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ON MARTINGALE TRANSFORMATIONS OF THE LINEAR BROWNIAN MOTION

Michael Mania Revaz Tevzadze

Abstract. We describe the classes of functions $f = (f(x), x \in R)$, for which processes $f(W_t) - Ef(W_t)$ and $f(W_t)/Ef(W_t)$ are martingales.

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1 Introduction. Let $W = (W_t, t \geq 0)$ be a standard Brownian Motion defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with filtration $F = (\mathcal{F}_t, t \geq 0)$ satisfying the usual conditions. A function $f = (f(x), x \in R)$ is called a semimartingale function of the process X if the transformed process $(f(X_t), t \geq 0)$ is a semimartingale. It was shown in [4] that every semimartingale function of Brownian Motion is locally difference of two convex functions. In [1], [3] the description of time-dependent semimartingale functions of Brownian Motion and diffusion processes in terms of generalized derivatives was given. All these results imply that if $f(W_t)$ is a right-continuous martingale, then f is a linear function.

We generalize this assertion in two directions. We show that a) if $f(W_t)$ is only a martingale (without assuming the regularity of paths), then $f(x)$ is equal to the linear function almost everywhere with respect to the Lebesgue measure and b) if $f(W_t)/Ef(W_t)$ (resp. $f(W_t) - Ef(W_t)$) is a right-continuous martingale, then the function f is of the form $f(x) = ae^{\lambda x} + be^{-\lambda x}$ (resp. $f(x) = ax^2 + bx + c$) for some constants a, b, c and $\lambda \in R$.

2 Main results. The following theorem is the main result of the paper.

Theorem 1. *Let $f = (f(x), x \in R)$ be a strictly positive function such that $f(W_t)$ is integrable for every $t \geq 0$.*

a) If the process

$$N_t = \frac{f(W_t)}{Ef(W_t)}, \quad t \geq 0,$$

is a right-continuous (P-a.s.) martingale, then the function f is of the form

$$f(x) = ae^{\lambda x} + be^{-\lambda x}, \quad \text{for some} \quad \lambda, a, b \in R. \tag{1}$$

b) If the process N_t is a martingale, then the function $f(x)$ coincides with the function $ae^{\lambda x} + be^{-\lambda x}$ (for some $\lambda, a, b \in R$) almost everywhere.

Proof. a) Let $g(t) \equiv Ef(W_t)$. Since $E|f(W_t)| < \infty$ for all $t \geq 0$ the function $g(t)$ will be continuous for any $t > 0$. Since N_t is right-continuous and $g(t)$ is continuous, the process $f(W_t)$ will be also right-continuous. This implies that $f(x)$ is a continuous function.

Since $f(W_t)/g(t)$ is a martingale, we have

$$\frac{f(W_t)}{g(t)} = \frac{1}{g(T)} E(f(W_T)/\mathcal{F}_t) \tag{2}$$

$P - a.s.$ for all $t \leq T$.

Let

$$u(t,x) = E(f(W_T)/W_t = x).$$

Since f is positive, $u(t,x)$ will be of the class $C^{1.2}$ on $(0,T) \times R$ and satisfies the backward Kolmogorov equation (see, e.g. [2] page 257)

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < t < T, x \in R. \tag{3}$$

By the Markov property $u(t,W_t) = E(f(W_T)/\mathcal{F}_t)$ and from (2) we have

$$f(W_t) = \frac{g(t)}{g(T)} u(t,W_t) \quad a.s.$$

Therefore,

$$\int_R \left| f(x) - \frac{g(t)}{g(T)} u(t,x) \right| \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx = 0$$

which implies that for any $0 < t \leq T$

$$f(x) = \frac{g(t)}{g(T)} u(t,x) \quad a.e. \tag{4}$$

with respect to the Lebesgue measure. Since f and u are continuous, we obtain that for any $0 < t < T$

$$f(x) = \frac{g(t)}{g(T)} u(t,x) \quad \text{for all} \quad x \in R.$$

Since $g(t) > 0$ for all t, this implies that $g(t)$ is differentiable, $f(x)$ is two-times differentiable and for any $0 < t < T$

$$u(t,x) = \frac{g(T)}{g(t)} f(x) \quad \text{for all} \quad x \in R. \tag{5}$$

Therefore, it follows from (3) and (5) that

$$\frac{1}{2}\frac{g(T)}{g(t)} f''(x) - \frac{g(T)g'(t)}{g^2(t)} f(x) = 0,$$

which implies that

$$\frac{f''(x)}{f(x)} = 2\frac{g'(t)}{g(t)}. \tag{6}$$

Since the left-hand side of (6) does not depend on t and the right- hand side on x, both parts of (6) are equal to a constant which should be positive, since f and g are strictly positive (hence f'' and g' have the same sign). Therefore, we obtain

$$f''(x) = \lambda^2 f(x) \quad \text{and} \quad g'(t) = \frac{\lambda^2}{2} g(t).$$

