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THE RELATION BETWEEN THE BASIC AND CONDITIONAL UTILITY MAXIMIZATION PROBLEMS

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Abstract. We study basic and conditional utility maximization problem in incomplete markets for utility functions defined on the whole real line and establish relations between optimal strategies of these problems.

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1. Introduction

We consider an incomplete financial market model, where the dynamics of asset prices is described by the continuous semimartingale S defined on the filtered probability space $(\Omega, \mathcal{F}, F = (F_t)_{t\in[0,T]}, P)$ satisfying the usual conditions, where $\mathcal{F} = F_T$ and $T < \infty$.

Denote by $\mathcal{M}^e$ (resp. $\mathcal{M}^a$) the set of probability measures Q equivalent (resp. absolutely continuous with respect) to P such that S is a local martingale under Q. We assume that $\mathcal{M}^e \neq \varnothing$.

Let $U = U(x) : R \to R$ be a utility function taking finite values at all points of real line R such that U is differentiable, strictly increasing, strictly concave and satisfies the Inada conditions (see [4] for details).

The predictable, $S-$integrable process π we call admissible if the stochastic integral $(\int_0^t \pi_u dS_u, t \in [0,T])$ is uniformly bounded from below.

We consider the utility maximization problem of terminal wealth and the value function V associated to the problem is defined by

$$V(x) = \sup_{\pi\in\Pi} E\left[U\left(x + \int_0^T \pi_u\, dS_u\right)\right], \tag{1}$$

where Π is the class of admissible strategies.

Let τ be a stopping time valued in $[0,T]$.

Denote by Π_τ the class of admissible processes, such that $\pi = \pi 1_{[\tau,T]}$. Define $\mathcal{Z}_{\tau,y} = \{Y : Y = y\frac{\rho_T}{\rho_\tau},\ \rho_T = \frac{dQ}{dP},\ Q \in \mathcal{M}^e(S)\}$.

The dynamic value functions of primal and dual problems are defined as

$$V(\tau, x) = \operatorname*{ess\,sup}_{\pi\in\Pi_\tau} E\left[U\left(x + \int_\tau^T \pi_u\, dS_u\right)\middle| F_t\right], \tag{2}$$

$$\widetilde{V}(\tau, y) = \operatorname*{ess\,inf}_{Y\in\mathcal{Z}_{\tau,y}} E\left[\widetilde{U}(Y) \mid F_t\right], \quad y > 0, \tag{3}$$

where $\widetilde{U} = \sup_x (U(x) - xy)$. For $V(0,x)$ and $\widetilde{V}(0,y)$ we use the notations $V(x)$ and $\widetilde{V}(y)$ respectively. Following [5] we make

Assumption 1. For each $y > 0$ the dual value function $\widetilde{V}(y)$ is finite and the minimizer $Q^*(y) \in \mathcal{M}^e$ (called the minimax martingale measure) exists. Let Π_x be the class of predictable S integrable processes π such that $U(x + (\pi \cdot S)_T) \in L^1(P)$ and $\pi \cdot S$ is a supermartingale under each $Q \in \mathcal{M}^a$ with finite $\widetilde{U}$-expectation $E\widetilde{U}(\frac{dQ}{dP})$, where the notation $\pi \cdot S$ stands for the stochastic integral.

Denote $Q(x) = Q^*(y) = Q^*(V'(x))$.

It was proved in [4] that optimal strategy $\pi(x) \in \Pi_x$ of problem (1) exists, is unique and $V(x) = EU(X_T(x))$, where the optimal wealth $X_T(x) = x + \int_0^T \pi_u(x)\,dS_u$ is a uniformly integrable $Q(x)$-martingale.

Besides, the following duality relations hold true almost surely (see [5] for the dynamic version)

$$U^{'}(X_T(x)) = Z_T(y), \quad y = V^{'}(x), \tag{4}$$

$$V^{'}\left(t, x + \int_0^t \pi_u(x)\,dS_u\right) = Z_t(y), \quad t \in [0,T]. \tag{5}$$

Our aim is to investigate whether Assumption 1 implies an existence of an optimal strategy to the conditional maximization problem (2) and how is this strategy related to the optimal strategy of the basic problem (1).

