Proceedings of A. Razmadze
Mathematical Institute
Vol. **130** (2002), 39–72

BACKWARD STOCHASTIC PDE AND HEDGING IN INCOMPLETE MARKETS

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Abstract. We consider a problem of minimization of a hedging error in an incomplete financial market model. The hedging error is measured by a positive strictly convex random function and the dynamics of asset price is given by a continuous one dimensional semimartingale defined on a complete probability space with continuous filtration. Under some regularity assumptions we derive a backward stochastic PDE for the value function of the problem and show that the strategy is optimal if and only if the corresponding wealth process satisfies a certain forward-SDE. As an example the case of mean-variance hedging is considered.

1. Introduction

Let S be a semimartingale with the decomposition

$$S_t = M_t + \int_0^t \lambda_s d\langle M\rangle_s, \tag{1.1}$$

where M is a continuous local martingale and λ is a predictable process. The process S is defined on a complete filtered probability space with continuous filtration and describes the dynamics of asset prices in an incomplet financial market.

2000 *Mathematics Subject Classification.* 90A09, 60H30, 90C39.
Key words and phrases. Backward stochastic partial differential equation, semimartingale market model, incomplete markets, mean-variance hedging.

We consider an optimization problem

$$\text{to minimize} \quad E\big[U(X_T^{x,\pi})\big] \quad \text{over all} \quad \pi\in\Pi, \tag{1.2}$$

where $X_t^{x,\pi}=x+\int\limits_0^t \pi_s dS_s$ is the wealth process starting from initial capital x, determined by the self-financing trading strategy π. U is an objective function eventually dependent on ω, which can be interpreted as a function which measures a hedging error and is assumed to be positive and strictly convex a.e..

Let us introduce the value function of the problem defined as

$$V(t,x)=\operatorname*{ess\,inf}_{\pi\in\Pi} E\bigg(U\Big(x+\int\limits_t^T \pi_s\,dS_s\Big)/\mathcal{F}_t\bigg), \tag{1.3}$$

where Π is a class of F-predictable $S-$integrable processes.

Under some regularity assumptions on the value function (sufficient for the application of the Itô–Ventzell formula) we show that this function satisfies the following backward stochastic partial differential equation (BSPDE)

$$\begin{aligned} V(t,x)=V(0,x)+\frac{1}{2}\int\limits_0^t \frac{(\varphi_x(s,x)+\lambda(s)V_x(s,x))^2}{V_{xx}(s,x)} d\langle M\rangle_s+\\ +\int\limits_0^t \varphi(s,x)dM_s+m(t,x), \quad V(T,x)=U(x), \end{aligned} \tag{1.4}$$

where $m(\cdot,x)$ is a local martingale orthogonal to M for all $x\in R$ and subscripts φ_x, V_x, V_{xx} stand for partial derivatives. Moreover, we show that the strategy π^* is optimal if and only if the corresponding wealth process $X^*=(X_t^*, t\in[0,T])$ satisfies the forward-SDE

$$X_t^*=x-\int\limits_0^t \frac{\varphi_x(u,X_u^*)+V_x(u,X_u^*)\lambda(s)}{V_{xx}(u,X_u^*)}\,dS_u. \tag{1.5}$$

If $U(x)=(x-H)^2$, where H is a contingent claim due at time T, then (1.2) corresponds to the well-known mean-variance hedging problem

$$\text{to minimize} \quad E(X_T^{x,\pi}-H)^2 \quad \text{over all} \quad \pi\in\Pi \tag{1.6}$$

first studied by Föllmer and Sondermann [8] (see, e.g., [5], [12], [22], [23], [21], [9], [11], for further generalizations and related results). We show that in this case $V(t,x)$ is a quadratic trinomial of the form $V(t,x)=V_0(t)-$

$2V_1(t)x + V_2(t)x^2$ and equation (1.4) gives a triangle system of backward equations for the coefficients V_i, $i = 0, 1, 2$, of the value function

$$V_2(t) = V_2(0) + \int_0^t \frac{(\varphi_2(s) + \lambda(s)V_2(s))^2}{V_2(s)} d\langle M\rangle_s + $$
$$+ \int_0^t \varphi_2(s)dM_s + L_2(t), \quad V_2(T) = 1,$$
$$V_1(t) = V_1(0) + \int_0^t \frac{(\varphi_2(s) + \lambda(s)V_2(s))(\varphi_1(s) + \lambda(s)V_1(s))}{V_2(s)} d\langle M\rangle_s +$$
$$+ \int_0^t \varphi_1(s)dM_s + L_1(t), \quad V_1(T) = H,$$
$$V_0(t) = V_0(0) + \int_0^t \frac{(\varphi_1(s) + \lambda(s)V_1(s))^2}{V_2(s)} d\langle M\rangle_s +$$
$$+ \int_0^t \varphi_0(s)dM_s + L_0(t), \quad V_0(T) = H^2,$$

where L_0, L_1 and L_2 are local martingales orthogonal to M.

Besides, equation (1.5) is transformed into a linear one

$$X_t^* = x + \int_0^t \frac{\varphi_1(s) + \lambda(s)V_1(s)}{V_2(s)} \, dS_s -$$
$$\int_0^t \frac{\varphi_2(s) + \lambda(s)V_2(s)}{V_2(s)} X_s^* \, dS_s \tag{1.7}$$

for the optimal wealth process.

Note that (1.7) gives an alternative equivalent form to the well-known feedback form solution of problem (1.6), usually derived using the density process of the variance-optimal martingale measure [11] (see also [12], [19], [20], [24]). At the end of Section 4 we establish relations between equation (1.7) and equation (4.9) derived in [11] as well, as between equations for V_2 and for the value process of the variance-optimal martingale measure (see [14], [18]).

2. Basic assumptions and some auxiliary facts

We consider an incomplete financial market model, where the dynamics of asset prices are described by a continuous semimartingale S defined on a complete filtered probability space $(\Omega, F, \mathcal{F}) = (\mathcal{F}_t, t \in [0,T], P)$, where $F = \mathcal{F}_T$ and $T < \infty$ is a fixed time horizon. For all unexplained notations from the martingale theory, we refer to [4] and [13].

Denote by $\mathcal{M}^e$ the set of martingale measures, i.e., the set of measures Q equivalent to P on $\mathcal{F}_T$ such that S is a local martingale under Q. Let $Z_t(Q)$ be the density process of Q with respect to the basic measure P, which is a strictly positive uniformly integrable martingale. For any $Q \in \mathcal{M}^e$ there is a P-local martingale M^Q such that $Z(Q) = \mathcal{E}(M^Q) = (\mathcal{E}_t(M^Q), t \in [0,T])$, where $\mathcal{E}(M)$ is the Doleans-Dade exponential of M.

We shall say that a measure Q satisfies the Reverse Hölder inequality $R_p(P)$, $p > 1$ if there exists a constant C such that

$$E\big(\mathcal{E}^p_{\tau T}(M^Q)/\mathcal{F}_\tau\big) \le C, \quad P-\text{a.s.}$$

for every stopping time τ, where $\mathcal{E}_{\tau T}(M^Q) = \frac{\mathcal{E}_T(M^Q)}{\mathcal{E}_\tau(M^Q)}$.

Troughout the paper, we shall use the following assumptions:

A 1) all P-local martingales are continuous;

A 2) the set of equivalent martingale measures $\mathcal{M}^e$ is not empty.

Remark 2.1. Condition $A1$) is equivalent to the continuity of the filtration and implies that any $\mathcal{F}$-semimartingale is special.

Remark 2.2. Since S is continuous, the existence of an equivalent martingale measure implies that the structure condition is satisfied, i.e., S admits the decomposition (1.1) and $\int_0^T \lambda_u^2 d\langle M\rangle_u < \infty$ a.s.

Let Π^p, $p \ge 1$, be the space of all predictable S-integrable processes π such that the stochastic integral

$$(\pi \cdot S)_t = \int_0^t \pi_u \, dS_u, \quad t \in [0,T],$$

is in the $\mathcal{S}^p$ space of semimartingales, i.e.,

$$E\bigg(\int_0^T \pi_s^2 d\langle M\rangle_s\bigg)^{p/2} + E\bigg(\int_0^T \big|\pi_s \, dA_s\big|\bigg)^p < \infty.$$

Define G_T^p as the space of terminal values of stochastic integrals, i.e.,

$$G_T^p(\Pi) = \big\{(\pi \cdot S)_T : \pi \in \Pi^p\big\}.$$

For convenience we give some assertions from Theorem 4.1 of [10] (previously proved in [2] for the case $p = 2$), which establishes nessecary and sufficient conditions for the closedness of the space G_T^p in L^p.

Proposition 2.1. *Let S be a continuous semimartingale. Let $p > 1$ and let q be conjugate to p. Then the following assertions are equivalent:*

(1) *There is a martingale measure $Q \in \mathcal{M}^e$ and G_T^p is closed in L^p.*

(2) *There is a martingale measure Q that satisfies the Reverse Hölder condition $R_q(P)$.*

(3) *There is a constant C such that for all $\pi \in \Pi^p$ we have*

$$\big\| \sup_{t\le T}(\pi\cdot S)_t \big\|_{L^p(P)} \le C\big|(\pi\cdot S)_T\big\|_{L^p(P)}.$$

(4) *There is a constant c such that for every stopping time τ, every $A \in \mathcal{F}_\tau$ and for every $\pi \in \Pi^p$ with $\pi = \pi I_{]\tau,T]}$ we have*

$$\big\| I_A - (\pi\cdot S)_T \big\|_{L^p(P)} \ge cP(A)^{1/p}.$$

Remark 2.3. Assertion (4) implies that for every stopping time τ and for every $\pi \in \Pi^p$ we have

$$E\bigg(\Big|1-\int_\tau^T \pi_u\,dS_u\Big|^p/\mathcal{F}_\tau\bigg) \ge c^p.$$

Suppose that the objective function $U(x) = U(\omega, x)$ satisfies the following conditions:

B1) $U(x)$ is non-negative and $EU(x) < \infty$,

B2) $U(x)$ is strictly convex function P -a.s.,

B3) optimization problem (1.3) admits a solution, i.e., for any t and x there is a strategy $\pi^*(t, x)$ such that

$$V(t,x) = E\bigg(U\Big(x+\int_t^T \pi_s^*(t,x)\,dS_s\Big)/\mathcal{F}_t\bigg). \tag{2.1}$$

One sufficient condition for B3) is given in Appendix C.