for some constant $\lambda \in R$. Therefore,

$$f(x) = ae^{\lambda x} + be^{-\lambda x}, \quad g(t) = Ef(W_t) = (a+b)e^{\frac{\lambda^2}{2}t}.$$

b) Let $\tilde{f}(x) = \frac{g(0)}{g(T)} u(0,x)$.
It follows from (4) that

$$\lambda(x : f(x) \neq \tilde{f}(x)) = 0, \tag{7}$$

where by definition of $u(t,x)$ the function $\tilde{f}(x)$ is continuous. It follows from (7) that $P(f(W_t) = \tilde{f}(W_t)) = 1$ for any $t \geq 0$ and since $Ef(W_t) = E\tilde{f}(W_t)$, we obtain that for any $t \geq 0$

$$P\left(\frac{f(W_t)}{Ef(W_t)} = \frac{\tilde{f}(W_t)}{E\tilde{f}(W_t)}\right) = 1.$$

This implies that the process $\tilde{f}(W_t)/E\tilde{f}(W_t)$ is a continuous martingale and it follows from part a) of this theorem that $\tilde{f}(x)$ is of the form (1). Therefore, $f(x)$ coincides with the function $ae^{\lambda x} + be^{-\lambda x}$ almost everywhere. □

Theorem 2. *Let $f(W_t)$ be integrable for every $t \geq 0$.*

a) If the process $M = (f(W_t) - Ef(W_t), t \geq 0)$ is a right-continuous (P-a.s.) martingale, then the function f is of the form

$$f(x) = ax^2 + bx + c \quad \textit{for some} \quad \alpha, b \quad \textit{and} \quad c \in R. \tag{8}$$

b) If the process M_t is a martingale, then $f(x)$ coincides with the function $ax^2 + bx + c$ (for some $a, b, c \in R$) almost everywhere w. r. t. the Lebesgue measure.

Proof. a) Let $g(t) = Ef(W_t)$ and $F(t,x) = f(x) - g(t), t \geq 0, x \in R$. Similarly to the proof of Theorem 2 one can show that for any $t \leq T$

$$f(x) - g(t) + g(T) = u(t,x), \quad \text{a.e} \tag{9}$$

and by continuity of $f(x)$

$$f(x) - g(t) + g(T) = u(t,x), \quad \text{for all} \quad 0 \leq t \leq T, x \in R, \tag{10}$$

where $u(t,x) = E(f(W_T)/W_t = x)$ is a solution of the Kolmogorov backward equation (3). This implies that $g(t)$ is differentiable, $f(x)$ is two-times differentiable and it follows from (3) and (10)

$$\frac{1}{2}f''(x) = g'(t). \tag{11}$$

Since the left-hand side of (11) does not depend on t and the right-hand side on x, both parts of (6) are equal to a constant. Therefore, we obtain

$$f''(x) = 2a \quad \text{and} \quad g'(t) = a \quad \text{for some} \quad a \in R. \tag{12}$$

The solutions of these equations are

$$f(x) = ax^2 + bx + c \quad \text{and} \quad g(t) = at + c \tag{13}$$

respectively. The part b) is proved similarly to corresponding assertion of Theorem 1. □

Corollary. *Let $f = (f(x), x \in R)$ be a function of one variable.*

a) If the process $(f(W_t), \mathcal{F}_t, t \geq 0)$ is a right-continuous martingale, then

$$f(x) = bx + c \quad \textit{for all} \quad x \in R \quad \textit{for some} \quad b, c \in R. \tag{14}$$

b) If the process $(f(W_t), \mathcal{F}_t, t \geq 0)$ is a martingale, then $f(x) = bx + c$ (for some constants $b, c \in R$) almost everywhere.

Proof. If the process $f(W_t)$ is a martingale, then $g(t) = Ef(W_t)$ is a constant and the constant a in (13) is equal to zero. Therefore, this corollary follows from Theorem 2. □

REFERENCES

1. Chitashvili, R., Mania, M. On functions transforming Wiener process into semimartingale. *Probab. Theory Related Fields*, **109**, (1997), 57–76.
2. Karatzas, I., Shreve, S.E. Brownian Motion and Stochastic Calculus. *Springer*, 1991.
3. Mania, M., Tevzadze, R. Semimartingale functions for a class of diffusion processes. *Theory Probab. Appl.*, **45**, 2 (2000), 337–343.
4. Wang, A.T. Generalized Itô's formula and additive functionals of Brownian motion. *Zeit. für Warscheinlichkeit.*, **41** (1977), 153–159.

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Author(s) address(es):

Michael Mania
A. Razmadze Mathematical Institute
I. Javakhishvili Tbilisi State University
Tamarashvili str. 6, 0177 Tbilisi, Georgia
E-mail: misha.mania@gmail.com

Revaz Tevzadze
Georgian American University
M. Alexidze str. 8, 0193 Tbilisi, Georgia
E-mail: rtevzadze@gmail.com