; *that is, A is a right H-comodule and the H-comodule coaction $\alpha_A : A \to A \otimes H$ is a morphism of algebras in $\mathcal{V}$ from the algebra $\mathbb{A} = (A, e_A, m_A)$ to the algebra $A \otimes \bar{H} = (A \otimes H, e_A \otimes e_H, m_{A\otimes H})$.*

Suppose now that $\mathbb{A} = (A, e_A, m_A)$ is a right H-comodule algebra with H-coaction $\alpha_A : A \to A \otimes H$. By the previous proposition, A is an algebra in the monoidal category $\mathcal{V}^{\underline{H}}$, and thus defines a monad $\mathbf{T}_H^A = (T_H^A, \eta_H^A, \mu_H^A)$ on $\mathcal{V}^{\underline{H}}$ as follows:

- $T_H^A(X, \delta_X) = (X, \delta_X) \otimes (A, \alpha_A)$;
- $(\eta_H^A)_{(X,\delta_X)} = X \otimes e_A$;
- $(\mu_H^A)_{(X,\delta_X)} = X \otimes m_A$.

It is easy to see that the monad $\mathbf{T}_H^A$ extends the monad $\mathbf{T}^A$; and it follows from Theorem 2.1 that there exists a distributive law $\lambda_\alpha : \mathbf{T}^{\mathbb{A}} \cdot \mathbf{G}_{\underline{H}} \to \mathbf{G}_{\underline{H}} \cdot \mathbf{T}^{\mathbb{A}}$ from the monad $\mathbf{T}^{\mathbb{A}}$ to the comonad $\mathbf{G}_{\underline{H}}$, and hence an entwining structure $(\underline{H}, \mathbb{A}, \lambda_{(A,\alpha_A)})$, where $\lambda_{(A,\alpha_A)} = (\lambda_\alpha)_I$.

Therefore we have:

6.2. Theorem. *Every right $\mathbb{H}$-comodule algebra $\mathbb{A} = ((A, \alpha_A), m_A, e_A)$ defines an entwining structure $(\underline{H}, \mathbb{A}, \lambda_{(A,\alpha_A)} : H \otimes A \to A \otimes H)$.*

6.3. Proposition. *Let $\mathbb{A} = ((A, \alpha_A), m_A, e_A)$ be a right $\mathbb{H}$-comodule algebra. Then the entwining structure $\lambda_{A,\alpha_A} : H \otimes A \to A \otimes H$ is given by the composite*:

$$H \otimes A \xrightarrow{H\otimes\alpha_A} H \otimes A \otimes H \xrightarrow{\sigma_{H,A}\otimes H} A \otimes H \otimes H \xrightarrow{A\otimes m_H} A \otimes H.$$

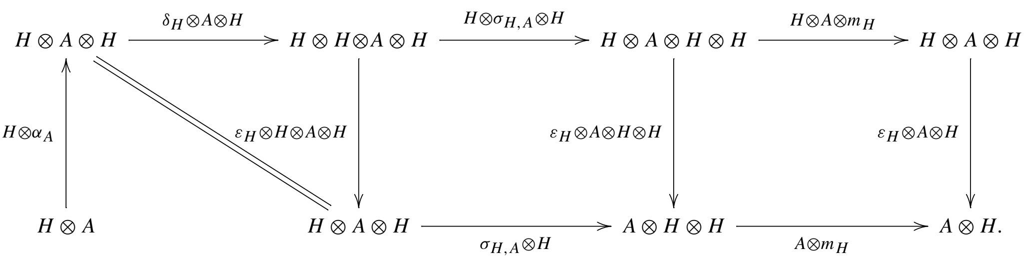
Proof. Since $(A, \alpha_A), (H, \delta_H) \in \mathcal{V}^{\underline{H}}$, the pair $(A \otimes H, \delta_{A\otimes H})$, where $\delta_{A\otimes H}$ is the composite