Remark 2.4. Condition B2) implies that the optimal strategy is unique if it exists. Indeed, if there exist two optimal strategies π^1 and π^2, then by convexity of U the strategy $\bar\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$ is also optimal. Therefore,

$$\frac{1}{2}E\Big[U\Big(x+\int_t^T \pi_s^1\,dS_s\Big)\Big|\mathcal{F}_t\Big] + \frac{1}{2}E\Big[U\Big(x+\int_t^T \pi_s^2\,dS_s\Big)|\mathcal{F}_t\Big] =$$

$$= E\Big[U\Big(x+\int_t^T \bar\pi_s\,dS_s\Big)|\mathcal{F}_t\Big]$$

and

$$\frac{1}{2}U\Big(x+\int_t^T \pi_s^1\,dS_s\Big)+\frac{1}{2}U\Big(x+\int_t^T \pi_s^2\,dS_s\Big)=$$
$$=U\Big(x+\int_t^T \bar{\pi}_s\,dS_s\Big)\quad P-\text{a.s.}$$

Now strict convexity of U leads to the equality $X^{\pi^1}=X^{\pi^2}$.

Denote by Π_x the class of strategies $\pi\in\Pi$ such that $EU(x+\int_0^T \pi_u dS_u)<\infty$.

Lemma 2.1. *Let Condition $B1$) be satisfied. Then*

$$V(t,x)=\operatorname*{ess\,inf}_{\pi\in\Pi_x}E\Big(U\Big(x+\int_t^T \pi_s\,dS_s\Big)/\mathcal{F}_t\Big). \tag{2.2}$$

Proof. It is sufficient to show that for any $\pi\in\Pi$ there exists $\tilde{\pi}\in\Pi_x$ such that

$$E\Big(U\Big(x+\int_t^T \tilde{\pi}_s\,dS_s\Big)/\mathcal{F}_t\Big)\le E\Big(U\Big(x+\int_t^T \pi_s\,dS_s\Big)/\mathcal{F}_t\Big). \tag{2.3}$$

Let $\tilde{\pi}_u=I_{(u>t)}I_B\pi_u$, where

$$B=\Big\{\omega: E\Big(U\Big(x+\int_t^T \pi_s\,dS_s\Big)/\mathcal{F}_t\Big)\le E\big(U(x)/\mathcal{F}_t\big)\Big\}$$

By the lattice property (see Appendix A) $\tilde{\pi}\in\Pi$ and

$$E\Big(U\Big(x+\int_t^T \tilde{\pi}_s\,dS_s\Big)/\mathcal{F}_t\Big)=E\big(U(x)/\mathcal{F}_t\big)\wedge E\Big(U\Big(x+\int_t^T \pi_s\,dS_s\Big)/\mathcal{F}_t\Big).$$

It is evident that (2.3) is satisfied and by Condition B1)

$$E\Big(U\Big(x+\int_0^T \tilde{\pi}_s\,dS_s\Big)\Big)=E\Big(U\Big(x+\int_t^T \tilde{\pi}_s\,dS_s\Big)\Big)\le EU(x)<\infty,$$

hence $\tilde{\pi}\in\Pi_x$. □

Note that, if B3) is assumed then (2.3) holds for $\tilde{\pi}=\pi^*$ and (2.2) is automatically satisfied. For convenience we give also the proof of the following known statement.

Lemma 2.2. *Under conditions B1)–B3) the value function $V(t,x)$ is a strictly convex function with respect to x.*

Proof. The convexity of $V(t,x)$ follows from B3), since for any $\alpha,\beta\in[0,1]$ with $\alpha+\beta=1$ and any $x_1,x_2\in R$ we have

$$\alpha V(t,x_1)+\beta V(t,x_2)=$$

$$=\alpha E\Big[U\Big(x_1+\int_t^T \pi_u^*(t,x_1)\,dS_u\Big)\big|\mathcal{F}_t\Big]+\beta E\Big[U\Big(x_2+\int_t^T \pi_u^*(t,x_2)\,dS_u\Big)\big|\mathcal{F}_t\Big]\geq$$

$$\geq E\Big[U\Big(\alpha x_1+\beta x_2+\int_t^T \big(\alpha\pi_u^*(t,x_1)+\beta\pi_u^*(t,x_2)\big)\,dS_u\big|\mathcal{F}_t\Big]\geq$$

$$\geq V(t,\alpha x_1+\beta x_2). \tag{2.4}$$

To show that $V(t,x)$ is strictly convex we must verify that if the equality

$$\alpha V(t,x_1)+\beta V(t,x_2)=V\big(t,\alpha x_1+\beta x_2\big) \tag{2.5}$$

is valid for some $\alpha,\beta\in(0,1)$ with $\alpha+\beta=1$, then $x_1=x_2$. □

Indeed, if equality (2.5) holds, then from (2.4) and the strict convexity of U follows that P-a.s.

$$x_1+\int_t^T \pi_u^*(t,x_1)\,dS_u=x_2+\int_t^T \pi_u^*(t,x_2)\,dS_u,$$

which implies that $x_1=x_2$.

Ito-Ventzell's formula. Let $(Y(t,x),\ t\in[0,T],\ x\in R)$ be a family of special semimartingales with the decomposition

$$Y(t,x)=Y(0,x)+B(t,x)+N(t,x), \tag{2.6}$$

where $B(\cdot,x)\in\mathcal{A}_{\text{loc}}$ and $N(\cdot,x)\in\mathcal{M}_{\text{loc}}$ for any $x\in R$. By the Galtchouk–Kunita–Watanabe (G-K-W) decomposition of $N(\cdot,x)$ with respect to M a parametrized family of semimartingales Y admits the representation

$$Y(t,x)=Y(0,x)+B(t,x)+\int_0^t \psi(s,x)dM_s+L(t,x), \tag{2.7}$$

where $L(\cdot,x)$ is a local martingale strongly orthogonal to M for all $x\in R$.

Assume that:

C1) there exists a predictable increasing process $(K_t,t\in[0,T])$ such that $B(\cdot,x)$ and $\langle M\rangle$ are absolutely continuous with respect to K, i.e., there is a measurable function $b(t,x)$ predictable for every $x\in R$ and a predictable process ν_t such that

$$B(t,x)=\int_0^t b(s,x)dK_s,\quad \langle M\rangle_t=\int_0^t \nu_s dK_s.$$

Note that, by continuity of M the square charateristic $\langle M\rangle$ is absolutely continuous with respect to the continuous part K^c of the process K and

$$\langle M\rangle_t=\int_0^t \nu_s\,dK_s^c=\int_0^t \nu_s\,dK_s.$$

Without loss of generality one can assume that ν is bounded.

C2) the mapping $x\to Y(t,x)$ is twice continuously differentiable for all (ω,t),

C3) the first derivative $Y_x(t,x)$ is a semimartingale for any x admitting the decomposition

$$Y_x(t,x)=Y_x(0,x)+B_{(x)}(t,x)+\int_0^t \psi_x(s,x)\,dM_s+L_{(x)}(t,x),\tag{2.8}$$

where $B_{(x)}(\cdot,x)\in\mathcal{A}_{\rm loc}$, $L_{(x)}(\cdot,x)$ is a local martingale orthogonal to M for all $x\in R$ and ψ_x is the partial derivative of ψ at x (note that $A_{(x)}$ and $L_{(x)}$ are not assumed to be derivatives of A and L respectively, whose existence does not necessarely follow from condition C2)),

C4) $Y_{xx}(t,x)$ is RCLL process for every $x\in R$,

C5) the functions $b(s,\cdot)$, $\psi(s,\cdot)$ and $\psi_x(s,\cdot)$ are continuous at x μ^K-a.e.,

C6) for any $c>0$

$$E\int_0^T \sup_{|x|\le c} g(s,x)\,dK_s<\infty$$

for g equal to $|b|$, $|\psi|^2$ and $|\psi|_x^2$.

In what follows we shall need the following version of Ito-Ventzell's formula

Proposition 2.2. *Let $(Y(\cdot,x),\ x\in R)$ be a family of special semimartingales satisfing conditions C1)–C6) and $X^\pi=x+\pi\cdot S$. Then the transformed process $(Y(t,X_t^\pi),\ t\in[0,T])$ is a special semimartingale with the decomposition*

$$Y(t,X_t^\pi)=Y(0,c)+B_t+N_t,$$

where

$$B_t = \int_0^t [Y_x(s, X_s^\pi)\lambda_s \pi_s d\langle M\rangle_s + \psi_x(s, X_s^\pi)\pi_s d\langle M\rangle_s + \\ + \frac{1}{2} Y_{xx}(s, X_s^\pi)\pi_s^2 d\langle M\rangle_s] + \int_0^t b(s, X_s^\pi)\, dK_s \quad (2.9)$$

and N is a continuous local martingale.

One can derive this assertion from Theorem 1.1 of [15] or from Theorem 2 of [1]. Here we don't require any condition on $L(t,x)$ imposed in [15] and [1], since the martingale part of substituted process X^π is orthogonal to $L(\cdot, x)$ and since we don't give an explicit expression of the martingale part N, which is not nessecary for our purposes.

Remark 2.5. Since the filtration is assumed to be continuous and the semimartingale S is of the form (1.1), only the latter term of (2.9) may have the jumps, i.e., the process K is not continuous in general.

3. The BSPDE for the value function

Denote by $\mathcal{V}^{1,2}$ the class of functions $Y : \Omega \times [0,T] \times R \to R$ satisfying conditions C1)-C6).