It was shown in [5] that if we start at time τ with the optimal wealth $X_\tau(x)$ then the optimal value in (2) is attained by $\pi(\tau,x) = \pi(0,x)I_{]\tau,T]}$, i.e.,

$$E[U(X_T(x))|F_\tau] \geq E[U(X_\tau(x) + \int_\tau^T \pi_u dS_u) \mid F_\tau], \ \pi \in \Pi_\tau,$$

which is well understood from the Bellman Principle.

We shall show that if we start at time τ with the wealth equal to arbitrary amount x, then the optimal strategy $\pi(\tau,x)$ of (2) is expressed in terms of the optimal strategy $\pi(x) = \pi(0,x)$ and the optimal wealth $X_\tau(x) = X_\tau(0,x)$ of (1) at time τ by the equality

$$\pi(\tau,x) = \pi(X_\tau^{-1}(x)), \quad \mu^{\langle S \rangle} - a.e.$$

To this end we first give some definitions and auxiliary assertions.

We shall say that an adapted stochastic process $(X_t, t \in [\tau,T])$ is a generalized martingale (resp. supermartingale) if

1) $E(|X_t|/F_\tau) < \infty$, for any $t \in [\tau,T]$

2) $E(X_t/F_{t'}) = X_{t'}$ (resp. $\leq X_{t'}$) for any $t' \leq t$, where $t', t \in [\tau,T]$

(see the definition of generalized conditional expectations and of generalized supermartingales for discrete time in [7])

Definition. A predictable S integrable process π is in $\Pi_{x,\tau}$, if $E(U(x + \int_\tau^T \pi_u dS_u)/F_\tau)$ is finite and $((\pi \cdot S)_t, t \geq \tau)$ is a generalized supermartingale under each $Q \in \mathcal{M}^a$ with finite $\widetilde{U}$-expectation $E\widetilde{U}(\frac{dQ}{dP})$.

We shall also need two complementary assumptions

Assumption 2. The filtration F is continuous and $\liminf_{y\to\infty} Z_T(y)/y > 0$ for the process $Z_T(y) = y\frac{dQ^*(y)}{dP} = y\rho^*_T(y)$.

Assumption 3. The utility function U is two times differentiable and there are constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 < R(x) = -\frac{U''(x)}{U'(x)} < c_2, \quad x \in R. \tag{6}$$

The last condition is similar to the condition on relative risk-aversion introduced in [1] for utility functions defined on the positive real line.

The proof of the following assertion follows from Theorem 4.1 and Proposition 3.1 of [3].

Proposition 1. Let Assumptions 1-3 are satisfied.

Then for any $t \in [0, T]$ there exists a modification of the optimal wealth process $(X_t(x), x \in R)$ (resp. of $Z_t(y)$) almost all paths of which are strictly increasing and absolutely continuous with respect to dx (resp. dy). Besides

$$X'_t(x) > 0, \quad E^{Q(x)}(X'_T(x))^2 \le C, \tag{7}$$

$$\lim_{x\to\infty} X_t(x) = \infty, \quad \lim_{x\to-\infty} X_t(x) = -\infty \tag{8}$$

P-a.s. for any $t \in [0, T]$ and the adapted inverse $X_t^{-1}(x)$ (resp. $Z_t^{-1}(y)$) of the optimal wealth process exists.

We shall need also the continuity properties of the square characteristics $\langle X(x) - X(y)\rangle$ which can be deduced from Proposition 1.

Lemma 1. *Let conditions of Proposition 1 be satisfied. Then, for any $t \in [0, T]$ the random field $(\langle X(x) - X(y)\rangle_t, x, y \in R)$ admits a continuous modification.*

Proof. It follows from Proposition 1 that $X_t(b) - X_t(a) = \int_a^b X'_t(x)dx$ and

$$\int_a^b E^{Q(x)}\langle X'(x)\rangle_T dx = \int_a^b E^{Q(x)} X'_T(x)^2 dx < \infty$$

and by the Fubini theorem $\int_a^b \frac{U'(X_T(x))}{V'(x)}\langle X'(x)\rangle_T dx < \infty,\ P-a.s.$ Thus by continuity of $\frac{V'(x)}{U'(X_T(x))}$ we obtain

$$\int_a^b \langle X'(x)\rangle_T dx \le \max_{x\in[a,b]} \frac{V'(x)}{U'(X_T(x))} \int_a^b \frac{U'(X_T(x))}{V'(x)}\langle X'(x)\rangle_T dx < \infty,\ P-a.s.$$