$$H \otimes A \xrightarrow{\delta_H\otimes\alpha_A} H \otimes H \otimes A \otimes H \xrightarrow{H\otimes\sigma_{H,A}\otimes H} H \otimes A \otimes H \otimes H \xrightarrow{H\otimes A\otimes m_H} H \otimes A \otimes H,$$

is also an object of $\mathcal{V}^{\underline{H}}$, and it follows from Theorem 2.1 that $\lambda_{(A,\alpha_A)}$ is the composite

$$H \otimes A \xrightarrow{\delta_{A\otimes H}} H \otimes A \otimes H \xrightarrow{\varepsilon_H\otimes A\otimes H} A \otimes H.$$

Consider now the following diagram



Since in this diagram

- the triangle commutes because ε_H is the counit for δ_H;
- the left square commutes by naturality of σ;
- the right square commutes because $- \otimes -$ is a bifunctor,

it follows that

$$\lambda_{(A,\alpha_A)} = (A \otimes m_H)(\sigma_{H,A} \otimes H)(H \otimes \alpha_A). \qquad \square$$

Note that the morphism $e_H : I \to H$ is a group-like element for the coalgebra $\underline{H} = (H, \varepsilon_H, \delta_H)$.

6.4. Proposition. *Let $\mathbb{H} = (\bar{H} = (H, e_H, m_H), \underline{H} = (H, \varepsilon_H, \delta_H))$ be a bialgebra in $\mathcal{V}$, and let $\mathbb{A} = ((A, \alpha_A), e_A, m_A)$ be a right $\mathbb{H}$-comodule algebra. Then the right $\underline{H}$-comodule structure on A corresponding to the group-like element $e_H : I \to H$ as in Proposition* 4.1 *coincides with α_A.*

Proof. We have to show that

$$(A \otimes m_H)(\sigma_{H,A} \otimes H)(H \otimes \alpha_A)(e_H \otimes A) = \alpha_A.$$

But since

- clearly $(H \otimes \alpha_A)(e_H \otimes A) = (e_H \otimes A \otimes H) \cdot \alpha_A$;
- $(\sigma_{H,A} \otimes H) \cdot (e_H \otimes A \otimes H) = A \otimes e_H \otimes H$ by naturality of σ;
- $(A \otimes m_H) \cdot (A \otimes e_H \otimes H) = 1_{A \otimes H}$ since e_H is the identity for m_H,

we have that

$$\begin{aligned}
&(A \otimes m_H)(\sigma_{H,A} \otimes H)(H \otimes \alpha_A)(e_H \otimes A) \\
&\quad = (A \otimes m_H)(\sigma_{H,A} \otimes H)(e_H \otimes A \otimes H)\alpha_A \\
&\quad = (A \otimes m_H)(A \otimes e_H \otimes H)\alpha_A \\
&\quad = 1_{A \otimes H} \cdot \alpha_A = \alpha_A. \qquad \square
\end{aligned}$$

It now follows from Proposition 5.3 that

6.5. Proposition. *Let $\mathbb{H} = (\bar{H} = (H, e_H, m_H), \underline{H} = (H, \varepsilon_H, \delta_H))$ be a bialgebra in $\mathcal{V}$, and let $\mathbb{A} = ((A, \alpha_A), e_A, m_A)$ be a right $\mathbb{H}$-comodule algebra. Then*

$$\mathbb{A} = (A, e_A, m_A) \in \mathcal{V}_{\mathbb{A}}^{\underline{H}}(\lambda_{A,\alpha_A}).$$

Recall that for any $(X, \alpha_X) \in \mathcal{V}^{\underline{H}}$, the algebra $X^{\underline{H}} = (X, \alpha_X)^{\underline{H}}$ is the equalizer of the morphisms

$$X \underset{X \otimes e_H}{\overset{\alpha_X}{\rightrightarrows}} X \otimes H.$$