Let us consider the following backward stochastic partial differential equation (BSPDE)

$$Y(t,x) = Y(0,x) + \frac{1}{2} \int_0^t \frac{(\psi_x(s,x) + \lambda(s) Y_x(s,x))^2}{Y_{xx}(s,x)} d\langle M\rangle_s + \\ + \int_0^t \psi(s,x)\, dM_s + L(t,x), \quad L(\cdot,x) \perp M, \quad (3.1)$$

with the boundary condition

$$Y(T,x) = U(x). \quad (3.2)$$

We shall say that Y solves equation (3.1), (3.2) if:

(i) $Y(\omega,t,x)$ is twice continuously differetiable for each (ω,t) and satisfies the boundary condition (3.2),

(ii) $Y(t,x)$ and $Y_x(t,x)$ are special semimartingales admitting decompositions (2.7) and (2.8) respectively, where ψ_x is the partial derivative of ψ at x and

(iii) $P-$ a.s. for all $x \in R$

$$B(t,x) = \frac{1}{2}\int_0^t \frac{(\psi_x(s,x) + \lambda(s)Y_x(s,x))^2}{Y_{xx}(s,x)} d\langle M\rangle_s. \tag{3.3}$$

Remark 3.1. If we substitute expression of $B(t,x)$, given by equality (3.3), in the canonical decomposition (2.7) for Y we obtain equation (3.1). Note also that condition A1) and equality (3.3) imply that for any $x \in R$ the process $Y(\cdot,x)$ is a continuous semimartingale for any solution Y of (3.1).

According to Proposition A1 the value process $V(t,x)$ is a submartingale for any $x \in R$, which admits the canonical decomposition

$$(t,x) = V(0,x) + A(t,x) + \int_0^t \varphi(s,x)dM_s + m(t,x), \tag{3.4}$$

where $A(\cdot,x) \in \mathcal{A}^+$ and $m(\cdot,x)$ is a local martingale strongly orthogonal to M for all $x \in R$.

Assume that $V \in \mathcal{V}^{1,2}$. This implies that $V_x(t,x)$ is a special semimartingale with the decomposition

$$V_x(t,x) =$$
$$= V_x(0,x) + A_{(x)}(t,x) + \int_0^t \varphi_x(s,x)\, dM_s + m_{(x)}(t,x), \tag{3.5}$$

where $A_{(x)}(\cdot,x) \in \mathcal{A}_{\text{loc}}$, $m_{(x)}(\cdot,x)$ is a local martingale orthogonal to M for all $x \in R$ and φ_x coincides with the partial derivative of φ μ^K-a.e. Besides

$$A(t,x) = \int_0^t a(s,x)\, dK_s,$$

for a measurable function $a(t,x)$.

Proposition 3.1. *Assume that conditions $B1$), $B2$) are satisfied and the value function $V(t,x)$ belongs to the class $\mathcal{V}^{1,2}$. Then the following inequality holds*

$$a(s,x) \geq \frac{1}{2}\frac{(\varphi_x(s,x) + \lambda(s)V_x(s,x))^2\nu_s}{V_{xx}(s,x)} \quad \textit{for all } x \in R \quad \mu^K - \textit{a.e.}. \tag{3.6}$$

Moreover, if the strategy π^* is optimal then the corresponding wealth process X^{π^*} is a solution of the following forward SDE

$$X_t^{\pi^*} = X_0^{\pi^*} - \int_0^t \frac{\varphi_x(s,X_s^{\pi^*}) + \lambda(s)V_x(s,X_s^{\pi^*})}{V_{xx}(s,X_s^{\pi^*})}\, dS_s. \tag{3.7}$$

Proof. Using Ito-Ventzell's formula (Proposition 2.2) for the function $V(t,x,\omega)\in\mathcal{V}^{1,2}$ and for the process $(x+\int_s^t \pi_u\,dS_u,\ s\le t\le T)$ we have

$$V\Big(t,x+\int_s^t \pi_u\,dS_u\Big)=V(s,x)+\int_s^t a\Big(u,x+\int_s^u \pi_v\,dS_v\Big)\,dK_u+$$

$$+\int_s^t G\Big(u,\pi_u,x+\int_s^u \pi_v\,dS_v\Big)dK_u+N_t-N_s, \tag{3.8}$$

where

$$G(t,p,x,\omega)=V_x(t,x)p\nu_t\lambda(t)+p\nu_t\varphi_x(t,x)+\frac{1}{2}V_{xx}(t,x)p^2\nu_t \tag{3.9}$$

and N is a martingale. Since by Proposition A1) the process $(V(t,x+\int_s^t \pi_u\,dS_u),\ t\in[s,T])$ is a submartingale for all $s\ge 0$ and $\pi\in\Pi$, the process

$$\int_s^t\Big[a\Big(u,x+\int_s^u \pi_v\,dS_v\Big)+G\Big(u,\pi_u,x+\int_s^u \pi_v\,dS_v\Big)\Big]dK_u,$$

is increasing for any $s\ge 0$. Hence, the process

$$\int_s^t\Big[a\Big(u,x+\int_s^u \pi_v\,dS_v\Big)+G\Big(u,\pi_u,x+\int_s^u \pi_v\,dS_v\Big)\Big]\,dK_u^c,$$

is also increasing for any $s\ge 0$, where $K=K^c+K^d$ is a decomposition of K into continuous and purely discontinuous increasing processes. Therefore, taking $\tau_s(\varepsilon)=\inf\{t\ge s: K_t^c-K_s^c\ge\varepsilon\}$ instead of t we have that for any $\varepsilon>0$ and $s\ge 0$

$$\frac{1}{\varepsilon}\int_s^{\tau_s(\varepsilon)} a\Big(u,x+\int_s^u \pi_v\,dS_v\Big)\,dK_u^c\ge$$

$$\ge-\frac{1}{\varepsilon}\int_s^{\tau_s(\varepsilon)} G\Big(u,\pi_u,x+\int_s^u \pi_v\,dS_v\Big)\,dK_u^c. \tag{3.10}$$

Passing to the limit in (3.10) as $\varepsilon\to 0$, from Proposition B of Appendix we obtain that

$$a(s,x)\ge -G(s,\pi_s,x)\quad \mu^{K^c}-\text{a.e.}$$

for all $\pi\in\Pi$. Thus,

$$a(t,x)\ge-\operatorname*{ess\,inf}_{\pi\in\Pi}G(t,\pi_t,x);\quad \mu^{K^c}-\text{a.e.}. \tag{3.11}$$

On the other hand,

$$\operatorname*{ess\,inf}_{\pi\in\Pi} G(t,\pi_t,x) = -\frac{(V_x(t,x)\lambda(t)+\varphi_x(t,x))^2\nu_t}{2V_{xx}(t,x)} +$$
$$+\operatorname*{ess\,inf}_{\pi\in\Pi}\Big(\pi_t + \frac{V_x(t,x)\lambda(t)+\varphi_x(t,x)}{V_{xx}(t,x)}\Big)^2\nu_t = -\frac{(V_x(t,x)\lambda(t)+\varphi_x(t,x))^2\nu_t}{2V_{xx}(t,x)}.$$

Since V_{xx} is strictly positive (by Lemma 2.2), conditions C3), C4), C6) and the continuity of the filtration imply that there is a sequence of stopping times $\tau_n = (\tau_n(x), n \geq 1)$ with $\tau_n(x) \uparrow T$ for any $x \in R$ such that

$$\big|V_x(t\wedge\tau_n,x)\big| \leq n, \quad V_{xx}\big(t\wedge\tau_n,x\big) \geq \frac{1}{n}, \quad \int_0^{t\wedge\tau_n} \lambda(s)^2 d\langle M\rangle_s \leq n,$$
$$\int_0^{t\wedge\tau_n} \varphi_x(s,x)^2 d\langle M\rangle_s \leq n,$$

which implies that the strategy $\pi_t^n = -I_{[0,\tau_n(x)]}(t)\frac{V_x(t,x)\lambda(t)+\varphi_x(t,x)}{V_{xx}(t,x)} \in \Pi$ (moreover $\pi^n \in \Pi^2$) for each $n \geq 1$ and, hence

$$\operatorname*{ess\,inf}_{\pi\in\Pi}\Big|\pi_t + \frac{V_x(t,x)\lambda(t)+\varphi_x(t,x)}{V_{xx}(t,x)}\Big|^2 \leq$$
$$\leq \frac{|V_x(t,x)\lambda(t)+\varphi_x(t,x)|^2}{V_{xx}^2(t,x)} I_{(\tau_n(x)\leq t)} \to 0; \quad \mu^{K^c} \quad \text{a.e.}$$

as $n \to \infty$.

Thus, for every $x \in R$ we have that

$$a(t,x) \geq \frac{(V_x(t,x)\lambda(t)+\varphi_x(t,x))^2\nu_t}{2V_{xx}(t,x)}, \quad \mu^{K^c} \text{ a.e..}$$

Since μ^K- a.e. $a(t,x) \geq 0$ and $\mu^{K^d}\{\nu \neq 0\} = 0$ we obtain that

$$a(t,x) \geq \frac{(V_x(t,x)\lambda(t)+\varphi_x(t,x))^2\nu_t}{2V_{xx}(t,x)}, \quad \mu^K \text{ a.e..} \tag{3.12}$$

Conditions C2) and C5) imply that inequality (3.12) holds μ^K-a.e. for all $x \in R$.

Let us show now that if the strategy π^* is optimal then the corresponding wealth process X^{π^*} is a solution of equation (3.7). Let $\pi^*(s,x)$ be the optimal strategy and denote by $X_t^*(s,x) = x + \int_s^t \pi_u^*(s,x)dS_u$ the corresponding wealth process.

By the optimality principle the process $V(t, x + \int_s^t \pi_u^*(s,x)dS_u)$ is a martingale on the time interval $[s,T]$ and the Ito-Ventzell formula implies that μ^K-a.s.

$$a\big(t,X_t^*(s,x)\big) + \lambda_t\nu_t\pi_t^* V_x\big(t,X_t^*(s,x)\big) +$$

$$+\varphi_x\big(t, X_t^*(s,x)\big)\nu_t\pi_t^*(s,x) + \frac{1}{2}\pi_t^*(s,x)^2\nu_t V_{xx}\big(t, X_t^*(s,x)\big) = 0. \quad (3.13)$$

It follows from (3.12) and (3.13) that μ^K-a.e.

$$V_{xx}(t, X_t^*(s,x))\Big(\pi_t^*(s,x) + \frac{\varphi_x(t, X_t^*(s,x)) + \lambda(t)V_x(t, X_t^*(s,x))}{V_{xx}(t, X_t^*(s,x))}\Big)^2\nu_t \leq 0.$$

Since $V_{xx} > 0$ and $\nu_t > 0$ $\mu^{\langle M\rangle}$- a.e., we obtain that

$$\pi_t^*(s,x) = -\frac{\varphi_x(t, X_t^*(s,x)) + \lambda(t)V_x(t, X_t^*(s,x))}{V_{xx}(t, X_t^*(s,x))} \quad \mu^{\langle M\rangle} - \text{a.e.} \quad (3.14)$$

and integrating both parts of (3.14) by dS_t (the structure condition 1.1 implies that the corresponding stochastic integrals are indistinguishable) we obtain that the wealth process of π^* satisfies equation

$$X_t^*(s,x) = x - \int_s^t \frac{\varphi_x(u, X_u^*(s,x)) + \lambda(u)V_x(u, X_u^*(s,x))}{V_{xx}(u, X_u^*(s,x))}\, dS_u \quad (3.15)$$

which gives equation (3.7) for $s = 0$. □

Under additional condidion

C*) $(X_t^*(s,x),\ t \geq s)$ is a continuous function of (s,x) $P-$ a.s., for each $t \in [s,T]$, we shall show that the value function V satisfies equation (3.1)-(3.2).

