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Applications of Stochastic Processes and Mathematical Statistics to Financial Economics and Social Sciences IV

Tbilisi Science and Innovation Festival 2019

Conference Materials, 25-26 September, 2019, Tbilisi, GAU

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Content

Martingale Method of Solving Lobachevsky's Functional Equation

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Abstract. The aim of this paper is to give a probabilistic (martingale) method to find the general measurable solution of the Lobachevsky functional equation. We show that to find general solution of this equation is equivalent to establish that a space-transformation of a Brownian Motion by suitable function is a martingale. This method can be applied for Cauchy's, Jensen's, Pexider's and other functional equations.

1 Introduction

We consider the Lobachevsky functional equation

$$f(x)(f(y) = f^2\big(\frac{x+y}{2}\big), \quad \text{for all} \quad x, y \in R, \tag{1}$$

where $f = (f(x), x \in R)$ is a real valued function (see, e.g. Aczel [1] about this and other functional equations and related results). It was shown by Neamptu [5] that if $f, f(0) \neq 0$ is a solution of (1) bounded on a neighborhood $(-r, r)$ of zero, then $f(x) = f(0)e^{\lambda x}$, for some $\lambda \in R$.

We give a martingale method to find the general measurable solution of the Lobachevsky functional equation. We don't require the boundedness on $(-r, r)$ for solutions of (1), but consider measurable solutions. We show that

f is a measurable solution of functional equation (1) if and only if the process $M_t = \ln f(0)f(W_t), t \geq 0$ is a martingale.

To this end we are using two facts from probability theory:

The first one is the Bernstein theorem ([3]) (see also [6] for definitive form) according to which if X and Y are independent random variables such that the random variables $X+Y$ and $X-Y$ are also independent, then X and Y admit normal distribution. Bernstein's theorem was used by S. Smirnov [7] to show that any measurable solution of Caushy's functional equation is locally integrable. We use this idea from [7] to show the integrability of the transformed process of Brownian motion $f(W_t)$.

The second assertion we used is that if the transformed process $g(W_t)$ is a martingale, then the function g is linear

$$g(x) = ax + b, \quad \text{for some constants} \quad a \in R, b \in R.$$

This fact follows from results of [2] or [4], where the semimartingale functions of Brownian motion are studied.

2 The proof of the main result

First we mention some simple properties of equation (1) which will be used in what follows. It is obvious and well known (see [5]) that a solution of (1) is either everywhere or nowhere 0 and if $f(0) \neq 0$ then

$$sign f(x) = sign f(0). \tag{2}$$

Indeed, if $f(x_0) = 0$ for some $x_0 \in R$, then $f^2(x) = f(x_0)f(2x - x_0) = 0$ for all $x \in R$ and if $f(0) \neq 0$ it follows from (1) by taking $y = 0$ that

$$f(0)f(x) = f^2\big(\frac{x}{2}\big) > 0, \tag{3}$$

which implies (2).

It is easy to see that the function g defined by $g(x) = \ln \frac{f(x)}{f(0)}$ is odd, since for $y = -x$ we have $f(x)f(-x) = f^2(0)$, which is equivalent to $\frac{f(x)}{f(0)}\frac{f(-x)}{f(0)} = 1$ and implies (since $f(x)/f(0) > 0$ for all $x \in R$) that

$$\ln \frac{f(x)}{f(0)} + \ln \frac{f(-x)}{f(0)} = 0. \tag{4}$$

Let $W = (W_t, t \geq 0)$ be a standard Brownian Motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and denote by $F = (\mathcal{F}_t, t \geq 0)$ the filtration generated by the Brownian Motion W. Assume that $\mathcal{F}_t$ is completed by sets from $\mathcal{F}$ having P-measure zero. As it is well known, all martingales with respect to such filtration are continuous. We consider martingales only with respect to this filtration.

Theorem 1. Let $(f(x), x \in R)$ be a measurable function and $f(0) \neq 0$. Then the following assertions are equivalent:

a) the function f is a solution of the functional equation (1).

b) The process $M_t = \ln f(0)f(W_t), t \geq 0$ is a martingale.

c) $f(x) = f(0)e^{\lambda x}$ for some constant $\lambda \in R$.

Proof. *a*) $\to$ *b*). Assume that f is a measurable solution of (1) with $f(0) \neq 0$. Let us show that the process $(\ln f(0)f(W_t), t \geq 0)$ is a martingale. Let first show that

$$E|lnf(0)f(W_t)| < \infty$$

for all $t \geq 0$. Let

$$X = f(0)f(W_t) \quad \text{and} \quad Y = f(0)f(B_t),$$

where B_t is a Brownian motion independent of W_t. It follows from (1) and (3) that

$$XY = f^2(0)f(W_t)f(B_t) = f^2(0)f\Big(\frac{W_t + B_t}{2}\Big) = f^3(0)f(W_t + B_t). \quad (5)$$