Applying Theorem 5.5 we get

6.6. Theorem. *Let $\mathbb{H} = (\bar{H} = (H, e_H, m_H), \underline{H} = (H, \varepsilon_H, \delta_H))$ be a bialgebra in $\mathcal{V}$, let $\mathbb{A} = ((A, \alpha_A), e_A, m_A)$ be a right $\mathbb{H}$-comodule algebra, and let $\lambda_{(A,\alpha_A)} : H \otimes A \to A \otimes H$ be the corresponding entwining structure. Then the functor*

$$\bar{F} : \mathcal{V}_{A^{\underline{H}}} \to \mathcal{V}_{\mathbb{A}}^{\underline{H}}(\lambda_{(A,\alpha_A)}),$$
$$(X, \nu_X) \to (X \otimes_{\mathbb{A}^{\underline{H}}} A, X \otimes_{\mathbb{A}^{\underline{H}}} m_A, X \otimes_{\mathbb{A}^{\underline{H}}} \alpha_A)$$

is an equivalence of categories iff the extension-of-scalars functor

$$F_{i_A} : \mathcal{V}_{A^{\underline{H}}} \to \mathcal{V}_A,$$
$$(X, \nu_X) \to (X \otimes_{\mathbb{A}^{\underline{H}}} A, X \otimes_{\mathbb{A}^{\underline{H}}} m_A)$$

is comonadic and A is $\underline{H}$-Galois (*in the sense that the canonical morphism*

$$\mathrm{can} : A \otimes_{A^{\underline{H}}} A \to A \otimes H$$

is an isomorphism).

Now applying Theorem 5.6 we get

6.7. Theorem. *Let $\mathbb{H} = (\bar{H} = (H, e_H, m_H), \underline{H} = (H, \varepsilon_H, \delta_H))$ be a bialgebra in $\mathcal{V}$, let $\mathbb{A} = ((A, \alpha_A), e_A, m_A)$ be a right $\mathbb{H}$-comodule algebra, and let $\lambda_{(A,\alpha_A)} : H \otimes A \to A \otimes H$ be the corresponding entwining structure. Suppose that H is flat. Then the following are equivalent*:

(i) *The functor*

$$\bar{F} : \mathcal{V}_{A^{\underline{H}}} \to \mathcal{V}_{\mathbb{A}}^{\underline{H}}(\lambda_{(A,\alpha_A)}),$$
$$(X, \nu_X) \to (X \otimes_{\mathbb{A}^{\underline{H}}} A, X \otimes_{\mathbb{A}^{\underline{H}}} m_A, X \otimes_{\mathbb{A}^{\underline{H}}} \alpha_A)$$

is an equivalence of categories.

(ii) *A is $\underline{H}$-Galois and ${}_{\mathbb{A}^{\underline{H}}}A$ is faithfully flat.*

Let $\mathbb{H} = (\bar{H} = (H, e_H, m_H), \underline{H} = (H, \varepsilon_H, \delta_H))$ be a bialgebra in $\mathcal{V}$, and let $\mathbb{A} = ((A, \alpha_A), e_A, m_A)$ be a right $\mathbb{H}$-comodule algebra. A right $(\mathbb{A}, \mathbb{H})$-module is a right A-module which is a right $\underline{H}$-comodule such that the $\underline{H}$-comodule structure morphism is a morphism of right A-modules. Morphisms of right $(\mathbb{A}, \mathbb{H})$-modules are right A-module right $\underline{H}$-comodule morphisms. We write $\mathcal{V}_{\mathbb{A}}^{\mathbb{H}}$ for this category. Note that the category $\mathcal{V}_{\mathbb{A}}^{\mathbb{H}}$ is the category $(\mathcal{V}^{\underline{H}})_{\mathbb{A}}$ of right $\mathbb{A}$-modules in the monoidal category $\mathcal{V}^{\underline{H}}$, and it follows from Theorem 2.1 that

6.8. Proposition. *In the situation of the previous theorem, $\mathcal{V}_{\mathbb{A}}^{\mathbb{H}} = \mathcal{V}_{\mathbb{A}}^{\underline{H}}(\lambda_{(A,\alpha_A)})$.*

The following is an immediate consequence of Theorem 6.6.