Theorem 3.1. *Let* $V \in \mathcal{V}^{1,2}$ *and assume that conditions* $B1)$*–*$B3)$*,* $C*)$ *are satisfied. Then the value function is a solution of BSPDE* (3.1)*–*(3.2)*, i.e.,*

$$V(t,x) = V(0,x) + \frac{1}{2}\int_0^t \frac{(\varphi_x(s,x) + \lambda(s)V_x(s,x))^2}{V_{xx}(s,x)}\, d\langle M\rangle_s +$$

$$+\int_0^t \varphi(s,x)dM_s + m(t,x), \quad V(T,x) = U(x). \quad (3.16)$$

Moreover, the strategy π^* *is optimal if and only if the corresponding wealth process* X^{π^*} *is a solution of the forward SDE* (3.7)*.*

Proof. Let $\pi^*(s,x)$ be the optimal strategy. By optimality principle $(V(t, X_t^*(s,x)), t \geq s)$ is a martingale. Therefore, using Ito-Ventzell's formula we have

$$\int_s^t \Big[a\big(u, X_u^*(s,x)\big) - g\big(u, X_u^*(s,x)\big) +$$

$$+\Big(\pi_u^*(s,x) + \frac{V_x(u, X_u^*(s,x))\lambda(u) + \varphi_x(u, X_u^*(s,x))}{V_{xx}(u, X_u^*(s,x))}\Big)^2\nu_u\Big]\, dK_u = 0,$$

$$\text{for all} \quad t \ge s \;\; P-\text{a.s.},$$

for all $t \ge s$ P-a.s., where

$$g(s,x) = \frac{1}{2}\frac{(\varphi_x(s,x)+\lambda(s)V_x(s,x))^2\nu_s}{V_{xx}(s,x)}.$$

It follows from (3.14) that

$$\Big(\pi^*_u(s,x) + \frac{V_x(u,X^*_u(s,x))\lambda(u)+\varphi_x(u,X^*_u(s,x))}{V_{xx}(u,X^*_u(s,x))}\Big)^2\nu_u = 0$$
$$\mu^K - \text{a.e.}$$

and by (3.6)

$$a(s,x) \ge g(s,x) \quad \mu^K - \text{a.e.}. \tag{3.17}$$

Thus,

$$\int_s^t \Big[a\big(u,X^*_u(s,x)\big) - g\big(u,X^*_u(s,x)\big)\Big]\,dK_u = 0, \quad t \ge s \quad P \quad \text{a.s.}$$

This implies that $(a(s,x) - g(s,x))(K_s - K_{s-}) = 0$ for any $s \in [0,T]$. Therefore,

$$a(s,x) = g(s,x) \quad \mu^{K^d} - \text{a.e.}. \tag{3.18}$$

On the other hand

$$\int_0^T \frac{1}{\varepsilon}\int_s^{\tau^\varepsilon_s} \Big[a\big(u,X^*_u(s,x)\big) - g\big(u,X^*_u(s,x)\big)\Big]\,dK^c_u\,dK^c_s = 0, \quad P-\text{a.s.}$$

and by Proposition B we obtain that

$$\int_0^T \big[a(s,x)-g(s,x)\big]\,dK^c_s = 0 \quad P-\text{a.s.}$$

Now (3.17), (3.18) and the latter relation result equality $a(s,x) = g(s,x)$ μ^K-a.e., hence

$$A(t,x) = \frac{1}{2}\int_0^t \frac{(\varphi_x(s,x)+\lambda(s)V_x(s,x))^2}{V_{xx}(s,x)}d\langle M\rangle_s$$

and $V(t,x)$ satisfies (3.1)–(3.2).

If $\hat\pi$ is a strategy such that the corresponding wealth process $X^{\hat\pi}$ satisfies equation (3.7), then $\hat\pi$ is optimal. Indeed, using the Ito-Ventzell formula and equation (3.7) we obtain that $V(t,X^{\hat\pi}_t)$ is a martingale, hence $\hat\pi$ is optimal by optimality principle. □

Remark 3.2. Thus, to give a construction of the optimal strategy we should: 1) first solve the backward equation (3.1),(3.2) (which determines V and φ simultaneously) and substitute corresponding derivatives of V and φ in equation (3.7), then 2) solve the forward equation (3.7) with respect to X^{π^*} and, finally, 3) reproduce the optimal strategy π^* from the corresponding wealth process X^{π^*}.

Recall that the process Z belongs to the class D if the family of random variables $Z_\tau I_{(\tau\le T)}$ for all stopping times τ is uniformly integrable.

Definition 3.1. We say that Y belongs to the class $D(\Pi)$ if:
i) there is a positive process c_t from the class D such that

$$Y(t,x)\ge -c_t,\quad \text{for all } x\in R,$$

ii) for any $x\in R$ the process $Y(t,x+\int_0^t\pi_u dS_u)$ is of class D for every $\pi\in\Pi_x$, i.e., for any $\pi\in\Pi$ with $EU(x+\int_0^T\pi_u dS_u)<\infty$.

Remark 3.3. Note that the value function $V(t,x)$ belongs to the class $D(\Pi)$, since for any $\pi\in\Pi_x$

$$0\le V\Big(t,x+\int_0^t\pi_u\,dS_u\Big)\le E\Big(U\Big(x+\int_0^T\pi_u\,dS_u\Big)/\mathcal{F}_t\Big) \tag{3.19}$$

and the right-hand-side of (3.19) is a uniformly integrable martingale.

Theorem 3.2. *Let conditions B1)–B3) be satisfied. If the pair $(Y,\mathcal{X})$ is a solution of the Forward-Backward Equation*

$$Y(t,x)=U(x)-\frac12\int_t^T\frac{((\psi_x(s,x)+\lambda(s)Y_x(s,x))^2}{Y_{xx}(s,x)}\,d\langle M\rangle_s-$$

$$-\int_t^T\psi(s,x)dM_s+L(T,x)-L(t,x), \tag{3.20}$$

$$\mathcal{X}_t=x-\int_0^t\frac{\psi_x(s,\mathcal{X}_s)+Y_x(s,\mathcal{X}_s)\lambda(s)}{Y_{xx}(s,\mathcal{X}_s)}\,dS_s, \tag{3.21}$$

and Y belongs to the class $\mathcal{V}^{1,2}\cap D(\Pi)$, then such solution is unique.

Proof. Using the Ito-Ventzell's formula for $Y(t,x+\int_s^t\pi_u dS_u)$ we have

$$Y\Big(t,x+\int_s^t\pi_u\,dS_u\Big)=Y(s,x)+\int_s^t b\Big(u,x+\int_s^u\pi_v\,dS_v\Big)\,dK_u+$$

$$+\int_s^t G\Big(u,\pi_u,c+\int_s^u \pi_v\,dS_v\Big)\,dK_u+N_t-N_s, \tag{3.22}$$

where

$$G(t,p,x,\omega)=Y_x(t,x)p\nu_t\lambda(t)+p\nu_t\psi_x(t,x)+\frac{1}{2}Y_{xx}(t,x)p^2\nu_t.$$

Since Y solves (3.20), then equality (3.3) is valid, which implies that $Y(t,x+\int_s^t\pi_u dS_u)$ is a local submartingale for each $\pi\in\Pi$.

Since Y is from the class $D(\Pi)$, then the process $Y(t,x+\int_s^t\pi_u\,dS_u)$ is a submartingale of class D for any $\pi\in\Pi_x$ and using the boundary condition we have that

$$Y(s,x)\le E\Big[U\Big(x+\int_s^T\pi_u\,dS_u\Big)/\mathcal{F}_s\Big]$$

which implies (taking Lemma 2.1 in mind) that

$$Y(s,x)\le V(s,x). \tag{3.23}$$

Using now the Ito-Ventzell's formula for $Y(t,\mathcal{X}_u)$ taking into account that Y satisfies (3.20) and $\mathcal{X}$ solves (3.21) we obtain that $Y(t,\mathcal{X}_u)$ is a local martingale and, hence, it is a supermartingale, since Y is bounded from below by the process of class D. Therefore, since $\mathcal{X}_0=x$, $Y(T,x)=U(x)$ we have that

$$Y(t,x)\ge E\big(Y(T,\mathcal{X}_T)/\mathcal{F}_t\big)=$$

$$=E\Big(U\Big(x+\int_t^T\frac{Y_x(u,\mathcal{X}_u)\lambda_u+\psi_x(u,\mathcal{X}_u)}{Y_{xx}(u,\mathcal{X}_u))}\,dS_u\Big)/\mathcal{F}_t\Big). \tag{3.24}$$

Applying inequalities (3.23) and (3.24) for $s=0$ we obtain

$$E\Big(U\Big(x+\int_0^T\frac{Y_x(u,\mathcal{X}_u)\lambda_u+\psi_x(u,\mathcal{X}_u)}{Y_{xx}(u,\mathcal{X}_u)}\,dS_u\Big)\Big)\le$$

$$\le Y(0,x)\le V(0,x)\le EU(x)<\infty. \tag{3.25}$$

Therefore, $\frac{\lambda(u)Y_x(u,\mathcal{X}_u)+\psi_x(u,\mathcal{X}_u)}{Y_{xx}(u,\mathcal{X}_u))}\in\Pi_x$ and it follows from (3.23) and (3.24) that

$$Y(t,x)=V(t,x), \tag{3.26}$$

hence solution of (3.20) is unique.