Therefore, using the Kunita-Watanabe and Hölder's inequalities we have

$$\langle X(b) - X(a)\rangle_t = \int_a^b\int_a^b \langle X'(x), X'(y)\rangle_t dxdy$$

$$\le \int_a^b\int_a^b \langle X'(x)\rangle_t^{1/2}\langle X'(y)\rangle_t^{1/2} dxdy = \left(\int_a^b \langle X'(x)\rangle_t^{1/2} dx\right)^2$$

$$\le (b-a)\int_a^b \langle X'(x)\rangle_t dx < \infty,\ P-a.s.$$

and it follows from inequality

$$\langle X(b') - X(a')\rangle_t - \langle X(b) + X(a)\rangle_t$$
$$\leq \langle X(b') - X(b)\rangle_t^{1/2} \langle X(b') - X(a') + X(b) - X(a)\rangle_t^{1/2}$$
$$+\langle X(a') - X(a)\rangle_t^{1/2} \langle X(b') - X(a') + X(b) - X(a)\rangle_t^{1/2}$$

that $\langle X(b_n) - X(a_n)\rangle_t \to \langle X(b) - X(a)\rangle_t,\ P - a.s.$ when $b_n \to b,\ a_n \to a$. Thus the stochastic field defined by

$$\langle X(x)-X(y)\rangle_t^* = \begin{cases} \lim_{r\to a, r'\to b} \langle X(r) - X(r')\rangle_t, & r, r' \ are\ rational, \\ 0, & if\ the\ limit\ does\ not\ exists \end{cases}$$

is continuous and stochastically equivalent to $\langle X(x) - X(y)\rangle_t$.

Theorem 1. Let Assumptions 1-3 be satisfied. Then there exist the maximizer of (2) and the minimizer of (3) in the classes $\Pi_{\tau,x}$ and $\mathcal{Z}_{\tau,y}$ respectively and equalities

$$X_T(\tau, x) = X_T(X_\tau^{-1}(x)),\ \pi_t(\tau, x) = \pi_t(X_\tau^{-1}(x)), t \geq \tau, \tag{9}$$

$$Y(\tau, y) = Z_T(Z_\tau^{-1}(y)),\ \rho_T^{Q*}(\tau, y) = \rho_\tau^*(y)\frac{Z_T(Z_\tau^{-1}(y))}{y} \tag{10}$$

are satisfied.

Moreover P-a.s.

$$V(\tau, x) = E\Big[U\Big(x + \int_\tau^T \pi_u(X_\tau^{-1}(x))dS_u\Big) \mid F_\tau\Big], \tag{11}$$

$$\widetilde{V}(\tau, y) = E\Big[\widetilde{U}(Z_T(Z_\tau^{-1}(y))) \mid F_\tau\Big],$$

the following duality relation holds

$$U'\left(x + \int_\tau^T \pi_u(X_\tau^{-1}(x))dS_u\right) = Z_T(Z_\tau^{-1}(y)),\ y = V'(\tau, x) \tag{12}$$

and the process

$$Z_t(Z_\tau^{-1}(y))X_t(X_\tau(x)^{-1}(x)),\quad t \in [\tau, T],\ \ where\ \ y = V'(\tau, x), \tag{13}$$

is a generalized martingale.

Proof. By the optimality principle (see, e.g. [2]) $V(t, X_t(x))$ is a martingale and since $V(T, x) = U(x)$ we have that for any $x \in R$

$$V(\tau, X_\tau(x)) = E\big(U(X_T(x))/\mathcal{F}_\tau\big)\quad P - a.s. \tag{14}$$