6.9. Theorem. *Let $\mathbb{H} = (\bar{H} = (H, e_{\mathbb{H}}, m_{\mathbb{H}}), \underline{H} = (H, \varepsilon_{\mathbb{H}}, \delta_{\mathbb{H}}))$ be a bialgebra in $\mathcal{V}$, and let $\mathbb{A} = ((A, \alpha_A), e_A, m_A)$ be a right $\mathbb{H}$-comodule algebra. Then the functor*

$$\bar{F} : \mathcal{V}_{A^{\underline{H}}} \to \mathcal{V}_{\mathbb{A}}^{\mathbb{H}}$$

is an equivalence of categories iff the extension-of-scalars functor

$$F_{i_A} : \mathcal{V}_{A^{\underline{H}}} \to \mathcal{V}_A$$

is comonadic and A is $\underline{H}$-Galois.

Let $\mathbb{H} = (\bar{H} = (H, e_H, m_H), \underline{H} = (H, \varepsilon_H, \delta_H), S)$ be a Hopf algebra in $\mathcal{V}$. Then clearly $\bar{H} = (H, e_H, m_H)$ is a right $\mathbb{H}$-comodule algebra.

6.10. Proposition. *In the above situation, the composite*

$$x : H \otimes H \xrightarrow{H \otimes \delta_H} H \otimes H \otimes H \xrightarrow{m_H \otimes H} H \otimes H$$

is an isomorphism.

Proof. We will show that the composite

$$y : H \otimes H \xrightarrow{H \otimes \delta_H} H \otimes H \otimes H \xrightarrow{H \otimes S \otimes H} H \otimes H \otimes H \xrightarrow{m_H \otimes H} H \otimes H$$

is the inverse for x. Indeed, consider the diagram

$$\begin{array}{ccccc}
H \otimes H & \xrightarrow{H \otimes \delta_H} & H \otimes H \otimes H & \xrightarrow{m_H \otimes H} & H \otimes H \\
{\scriptstyle H \otimes \delta_H}\downarrow & (1) & \downarrow{\scriptstyle H \otimes H \otimes \delta_H} & (2) & \downarrow{\scriptstyle H \otimes \delta_H} \\
H \otimes H \otimes H & \xrightarrow{H \otimes \delta_H \otimes H} & H \otimes H \otimes H \otimes H & \xrightarrow{m_H \otimes H \otimes H} & H \otimes H \otimes H \\
 & & {\scriptstyle H \otimes H \otimes S \otimes H}\downarrow & (3) & \downarrow{\scriptstyle H \otimes S \otimes H} \\
 & & H \otimes H \otimes H \otimes H & \xrightarrow{m_H \otimes H \otimes H} & H \otimes H \otimes H \\
 & & {\scriptstyle H \otimes m_H \otimes H}\downarrow & (4) & \downarrow{\scriptstyle m_H \otimes H} \\
 & & H \otimes H \otimes H & \xrightarrow[m_H \otimes H]{} & H \otimes H.
\end{array}$$

We have:

- Square (1) commutes because of coassociativity of δ_H;
- Square (2) commutes because of naturality of $m_H \otimes -$;
- Square (3) commutes because $- \otimes -$ is a bifunctor;
- Square (4) commutes because of associativity of m_H.