The equalities (3.26) and (3.21) imply that $\mathcal{X}$ satifies equation (3.7). Besides, according to Proposition 3.1 the solution of (3.7) is the optimal wealth process, hence $\mathcal{X}=X^{\pi*}$ by the uniqueness of the optimal strategy for the problem (1.3) (see Remark 2.4). □

4. Mean-Variance Hedging

Let us consider now the case $U(x) = (x - H)^2$, which corresponds to the mean-variance hedging problem (1.6), where H is a F_T-measurable random variable describing the net payoff at time T of a certain financial instrument.

Assume that

$A2^*$) there exists a martingale measure that satisfies the Reverse Hölder condition $R_2(P)$.

Theorem 4.1. *Let H be a square integrable $\mathcal{F}_T$-measurable random variable and let the objective function be of the form $U(x) = |H - x|^2$. Then the value function of the problem* (1.6) *admits a representation*

$$V(t,x) = V_0(t) - 2V_1(t)x + V_2(t)x^2, \tag{4.1}$$

where the processes $V_0(t)$, $V_1(t)$ and $V_2(t)$ satisfy the following system of backward equations

$$V_2(t) = V_2(0) + \int_0^t \frac{(\varphi_2(s) + \lambda(s)V_2(s))^2}{V_2(s)} d\langle M\rangle_s + \int_0^t \varphi_2(s)dM_s + L_2(t), \quad V_2(T) = 1, \tag{4.2}$$

$$V_1(t) = V_1(0) + \int_0^t \frac{(\varphi_2(s) + \lambda(s)V_2(s))(\varphi_1(s) + \lambda(s)V_1(s))}{V_2(s)} d\langle M\rangle_s + \int_0^t \varphi_1(s)dM_s + L_1(t), \quad V_1(T) = H, \tag{4.3}$$

$$V_0(t) = V_0(0) + \int_0^t \frac{(\varphi_1(s) + \lambda(s)V_1(s))^2}{V_2(s)} d\langle M\rangle_s + \int_0^t \varphi_0(s)dM_s + L_0(t), \quad V_0(T) = H^2, \tag{4.4}$$

where L_0, L_1 and L_2 are local martingales orthogonal to M.

If a triple (Y_0, Y_1, Y_2), where $Y_0 \in D, Y_1^2 \in D$ and $c \le Y_2 \le C$ for some constants $0 < c < C$, satisfies the system (4.2)-(4.4)*, then such solution is unique and coincides with the triple (V_0, V_1, V_2).*

Besides the optimal wealth process X^{π^*} satisfies the linear equation

$$X_t^{\pi^*} = x + \int_0^t \frac{\varphi_1(s) + \lambda(s)V_1(s)}{V_2(s)} dS_s -$$

$$-\int_0^t \frac{\varphi_2(s)+\lambda(s)V_2(s)}{V_2(s)} X_s^{\pi^*}\, dS_s. \qquad (4.5)$$

Proof. It is evident that $U(x)=|H-x|^2$ satisfies conditions B1), B2) and condition B3) follows from Proposition C of Appendix, since the function $U(x)=|H-x|^2$ satisfies condition B3') for $p=2$ and the space G_T^2 of stochastic integrals is closed by Proposition 2.1. Hence there exists optimal strategy $\pi^*(t,x)$ and $V(t,x)=E[|H-x-\int_t^T \pi_u^*(t,x)dS_u|^2|\mathcal{F}_t]$. Since $\int_t^T \pi_u^*(t,x)dS_u$ coincides with the orthogonal projection of $H-x\in L^2$ on the closed subspace of stochastic integrals, then the optimal strategy is linear with respect to x, i.e., $\pi_u^*(t,x)=\pi_u^0(t)+x\pi_u^1(t)$. This implies that the value function $V(t,x)$ is of the form (4.1), where

$$V_0(t)=E\Big[\Big|\int_t^T \pi_u^0(t)\,dS_u - H\Big|^2 |\mathcal{F}_t\Big],$$

$$V_1(t)=E\Big[\Big(1+\int_t^T \pi_u^1(t)\,dS_u\Big)\Big(\int_t^T \pi_u^0(t)\,dS_u - H\Big)|\mathcal{F}_t\Big],$$

$$V_2(t)=E\Big[\Big|\int_t^T \pi_u^1(t)\,dS_u + 1\Big|^2 |\mathcal{F}_t\Big]. \qquad (4.6)$$

It is evident that the function $U(x)=|x-H|^2$ satisfies all conditions of Proposition A2 and assertion (3) of Proposition 2.1 implies that $\tilde{\Pi}=\Pi^2$, where the class $\tilde{\Pi}$ is defined in Appendix A. Therefore, accoding to Proposition $A2$ of Appendix $V(t,x)$ is a RCLL submartingale for each $x\in R$. Thus $V_0(t)=V(t,0)$ is a RCLL submaringale. On the other hand for any $s\geq t$

$$E\big[V_2(t)|\mathcal{F}_s\big]=\lim_{x\to\infty}\frac{1}{x^2}E\big[V(t,x)|\mathcal{F}_s\big]\geq \lim_{x\to\infty}\frac{1}{x^2}V(s,x)=V_2(s)\ \ P-\text{a.s.}$$

and $V_2(t)$ is also a submartingale with RCLL trajectories (as a uniform limit of RCLL processes). Hence $V_1(t)=\frac{1}{2}(V_0(t)+V_2(t)-V(t,1))$ is a special semimartingale.

Because V_0 and V_2 are submartingales

$$V_2(t)\leq E\big(V_2(T)/\mathcal{F}_t\big)\leq 1,$$
$$V_0(t)\leq E\big(H^2/\mathcal{F}_t\big)$$

and since $V(t,x)=V_0(t)-2V_1(t)x+V_2(t)x^2\geq 0$ for all $x\in R$, we have that $V_1^2(t)\leq V_0(t)V_2(t)$, hence

$$V_1^2(t)\leq E\big(H^2/\mathcal{F}_t\big).$$

Since $V(t,x)$ is strictly convex and $V_{xx}(t,x) = 2V_2(t)$, the process V_2 is strictly positive. Moreover, it follows from Proposition 2.1 (see Remark 2.3) that there is a constant $c > 0$ such that $V_2(t) \geq c$.

Thus, V_0, V_1^2 belong to the class D and the process V_2 satisfies the two-sided inequality

$$c \leq V_2(t) \leq 1.$$

Let

$$V_i(t) = V_0(0) + A_i(t) + \int_0^t \varphi_i(u) dM_u + m_i(t)$$

be the canonical decomposition of V_i for $i = 0, 1, 2$, where m_i is a local martingale strongly orthogonal to M and $A_i \in \mathcal{A}_{\text{loc}}$ (moreover A_0 and A_2 are increasing processes). Taking

$$K(t) = A_0(t) + A_2(t) + Var(A_1)(t) + \langle M \rangle_t + t$$

it is evident that condition C1) is satisfied. It is easy to see that conditions C2)-C6) are also fulfilled. By Proposition 3.1 $X_t^*(s,x)$ is a solution of the forward equation (3.15), which coincides in this case with linear equation (4.5) and can be solved explicitly in terms of V_i, $i = 1, 2$. Therefore condition C^*) is also satisfied and we may apply Theorem 3.1. Equalizing the coefficients of the quadratic trinomial (4.1) in equation (3.16) we obtain that V_2, V_1 and V_0 satisfy equations (4.2), (4.3) and (4.4) respectively. The boundary conditions for these equations follow from equality (4.6).

The proof of uniqueness.

If a triple (Y_0, Y_1, Y_2) is a solution of system (4.2)-(4.4), then the function $Y(t,x) = Y_0(t) - 2Y_1(t)x + Y_2(t)x^2$ will be a solution of (3.1)-(3.2). By assertion (3) of Proposition 2.1 the process $(\int_0^t \pi_u \, dS_u)^2$ is of class D. Since $Y_1^2(t) \in D$ the Hölder inequality implies that the process $Y_1(t)(\int_0^t \pi_u dS_u)$ is of class D. Therefore, $Y(t, x + \int_0^t \pi_u dS_u)$ belongs to the class D for every $\pi \in \Pi_x$.

It is easy to see that $Y_2(t) > c$ implies

$$Y(t,x) = Y_0(t) - 2Y_1(t)x + Y_2(t)x^2 \geq -\frac{1}{c} Y_1^2$$

for all $x \in R$. Thus Y belongs to the class $D(\Pi)$ and $Y(t,x) = V(t,x)$ by Theorem 3.2, which implies that $Y_i = V_i$ for $i = 0, 1, 2$. □

Remark 4.1. In a similar way one can show that for $U(x) = |H - x|^p$ the optimal strategy is also linear w.r.t. x. Moreover if p is even, i.e., $p = 2n$, then the value function is polynomial of x, i.e., $V(t,x) = \sum_{j=0}^p V_j(t)x^j$ and (3.1),(3.2) are transformed into a system of Backward SDE's of order $2n+1$ for the processes $V_j(t)$.

Remark 4.2. Equation (4.3) is linear with respect to (V_1, φ_1) and V_1 is expressed explicitly in terms of (V_2, φ_2) as

$$V_1(t) = E\Big(H\mathcal{E}_{tT}\Big(-\Big(\frac{\varphi_2}{V_2}+\lambda\Big)\cdot S\Big)/\mathcal{F}_t\Big). \tag{4.7}$$

Now we give relations of equation (4.5) with the known feedback form solution of problem (1.6), expressed in terms of the variance-optimal martingale measure (see, e.g., [11]). To this end we recall the notion of the variance-optimal martingale measure.

The variance-optimal martingale measure is a signed measure such that its density with respect to the reference measure P is of minimal L^2 norm (see [2], [24] for exact definition and related results). According to [2], [24] the variance-optimal martingale measure Q^* always exists and is a probability measure equivalent to P, if S iscontinuous and if the subset $\mathcal{M}_2^e$ of equivalent martingale measures with square integrable densities is not empty. Moreover, as it was shown in [2], if Q^* is the variance-optimal martingale measure then the density Z_T^* of Q^* with respect to the basic measure P can be written as a constant plus a stochastic integral of S and the density process Z_t^* defined by $E^*(Z_T/\mathcal{F}_t)$ admits the same representation

$$Z_t^* = E^*Z_T + \int_0^t h^*(u)\,dS_u$$

for a predictable S-integrable process h^*.