Since for any τ the functions $V(\tau, x)$ and $X_\tau(x)$ are continuous for almost all $\omega \in \Omega$, the equality (14) holds P-a.s. for all $x \in R$ and substituting $X_\tau^{-1}(x)$ in this equality we obtain that

$$V(\tau, x) = E\big(U(X_T(X_\tau^{-1}(x)))/F_\tau\big)\quad P - a.s.,$$

which means the maximality of $X_T(X_\tau^{-1}(x))$. Let us show that $X_T(X_\tau^{-1}(x))$ is equal to the stochastic integral

$$X_T(X_\tau^{-1}(x)) = x + \int_\tau^T \pi_u(X_\tau^{-1}(x))dS_u \tag{15}$$

and that $\pi(X_\tau^{-1}(x))$ belongs to the class $\Pi_{\tau,x}$

In order to show equality (15) it is enough to show that $\int_\tau^T \pi_u(x)dS_u\big|_{x=\xi} = \int_\tau^T \pi_u(\xi)dS_u$, for $\xi = X_\tau^{-1}(x)$.

Let us consider the sequence of simple random variables $\xi_n = \sum_{k=-\infty}^{\infty} c_k 1_{A_k}$, where $A_k = (\frac{k}{n} \le \xi < \frac{k+1}{n}),\ c_k = \frac{k}{n}$. We have $\xi_n \to \xi$ uniformly and

$$\int_\tau^T \pi_u(\xi_n)dS_u = \sum_{k=-\infty}^{\infty} \int_\tau^T \pi_u(c_k)1_{A_k}dS_u$$

$$= \sum_{k=-\infty}^{\infty} 1_{A_k} \int_\tau^T \pi_u(c_k)dS_u = \int_\tau^T \pi_u(x)dS_u\big|_{x=\xi_n}.$$

On the other hand

$$\int_\tau^T \pi_u(x)dS_u\big|_{x=\xi_n} - \int_\tau^T \pi_u(x)dS_u\big|_{x=\xi}$$

$$= X_T(\xi_n) - X_\tau(\xi_n) - (X_T(\xi) - X_\tau(\xi)) \to 0,$$

as $n \to \infty$, since $X_t(x)$ is continuous and

$$\int_\tau^T (\pi_u(\xi_n) - \pi_u(\xi))^2 d\langle S\rangle_u =$$

$$= \langle X(x) - X(y)\rangle_T - \langle X(x) - X(y)\rangle_\tau|_{x=\xi_n, y=\xi} \to 0, P-a.s.$$

as $n \to \infty$, by continuity of $\langle X(x) - X(y)\rangle_t$. Hence $\int_\tau^T \pi_u(\xi_n)dS_u \to \int_\tau^T \pi_u(\xi)dS_u$ in probability and $\int_\tau^T \pi_u(x)dS_u\big|_{x=\xi} = \int_\tau^T \pi_u(\xi)dS_u - P.a.s.$.

Since $E|U(X_T(x))| < \infty$ and $E^Q|X_t(x)| < \infty, t \in [0,T]$ for any $Q \in \mathcal{M}^a$ and $X_\tau^{-1}(x)$ is F_τ-measurable we have that

$$E[|U(X_T(X_\tau^{-1}(x)))| \mid F_\tau] < \infty, \quad E^Q(|X_t(X_\tau^{-1}(x))|/F_\tau) < \infty \quad P-a.s., t \ge \tau$$

On the other hand, since for any $t \in [0,T]$ the function $(X_t(x), x \in R)$ is continuous and increasing, the supermartingale inequality $E^Q(X_t(x)/F_{t'}) \le X_{t'}(x), \quad t' \le t \le T$ implies that

$$E^Q(X_t(X_\tau^{-1}(x))/F_{t'}) \le X_{t'}(X_\tau^{-1}(x)), \quad \tau \le t' \le t \le T$$

for any $Q \in \mathcal{M}^a$, hence $\pi(\tau, x) = \pi(X_\tau^{-1}(x)$ belongs to the class $\Pi_{\tau,x}$ and the equality (11) holds. Similarly one can show the minimality of $Z_T(Z_\tau^{-1}(y))$, so conditional density of the minimax martingale measure to the problem (2) is $\frac{Z_T(Z_\tau^{-1}(y))}{y}$.

Since for any $t \in [0,T]$ the functions $V'(t,x), x \in R$ and $Z_t(y), y > 0$ are continuous and the inverse of $Z_t(y)$ exists, from (5) we have that P-a.s.

$$Z_\tau^{-1}(V'(\tau,x)) = V'(X_\tau^{-1}(x)) \tag{16}$$

which, together with (4) implies the conditional duality relation (12).