Then

$$\begin{aligned} yx &= (m_H \otimes H)(H \otimes S \otimes H)(H \otimes \delta_H)(m_H \otimes H)(H \otimes \delta_H) \\ &= (m_H \otimes H)(H \otimes m_H \otimes H)(H \otimes H \otimes S \otimes H)(H \otimes \delta_H \otimes H)(H \otimes \delta_H), \end{aligned}$$

but since

$$m_H(H \otimes S)\delta_H = e_H \cdot \varepsilon_H,$$

$$\begin{aligned} yx &= (m_H \otimes H)(H \otimes e_H\varepsilon_H \otimes H)(H \otimes \delta_H) \\ &= (m_H \otimes H)(H \otimes e_H \otimes H)(H \otimes \varepsilon_H \otimes H)(H \otimes \delta_H) \\ &= 1_{H\otimes H} \otimes 1_{H\otimes H} = 1_{H\otimes H}. \end{aligned}$$

Thus $yx = 1$. The equality $xy = 1$ can be shown in a similar way. □

6.11. Proposition. *In the situation of the previous proposition, there is an isomorphism*

$$(H, \delta_H)^{\underline{H}} \simeq (I, e_H).$$

Proof. We will first show that the diagram

$$\begin{array}{ccc} H & \underset{e_H \otimes H}{\overset{H\otimes e_H}{\rightrightarrows}} & H \otimes H \\ \| & & \downarrow{\scriptstyle x} \\ H & \underset{e_H \otimes H}{\overset{\delta_H}{\rightrightarrows}} & H \otimes H \end{array}$$

is serially commutative. Indeed, we have:

$$\begin{aligned} x(H \otimes e_H) &= (m_H \otimes H)(H \otimes \delta_H)(H \otimes e_H) = \text{since } \delta_H \text{ is an algebra morphism} \\ &= (m_H \otimes H)(H \otimes e_H \otimes e_H) = \text{since } e_H \text{ is the unit for } m_H \\ &= H \otimes e_H; \\ x(e_H \otimes H) &= (m_H \otimes H)(H \otimes \delta_H)(e_H \otimes H) = \text{since } e_H \text{ is a coalgebra morphism} \\ &= (m_H \otimes H)(e_H \otimes H)\delta_H = 1_H\delta_H = \delta_H. \end{aligned}$$

Thus, $(H, \delta_H, e_H)^{\underline{H}}$ is isomorphic to the equalizer of the pair $(H \otimes e_H, e_H \otimes H)$. But since $e_H : I \to H$ is a split monomorphism in $\mathcal{V}$, the diagram

$$I \xrightarrow{e_H} H \underset{e_H \otimes H}{\overset{H \otimes e_H}{\rightrightarrows}} H \otimes H$$

is an equalizer diagram. Hence $(H, \delta_H, e_H)^{\underline{H}} \simeq (I, e_H)$. □

The following result can be seen as an extension of the structure theorem on ordinary Hopf modules over a k-Hopf algebra, k being a field, (see [12, p. 84]) to braided monoidal categories.

6.12. Theorem. *Let $\mathbb{H} = (\bar{H} = (H, e_H, m_H), \underline{H} = (H, \varepsilon_H, \delta_H), S)$ be a Hopf algebra in $\mathcal{V}$. Then the functor*

$$\mathcal{V} \to \mathcal{V}_{\mathbb{H}}^{\mathbb{H}},$$
$$V \to V \otimes H$$

is an equivalence of categories.

Proof. It follows from Propositions 6.10 and 6.11 that H is $\underline{H}$-Galois, and according to Theorem 6.6, the functor $\mathcal{V} \to \mathcal{V}_{\mathbb{H}}^{\mathbb{H}}$ is an equivalence iff the functor $- \otimes H : \mathcal{V} \to \mathcal{V}_{\bar{H}}$ is comonadic. But since the morphism $e_H : I \to H$ is a split monomorphism in $\mathcal{V}$, the unit of the adjunction $F_{e_H} \dashv U_{e_H}$ is a split monomorphism, and since any category admitting equalizers is Cauchy complete, it follows from 3.16 of [11] that F_{e_H} is comonadic. This completes the proof. □

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