On the other hand, substituting $x = W_t - B_t, y = B_t$ in (1) we have from (3) that

$$f(W_t - B_t)f(B_s) = f^2\Big(\frac{W_t}{2}\Big) = f(0)f(W_t),$$

which implies that

$$\frac{X}{Y} = \frac{f(W_t)}{f(B_t)} = \frac{f(W_t - B_t)}{f(0)}. \quad (6)$$

Since $W_t + B_t$ and $W_t - B_t$ are independent, equations (5) and (6) imply that the random variables XY and $\frac{X}{Y}$ will be also independent. Therefore, it follows from Bernstein's theorem that $X = (f(0)f(W_t)$ (and $Y = f(0)f(B_t))$ will have the lognormal distribution and $lnf(0)f(W_t)$ admits the normal distribution, hence $lnf(0)f(W_t)$ is integrable for any $t \geq 0$.

Let us show now the martingale equality.

Substituting $x = W_t - W_s, y = W_s$ in (1) we have

$$f(W_t - W_s)f(W_s) = f^2(\frac{W_t}{2}) = f(0)f(W_t). \tag{7}$$

Multiplying both parts of (7) by $f^2(0)$ and taking logarithms we obtain that

$$\ln f(0)f(W_t - W_s) + \ln f(0)f(W_s) = \tag{8}$$

$$= \ln f^3(0)f(W_t) = \ln f^2(0) + \ln f(0)f(W_t)$$

which implies that

$$\ln f(0)f(W_t) - \ln f(0)f(W_s) = \ln f(0)f(W_t - W_s) - \ln f^2(0) =$$

$$= \ln \frac{f(W_t - W_s)}{f(0)}. \tag{9}$$

By independent increment property of the Brownian motion $\ln f\,(W_t - W_s)$ is independent of F_s and taking conditional expectation in (9) we have that $P - a.s.$

$$E\big(\ln f(0)f(W_t) - \ln f(0)f(W_s)/F_s\big) = E\big(\ln \frac{f(W_t - W_s)}{f(0)}/F_s\big) =$$

$$= E \ln \frac{f(W_t - W_s)}{f(0)}. \tag{10}$$

But $E \ln \frac{f(W_t - W_s)}{f(0)} = 0$ since by equality (4) the function $\ln \frac{f(x)}{f(0)}$ is odd and $W_t - W_s$ is symmetrically distributed.

Thus, for any $s, t, s \le t$

$$E\big(\ln f(0)f(W_t) - \ln f(0)f(W_s)/F_s\big) = 0; \quad P - \text{a.s}$$

and the process $(\ln f(0)f(W_t), t \ge 0)$ is a martingale.

$b) \to c)$ Now let us assume that process $\ln f(0)f\,(W_t)$ is a martingale and $f(0) \ne 0$. This implies that the function $\ln f(0)f(x)$ is linear

$$\ln f(0)f\,(x) = \lambda x + c, \tag{11}$$

for some constants $\lambda \in R$ and $c \in R$. Hence

$$f(0)f(x) = \exp\{\lambda x + c\} = e^c e^{\lambda x},$$

which implies (taking $x = 0$ in this equality) that $e^c = f^2(0)$ and since $f(0) \neq 0$, we obtain that $f(x) = f(0)e^{\lambda}$.

$c) \to a)$ It is easy to verify that the function $f(x) = f(0)e^{\lambda}$ satisfies equation (1).

Remark. Since $f(0) = 0$ implies that $f(x) = 0$ for all $x \in R$ and $f(x) = 0$ is a solution of (1),

$$f(x) = \alpha e^{\lambda x}, \quad \text{for some constants} \quad \alpha \in R, \lambda \in R$$

will be the most general solution of (1). □

References

[1] J.Aczel, Lectures on Functional Equations and their Applications,Academic Press 1966.

[2] Cinlar, E., Jacod, C., Protter,P. and Sharpe, M.J., Semimartingales and Markov processes. Z. Warscheinlichkeitstheor. Verw. Geb. V. 54, (1980), p.161-218.

[3] S. N. Bernstein, Ob odnom svoystve, harakterizuyushhem zakon Gaussa (in Russian) [On a characteristic property of the normal law]. Tr. leningr.Polytech. Inst.3, (1941), pp. 21-22

[4] M. Mania and R. Tevzadze, Semimartingale functions for a class of diffusion processes. (in Russian) Teor. Veroyatnostei i Primenenija. 45 (2000), No. 2, 374-380; English transl.: Theory Probab. Appl. Vol. 45, No. 2 (2000), pp. 337-343 .

[5] N. Neamptu, About some Classical Functional Equations, Tr. J. of Mathematics, 22, (1998), pp. 119-126.

[6] M. P. Quine, On Three Characterizations of the Normal Distribution, Probability Theory and Mathematical Statistics, Vol 14, Fasc 2, (1993), pp. 257-263.

[7] S. N. Smirnov, A probabilistic note on the Cauchy functional equation, Aequat. Math.Vol. 93, Issue 2, (2019), pp. 445-449,