Note also that since $Z_t(y)X_t(x)$ is a martingale (see Theorem 1 from [5]), by continuity of $X(x)$ and $Z(y)$ the process $(Z_t(V'(X_\tau^{-1}))X_t(X_\tau^{-1}(x)), t \geq \tau)$ will be a generalized martingale and by equality (16) this is equivalent to (13). □

Now we make additional

Assumption 4. Each F_t is countably generated and there exists a regular, conditional probability measure $Q^{\tau,\omega}$ of each measure Q given F_τ, where $\omega \in \Omega_Q$ for some $\Omega_Q \in F_\tau$ with $Q(\Omega_Q) = 1$. From Theorem 1.2.10 of [6] we have that $X_t(x),\ t \in [0,T]$ is a $Q-$martingale if and only if $X_{t\wedge\tau}(x),\ t \in [0,T]$ is a $Q-$martingale and $X_t(x) - X_{t\wedge\tau}(x),\ t \in [0,T]$ is $Q^{\tau,\omega}-$martingale.

Let us introduce classes

$$\begin{aligned}\Pi^1_{\tau,\omega,x} =&\{\pi : \pi \textit{ is predictable}, S-\textit{integrable}, E^{\tau,\omega}\left(U(x+\int_\tau^T \pi_u dS_u\right) < \infty \\ &\textit{and } Z_t(V'(X_\tau^{-1}(x)))\int_\tau^t \pi_u dS_u,\ t \geq \tau \textit{ is a } P^{\tau,\omega} - \textit{martingale}\}.\end{aligned}$$

$$\begin{aligned}\Pi^2_{\tau,\omega,x} =&\{\pi : \pi \textit{ is predictable}, S-\textit{integrable}, E^{\tau,\omega}(U(x+\int_\tau^T \pi_u dS_u) < \infty \\ &\textit{and } \frac{\rho_t^Q}{\rho_\tau^Q}\int_\tau^t \pi_u dS_u,\ t \geq \tau \textit{ is a } P^{\tau,\omega} - \textit{supermartingale},\ Q \in \mathcal{M}^a\}.\end{aligned}$$

It is evident that

$$V(\tau,x,\omega) = \operatorname*{ess\,sup}_{\pi\in\Pi^1_{\tau,\omega,x}} E^{\tau,\omega}U\left(x + \int_\tau^T \pi_u\, dS_u\right),\ \omega \in \Omega_P. \tag{17}$$

Since $\frac{\rho_T^Q}{\rho_\tau^Q} = \frac{dQ^{\tau,\omega}}{dP^{\tau,\omega}}$ for $Q \in \mathcal{M}^e(S),\ \textit{with } \rho_T^Q = \frac{dQ}{dP}$, we can define

$$\mathcal{Z}_{\tau,\omega,y} = \{Y : Y = y\frac{dQ^{\tau,\omega}}{dP^{\tau,\omega}},\ Q \in \mathcal{M}^e(S)\}.$$

Similarly we have

$$\widetilde{V}(\tau,y,\omega) = \operatorname*{ess\,inf}_{Y\in\mathcal{Z}_{\tau,\omega,y}} E^{\tau,\omega}\widetilde{U}(Y). \tag{18}$$

Hence we obtain

Theorem $1'$**.** *Let Assumptions 1-4 are satisfied. Then equalities (9), (10) are valid and*

$$\sup_{\pi\in\Pi_{\tau,\omega,x}} E^{\tau,\omega}\Big[U\Big(x+\int_\tau^T \pi_u\,dS_u\Big)\Big] = E^{\tau,\omega}\Big[U\Big(X_T(X_\tau^{-1}(x))\Big)\Big],$$

$$\inf_{Q\in\mathcal{M}^e} E^{\tau,\omega}\Big[\widetilde{U}\Big(V'(\tau,x)\frac{\rho_T^Q}{\rho_\tau^Q}\Big)\Big] = E^{\tau,\omega}\Big[\widetilde{U}(Z_T(Z_\tau^{-1}(V'(\tau,x))))\Big].$$

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