Let $V_t^H = E^*(H/\mathcal{F}_t)$ and let

$$V_t^H = E^*H + \int_0^t \xi^H(u)\,dS_u + L_t^H, \quad \langle L^H, X\rangle = 0, \tag{4.8}$$

be the Galtchouk–Kunita–Watanabe decomposition of V_t^H with respect to the variance-optimal martingale measure Q^*.

It was shown in [11] (see also [12], [19], [20], [24]) that the optimal mean-variance hedging strategy is expressed in the feedback form

$$\pi_t^* = \xi_t^H - \frac{h_t^*}{Z_t^*}\Big(V_t^H - c - \int_0^t \pi_u^*\,dS_u\Big).$$

Integrating both part with respect to dS_u we obtain the linear equation for the optimal wealth process

$$X_t^{\pi^*} = x + \int_0^t \Big[\xi_s^H - \frac{h_s^*}{Z_s^*}V_s^H\Big]\,dS_s + \int_0^t \frac{h_s^*}{Z_s^*}X_s^{\pi^*}\,dS_s. \tag{4.9}$$

To show that equations (4.9) and (4.5) are equivalent we need the following assertion proved in [17–18]. Under the present assumptions the variance-optimal martingale measure is the solution of the optimization problem

$$\inf_{Q\in\mathcal{M}_2^e} EZ_T^2(Q)$$

and let

$$V_t = \operatorname*{ess\,inf}_{Q\in\mathcal{M}_2^e} E\Big(\frac{Z_T^2(Q)}{Z_t^2(Q)}/\mathcal{F}_t\Big)$$

be the value process of the problem.

Proposition 4.1. *Let conditions A1) and A2∗) be satisfied. Then, the value process V is a unique solution of the semimartingale backward equation*

$$V_t = V_0 - \int_0^t V_s\lambda_s^2 d\langle M\rangle_s + 2\int_0^t \lambda_s\varphi_s d\langle M\rangle_s + $$
$$+\frac{1}{V_s}d\langle m\rangle_s + \int_0^t \varphi_s dM_s + m_t, \quad V_T = 1, \tag{4.10}$$

in the class of semimartingales Y satisfying the two-sided inequality

$$c \le Y_t \le C. \tag{4.11}$$

Moreover, the martingale measure Q^* is variance-optimal if and only if the corrersponding density is represented as

$$Z_T^* = \mathcal{E}_T\Big(-\int_0^\cdot \lambda_s d\,M_s - \int_0^\cdot \frac{1}{V_s}\,dm_s\Big). \tag{4.12}$$

or, equivalently, iff

$$Z_T^* = c\mathcal{E}_T\Big(\Big(\frac{\varphi}{V} - \lambda\Big)\cdot S\Big). \tag{4.13}$$

The following Proposition shows that equations (4.9) and (4.5) are equivalent.

Proposition 4.2. *Let conditions A1) and A2∗) be satisfied. Then*

$$V(t) = \frac{1}{V_2(t)}, \quad \frac{h_t^*}{Z_t^*} = \frac{\varphi_2(t)}{V_2(t)} - \lambda_t, \quad V^H(t) = \frac{V_1(t)}{V_2(t)}$$

and the optimal wealth process X^* satisfies equation (4.9).

Proof. If we write the Itô formula for $\frac{1}{V_2(t)}$, taking in mind that $V_2(t)$ satisfies equation (4.2), we obtain that the semimartingale $\frac{1}{V_2(t)}$ satisfies equation (4.10) with $\varphi = -\frac{\varphi_2}{V_2^2}$, $m = -\frac{1}{V_2^2}\cdot L$ and by uniqueness of a solution (since $c \le V_2(t) \le 1$) we have that

$$V(t) = \frac{1}{V_2(t)}, \quad \frac{\varphi(t)}{V(t)} = -\frac{\varphi_2(t)}{V_2(t)}. \tag{4.14}$$

It follows from (4.13) that

$$Z_t^* = E^*(Z_T^*/\mathcal{F}_t) = V_0\mathcal{E}_t\Big(\Big(\frac{\varphi}{V} - \lambda\Big)\cdot S\Big)$$

and

$$h_t^* = V_0\Big(\frac{\varphi_t}{V_t} - \lambda_t\Big)\mathcal{E}_t\Big(\Big(\frac{\varphi}{V} - \lambda\Big)\cdot S\Big).$$

Therefore (4.13) and (4.14) imply that

$$\frac{h_t^*}{Z_t^*} = \frac{\varphi_t}{V_t} - \lambda_t = \frac{\varphi_2(t) + \lambda(t)V_2(t)}{V_2(t)}. \tag{4.15}$$

Let us show now that

$$\frac{\varphi_1(t) + \lambda(t)V_1(t)}{V_2(t)} = \xi^H(t) - \frac{h_t^*}{Z_t^*}V_t^H.$$

From (4.12) we have that

$$V^H(t) = E\Big(H\mathcal{E}_{tT}\Big(-\lambda\cdot M - \frac{\varphi}{V}\cdot m\Big)/\mathcal{F}_t\Big).$$

Therefore (4.7),(4.13) and the equality

$$\mathcal{E}_T\Big(-\lambda\cdot M - \frac{\varphi}{V}\cdot m\Big) = c\mathcal{E}_T\Big(\Big(\frac{\varphi}{V} - \lambda\Big)\cdot S\Big)$$

imply that

$$V^H(t) = cV_1(t)\frac{\mathcal{E}_t((\frac{\varphi}{V} - \lambda)\cdot S)}{\mathcal{E}_t(-\lambda\cdot M - \frac{\varphi}{V}\cdot m)} =$$

$$cV_1(t)\frac{E^*(\mathcal{E}_T((\frac{\varphi}{V} - \lambda)\cdot S)/\mathcal{F}_t)}{\mathcal{E}_t(-\lambda\cdot M - \frac{\varphi}{V}\cdot m)} = V_1(t)V(t) = \frac{V_1(t)}{V_2(t)},$$

hence $V_1(t) = V^H(t)V_2(t)$.

Using the formula of integration by parts and equalizing the martingale parts of $V_1(t)$ and $V^H(t)V_2(t)$ we obtain that μ^K- a.e.

$$\varphi_1(t) = \varphi_2(t)V^H(t) + \xi^H(t)V_2(t).$$

Therefore, (4.14), (4.15) and the latter equality imply that

$$\frac{\varphi_1(t) + \lambda(t)V_1(t)}{V_2(t)} = \frac{\varphi_2(t)V^H(t) + \xi^H(t)V_2(t) + \lambda(t)V_1(t)}{V_2(t)} =$$

$$= \xi^H(t) - V^H(t)\frac{\varphi(t)}{V(t)} + \lambda(t)V^H(t) = \xi^H(t) - V_t^H \frac{h_t^*}{Z_t^*},$$

hence (4.9) and (4.5) are equivalent. □

Remark 4.3. Note that in [11] equation (4.9) was derived assuming only that S is continuous and $\mathcal{M}_2^e \neq \emptyset$, i.e., without assumptions $A1)$ and $A2^*)$. It should be mentioned that the form of equations (3.15)-(3.16), (4.2)-(4.5) remain the same if $A1)$ is not required.

Remark 4.4. The condition $V \in \mathcal{V}^{1,2}$ is also satisfied in several other particular cases (e.g., in the case of exponential hedging, where $U(x) = \exp(H - x)$), but it is important to derive the required properties of the value function from the assumptions on the basic objects U and X, which we plan to do in future.

Let us consider now the optimiztion problem

$$\text{minimize } E\Big(c + \int_0^T \pi_s\, dS_s - H\Big)^2 \tag{4.16}$$

over all $c \in R$ and $\pi \in \Pi$. Then for any $c \in R$

$$E\Big(c + \int_0^T \pi_s\, dS_s - H\Big)^2 \geq E\Big(c + \int_0^T \pi_s^*(c)\, dS_s - H\Big)^2 =$$
$$= V(0,c) = V_0(0) - 2V_1(0) + c^2 V_2(0). \tag{4.17}$$

The infimum on the rigth-hand side of (4.17) is attained for $c = \frac{V_1(0)}{V_2(0)}$. It follows from Propositoin 4.2 that

$$\frac{V_1(0)}{V_2(0)} = V_0^H = E^* H,$$

where E^* is an expectation with respect to the variance-optimal martingale measure.

Therefore,

$$E\Big(c + \int_0^T \pi_s\, dS_s - H\Big)^2 \geq E\Big(E^*H + \int_0^T \pi_s\, dS_s - H\Big)^2$$

for all c and π. Thus, if (c^*, π^*) is a solution of (4.16) then $c^* = E^* H$, as proved by Schweizer in [24].

A. Appendix

Let us show that the family

$$\Lambda_t^\pi = E\Big(U\Big(x+\int_0^T \pi_u\,dS_u\Big)\big|\mathcal{F}_t\Big),\quad \pi\in\Pi(\tilde\pi,t,T), \tag{A.1}$$

satisfies the ε-lattice property (with $\varepsilon=0$) for any $t\in[0,T]$ and $\tilde\pi$. $\Pi(\tilde\pi,t,T)$ is a set of predictable S-integrable processes π such that

$$\pi_s=\tilde\pi_s I_{(0\le s<t)}.$$

We shall write $\Pi(t,T)$ instead of $\Pi(0,t,T)$ for the class of strategies corresponding to $\tilde\pi=0$ up to time t.

We must show that for any π^1, $\pi^2\in\Pi(\tilde\pi,t,T)$ there exists a strategy $\pi\in\Pi(\tilde\pi,t,T)$ such that

$$\Lambda_t^\pi=\min(\Lambda_t^{\pi^1},\Lambda_t^{\pi^2}). \tag{A.2}$$

For any π^1 and π^2 let us define the set

$$B=\big\{\omega:\Lambda_t^{\pi^1}\le\Lambda_t^{\pi^2}\big\}$$

and let

$$\pi_s=\tilde\pi_s I_{(0\le s<t)}+\pi_s^1 I_B I_{(s\ge t)}+\pi_s^2 I_{B^c} I_{(s\ge t)}.$$

Since B is $\mathcal{F}_t-$measurable we have

$$\Lambda_t^\pi=E\Big(U\Big(x+\int_0^T\pi_u\,dS_u\Big)\big|\mathcal{F}_t\Big)=$$

$$=E\Big(U\Big(x+\int_0^t\tilde\pi_u\,dS_u+I_B\int_t^T\pi_u^1\,dS_u+I_{B^c}\int_t^T\pi_u^2\,dS_u\Big)\big|\mathcal{F}_t\Big)=$$

$$I_BE\Big(U\Big(x+\int_0^t\tilde\pi_u\,dS_u+\int_t^T\pi_u^1\,dS_u\Big)\big|\mathcal{F}_t\Big)+$$

$$+I_{B^c}E\Big(U\Big(x+\int_\tau^t\tilde\pi_u\,dS_u+\int_t^T\pi_u^2\,dS_u\Big)\big|\mathcal{F}_t\Big)=$$

$$=I_BE\Big(U\Big(x+\int_0^T\pi_u^1\,dS_u\Big)\big|\mathcal{F}_t)+I_{B^c}E\Big(U\Big(x+\int_0^T\pi_u^2\,dS_u\Big)\big|\mathcal{F}_t\Big)=$$

$$=E\Big(U\Big(x+\int_0^T\pi_u^1\,dS_u\Big)\big|\mathcal{F}_t\Big)\wedge E\Big(U\Big(x+\int_0^T\pi_u^2\,dS_u\Big)\big|\mathcal{F}_t\Big),$$

hence (A.2) is satisfied.

Proposition A1) (Optimality principle). *Let condition $B1$) be satisfied.*

a) *For all $x \in R$, $\pi \in \Pi$ and $s \in [0,T]$ the process $(V(t, x + \int_s^t \pi_u \, dS_u), t \geq s)$ is a submartingale,*

b) $\pi^*(s,x)$ *is optimal iff $(V(t, x + \int_s^t \pi_u^* dS_u), t \geq s)$ is a martingale.*

c) *for all $s < t$*

$$V(s,x) = \operatorname*{ess\,inf}_{\pi \in \Pi(s,T)} E\Big(V\Big(t, x + \int_s^t \pi_u \, dS_u\Big) \big| \mathcal{F}_s\Big). \qquad (A.3)$$

Proof. a) For simplicity we shall take s equal to zero. $V(t,x)$ is integrable, since by condition $B1$)

$$V(t,x) \leq E\big(U(x)/\mathcal{F}_t\big).$$

Let us show that $Y_t = V(t, x + \int_0^t \tilde{\pi}_u \, dS_u)$ is submartingale for all x and $\tilde{\pi}$. Since

$$Y_t = \operatorname*{ess\,inf}_{\pi \in \Pi(t,T)} E\Big(U\Big(x + \int_0^t \tilde{\pi}_u \, dS_u + \int_t^T \pi_u \, dS_u\Big) \big| \mathcal{F}_t\Big)$$

using the lattice property of the family (A.1) from Lemma 16.A.5 of [7], taking Lemma 2.1 in mind, we have

$$E\big(Y_t | \mathcal{F}_s\big) = E\big(\operatorname*{ess\,inf}_{\pi \in \Pi(t,T)} E\Big(U\Big(x + \int_0^t \tilde{\pi}_u \, dS_u + \int_t^T \pi_u \, dS_u\Big) \big| \mathcal{F}_t\Big) \big| \mathcal{F}_s\Big) =$$

$$E\Big(\operatorname*{ess\,inf}_{\pi \in \Pi(\tilde{\pi},t,T)} E\Big(U\Big(x + \int_0^T \pi_u \, dS_u\Big) \big| \mathcal{F}_t\Big) \big| \mathcal{F}_s\Big) =$$

$$\operatorname*{ess\,inf}_{\pi \in \Pi(\tilde{\pi},t,T)} E\Big(U\Big(x + \int_0^T \pi_u \, dS_u\Big) \big| \mathcal{F}_s\Big). \qquad (A.4)$$

It is evident that $\Pi(\tilde{\pi}, t, T) \subseteq \Pi(\tilde{\pi}, s, T)$ for $s \leq t$, which implies the inequality

$$\operatorname*{ess\,inf}_{\pi \in \Pi(\tilde{\pi},t,T)} E\Big(U\Big(x + \int_0^T \pi_u \, dS_u\Big) \big| \mathcal{F}_s\Big) \geq$$

$$\operatorname*{ess\,inf}_{\pi \in \Pi(\tilde{\pi},s,T)} E\Big(U\Big(x + \int_0^T \pi_u \, dS_u\Big) \big| \mathcal{F}_t\Big) = V\Big(s, x + \int_0^s \tilde{\pi}_u \, dS_u\Big). \quad (A.5)$$

Thus (A.4) and (A.5) imply that $E(Y_t/F_s) \geq Y_s$.

b) If $V(t,x+\int_0^t \pi^*_u dS_u))$ is a martingale, then

$$\inf_{\pi\in\Pi} EU\Big(x+\int_0^T \pi_u\,dS_u\Big) = V(0,x) = EV(0,x) =$$

$$= EV\Big(T,x+\int_0^T \pi^*_u\,dS_u\Big) = EU\Big(x+\int_0^T \pi^*_u\,dS_u\Big),$$

hence, π^* is optimal.

Conversely, if π^* is optimal, then

$$EV(0,x) = \inf_{\pi\in\Pi} EU\Big(x+\int_0^T \pi_u\,dS_u\Big) =$$

$$= EU\Big(x+\int_0^T \pi^*_u\,dS_u\Big) = EV\Big(T,x+\int_0^T \pi^*_u\,dS_u\Big).$$

Since $V(t,x+\int_0^t \pi^*_u dS_u)$ is a submartingale, the latter equality implies that this process is also a martingale (it follows from Lemma 6.6 of [16]).

c) Since $Y_t = V(t,x+\int_s^t \tilde\pi_u\,dS_u)$ is a submartingale for any $\tilde\pi\in\Pi(s,T)$, $x\in R$ and $t\geq s$ we have

$$V(s,x) \leq E\Big(V\Big(t,x+\int_s^t \tilde\pi_u\,dS_u\Big)\big|\mathcal{F}_s\Big),$$

hence

$$V(s,x) \leq \operatorname*{ess\,inf}_{\tilde\pi\in\Pi(s,T)} E\Big(V\Big(t,x+\int_s^t \tilde\pi_u\,dS_u\Big)\big|\mathcal{F}_s\Big). \qquad (A.6)$$

On the other hand for any $\tilde\pi$

$$E\Big(V\Big(t,x+\int_s^t \tilde\pi_u\,dS_u\Big)\big|\mathcal{F}_s\Big) =$$

$$= E\Big(\operatorname*{ess\,inf}_{\pi\in\Pi(t,T)} E\Big(U\Big(x+\int_s^t \tilde\pi_u\,dS_u+\int_t^T \pi_u\,dS_u\Big)\big|\mathcal{F}_t)\mathcal{F}_s\Big) \leq$$

$$E\Big(U\Big(x+\int_s^T \tilde\pi_u\,dS_u\Big)\big|\mathcal{F}_t\Big)\mathcal{F}_s\Big) = E\Big(U\Big(x+\int_s^T \tilde\pi_u\,dS_u\Big)\big|\mathcal{F}_s\Big).$$

Taking essinf of the both parts we obtain

$$\operatorname*{ess\,inf}_{\tilde\pi\in\Pi(s,T)} E\Big(V\Big(t, x+\int_s^t \tilde\pi_u\,dS_u\Big)\big|\mathcal{F}_s\Big) \le$$

$$\le \operatorname*{ess\,inf}_{\tilde\pi\in\Pi(s,T)} E\Big(U\Big(x+\int_s^T \tilde\pi_u\,dS_u\Big)\big|\mathcal{F}_s\Big) = V(s,x). \qquad (A.7)$$

Thus the equality (A.3) follows from (A.6) and (A.7). □

An existence of an RCLL modification for the value process will be proved for a more restricted class of strategies and under additional assumptions on the function U, which is sufficient for the case $U(x)=(H-x)^2$ considered in Section 4.

Let $\tilde\Pi$ be a class of strategies $\pi\in\Pi$ such that

$$E\sup_{t\le T} U\Big(\int_0^t \pi_u\,dS_u\Big) < \infty$$

and let

$$\tilde V(t,x) = \operatorname*{ess\,inf}_{\pi\in\tilde\Pi(t,T)} E\Big(U\Big(x+\int_t^T \pi_u\,dS_u\Big)\big|\mathcal{F}_t\Big).$$

Proposition A2. *Let conditions $B1)$, $B2)$ be satisfied and let for any real number α there exist constants C_α, B_α and an integrable random variable η such that*

$$U(\alpha x)\le C_\alpha U(x) + B_\alpha\eta \quad \text{for all} \quad x\in R. \qquad (A.8)$$

Then for every $x\in R$ and every $\pi\in\tilde\Pi$ the process $\tilde V(t, x+\int_0^t \pi_u\,dS_u)$ is a submartingale admiting an RCLL modification. In particular, for any $x\in R$ there exists an RCLL submartingale (still denoted by $\tilde V(t,x)$) such that for all $t\in[0,T]$

$$\tilde V(t,x) = \operatorname*{ess\,inf}_{\pi\in\tilde\Pi(t,T)} E\Big(U\Big(x+\int_t^T \pi_u\,dS_u\Big)/\mathcal{F}_t\Big).$$

Proof. Let us show that the process $\tilde V(t, x+\int_0^t \tilde\pi_u dS_u)$ admits an RCLL modification for each $x\in R$ and $\pi\in\tilde\Pi$. According to Theorem 3.1 of [16] it is sufficient to prove that the function $E\tilde V(t, x+\int_0^t \tilde\pi_u\,dS_u)), t\in[0,T])$ is right-continuous for every $x\in R$.

Let $(t_n, n \geq 1)$ be a sequence of positive numbers such that $t_n \downarrow t$, as $n \to \infty$. Since $\tilde{V}(t, x + \int_0^t \tilde{\pi}_u dS_u)$ is a submartingale, we have

$$E\tilde{V}\Big(t, x + \int_0^t \tilde{\pi}_u \, dS_u\Big) \leq \lim_{n\to\infty} E\tilde{V}\Big(t_n, x + \int_0^{t_n} \tilde{\pi}_u \, dS_u\Big). \qquad (A.9)$$

Let us show the inverse inequality. For $s = 0$ equality (A.4) takes the form

$$E\tilde{V}\Big(t, x + \int_0^t \tilde{\pi}_u \, dS_u\Big) = \inf_{\pi \in \tilde{\Pi}(\tilde{\pi}, t, T)} E\Big(U\Big(x + \int_0^T \pi_u \, dS_u\Big)\Big). \quad (A.10)$$

Therefore, for any $\varepsilon > 0$ there exists a strategy π^ε such that

$$E\tilde{V}\Big(t, x + \int_0^t \tilde{\pi}_u \, dS_u\Big) \geq$$

$$\geq E\Big(U\Big(x + \int_0^t \tilde{\pi}_u \, dS_u + \int_t^T \pi_u^\varepsilon \, dS_u\Big)\Big) - \varepsilon. \qquad (A.11)$$

Let us define a sequence $(\pi^n, n \geq 1)$ of strategies

$$\pi_s^n = \tilde{\pi}_s I_{(s<t_n)} + \pi_s^\varepsilon I_{(s \geq t_n)}.$$

Using inequality (A.11), the continuity of U (it follows from B1) and B2)), the convergence of the stochastic integrals and Fatou's lemma, we have

$$E\tilde{V}\Big(t, x + \int_0^t \tilde{\pi}_u \, dS_u\Big) \geq E\Big(U\Big(x + \int_0^t \tilde{\pi}_u \, dS_u + \int_t^T \pi_u^\varepsilon \, dS_u\Big)\Big) - \varepsilon =$$

$$= E\Big(\lim_n U\Big(x + \int_0^{t_n} \tilde{\pi}_u \, dS_u + \int_{t_n}^T \pi_u^\varepsilon \, dS_u\Big)\Big) - \varepsilon \geq$$

$$\geq \overline{\lim_n} E\Big(E\Big(U\Big(x + \int_0^{t_n} \tilde{\pi}_u \, dS_u + \int_{t_n}^T \pi_u^\varepsilon \, dS_u\Big)/\mathcal{F}_{t_n})) - \varepsilon \geq$$

$$\geq \overline{\lim_n} E\Big(\operatorname*{ess\,inf}_{\pi \in \tilde{\Pi}(\tilde{\pi}, t_n, T)} E\Big(U\Big(x + \int_0^{t_n} \tilde{\pi}_u \, dS_u + \int_{t_n}^T \pi_u \, dS_u\Big)/\mathcal{F}_{t_n}\Big)\Big) - \varepsilon =$$

$$= \overline{\lim_{n\to\infty}} 2E\Big(\tilde{V}\Big(t_n, x + \int_0^{t_n} \tilde{\pi}_u \, dS_u\Big)\Big) - \varepsilon. \qquad (A.12)$$

Here we may use the Fatou lemma, since by convexity of U and condition (A.8) we have that

$$U\Big(x+\int_0^{t_n}\tilde{\pi}_u\,dS_u+\int_{t_n}^{T}\pi_u^{\varepsilon}\,dS_u\Big)\le$$

$$\le \mathrm{Const}\bigg(U(x)+\sup_{s\le T}U\Big(\int_0^{s}\tilde{\pi}_u\,dS_u\Big)+\sup_{t\le s\le T}U\Big(\int_0^{t}\tilde{\pi}_u dS_u+\int_t^{s}\pi_u^{\varepsilon}\,dS_u\Big)+\eta\bigg)$$

and the right-hand-side of the latter inequality is integrable, since η is integrable and the strategies $\tilde{\pi}$ and $\pi=\tilde{\pi}_s I_{(s<t)}+\pi_s^{\varepsilon}I_{(s\ge t)}$ belong to the class $\tilde{\Pi}$.

Since ε is an arbitrary positive number, from (A.12) we obtain that

$$E\tilde{V}\Big(t,x+\int_0^{t}\tilde{\pi}_u\,dS_u\Big)\ge\lim_{n\to\infty}E\tilde{V}\Big(t_n,x+\int_0^{t_n}\tilde{\pi}_u\,dS_u\Big),\qquad (A.13)$$

which together with (A.9) implies that the function $(E\tilde{V}(t,x+\int_0^t\tilde{\pi}_u dS_u))$, $t\in[0,T])$ is right-continuous. □

Appendix B.

Let $(K(t),t\in R)$ be an increasing continuous function with continuous inverse $K^{-1}(t)$ and $K(\pm\infty)=\pm\infty$. Denote $\tau_t^{\varepsilon}=K^{-1}(K_t+\varepsilon)$, $\sigma_t^{\varepsilon}=K^{-1}(K_t-\varepsilon)$.

Lemma B. *For any K-integrable function F*

$$\int_R\frac{1}{\varepsilon}\int_s^{\tau_s^{\varepsilon}}\big|F(t)-F(s)\big|\,dK_t\,dK_s\to 0,\ \text{ as }\ \varepsilon\to 0.$$

Proof. Let first assume that F is a continuous and $F(t)=0$ if $|t|>T$ for some $T>0$. Then

$$\int_R\frac{1}{\varepsilon}\int_s^{\tau_s^{\varepsilon}}\big|F(s)-F(t)\big|\,dK_t\,dK_s\le\int_R\max_{t\le s\le\tau_t^{\varepsilon}}\big|F(s)-F(t)\big|\,dK_t\le$$
$$\le\max_{0\le s-t\le\tau_t^{\varepsilon}-t}\big|F(s)-F(t)\big|\quad\text{as }\ \varepsilon\le 0$$

since F is uniformly continuous on $[-T,T]$ and $\tau_t^{\varepsilon}-t\to 0$ as $\varepsilon\to 0$.

On the other hand

$$\int_R\frac{1}{\varepsilon}\int_s^{\tau_s^{\varepsilon}}\big|F(s)-F(t)\big|\,dK_t\,dK_s\le\big|F\big|_{L^1(R,dK)}+\int_R\frac{1}{\varepsilon}\int_s^{\tau_s^{\varepsilon}}\big|F(t)\big|\,dK_t\,dK_s\le$$

$$\big|F\big|_{L^1(R,dK)} + \int_R \frac{1}{\varepsilon}\int_{\sigma_t^\varepsilon}^{t} \big|F(t)\big|\, dK_s\, dK_t \le 2\big|F\big|_{L^1(R,dK)}, \qquad (B.1)$$

since by Fubini's theorem

$$\int_R \int_s^{\tau_s^\varepsilon} \big|F(t)\big|\, dK_t\, dK_s = \int_R \int_R 1_{(s\le t\le \tau_s^\varepsilon)}\big|F(t)\big|\, dK_s\, dK_t =$$

$$= \int_R \int_{\sigma_t^\varepsilon}^{t} \big|F(t)\big|\, dK_s\, dK_t \le \int_R \big|F(t)\big|(K_t - K_{\sigma_t^\varepsilon})dK_t \le \varepsilon\big|F\big|_{L^1(R,dK)}.$$

Using the inequality (B.1) we can approximate each function $F \in L^1(R,dK)$ by means of continuous functions with compact support. This completes the proof. □

Corollary. *For $F \in L^1(R,dK)$*

$$\int_R \bigg|\frac{1}{\varepsilon}\int_s^{\tau_s^\varepsilon} F(t)\, dK_t - F(s)\bigg|\, dK_s \to 0, \quad \text{as } \varepsilon \to 0$$

and if $\int_t^{\tau_t^\varepsilon} F(s)dK_s = 0$, dK-a.s., then $F_t = 0$ dK-a.s.

Proposition B. *Let $(f(t,x),(t,x)\in R^2)$ and $(X(t,s), t \ge s)$ be measurable functions such that the family $x \to f(\cdot,x)$ is continuous in $L^1(R,dK)$ and $X(s,t)$ is a continuous function on $\{(t,s); t \ge s\}$ with $X(s,s) = x$ for all $s \in R$ and some $x \in R$.*

Then

$$\int_R \bigg|\frac{1}{\varepsilon}\int_s^{\tau_s^\varepsilon} f\big(t, X(t,s)\big)\, dK_t - f(s,x)\bigg|\, dK_s \to 0, \quad \text{as } \varepsilon \to 0.$$

Proof. Denote by b_t^ε expression $\max_{\sigma_t^\varepsilon \le s \le t} |X(t,s) - x|$. Then

$$\int_R \frac{1}{\varepsilon}\int_s^{\tau_s^\varepsilon} \big|f(t,X(t,s)) - f(s,x)\big|\, dK_t\, dK_s \le$$

$$\int_R \frac{1}{\varepsilon}\int_{\sigma_t^\varepsilon}^{t} \big|f(t,X(t,s)) - f(t,x)\big|\, dK_s\, dK_t +$$

$$\int_R \frac{1}{\varepsilon}\int_s^{\tau_s^\varepsilon} \big|f(t,x) - f(s,x)\big|\, dK_t\, dK_s$$

The first term in the latter expression can be estimated by

$$\int\limits_R \max_{|x-y|\le b_t^\varepsilon} \big|f(t,x)-f(t,y)\big|\,dK_t.$$

Since $X(\cdot,\cdot)$ is continuous then $b_t^\varepsilon \to 0$ uniformly on each $[-T,T]$ as $\varepsilon\to 0$ and by continuity of the family $f(\cdot,x)\in L^1$ we get that the first summand tends to zero. The second summand tends to zero by Lemma B. $\square$

Remark. If the functions f and K are defined on the subsets $[0,T]\times R$ and $[0,T]$ respectively we can consider the functions

$$\tilde f(t,x)=\begin{cases} f(t,x),(t,x)\in[0,T\times R,\\ 0,\ (t,x)\in[0,T]\times R\end{cases},\quad \tilde K(t)=\begin{cases}K(t),t\in[0,T]\\ t+K(0),t<0\\ K(T)+t-T,t>T\end{cases}$$

and furter we can use the Proposition B.

Appendix C.

Assume that B3') there exist $\gamma>0$, a positive integrable random variable ξ and $p>1$ such that $U(x)\ge\gamma|x|^p-\xi$.

Note that function $U(x)=|H-x|^p$ for $H\in L^p$ satisfies B3') as well as conditions B1)-B2) of Section 2.

Proposition C. *Suppose that one of the assertions of Proposition* 2.1 *and conditions B*1)*, B*2) *and B*3$'$) *are satisfied. Then for any t and x the problem*

$$\operatorname*{ess\,inf}_{\pi\in\Pi} E\bigg(U\bigg(x+\int\limits_t^T \pi_s\,dS_s\bigg)/\mathcal{F}_t\bigg)$$

admits a unique solution with p-integrable wealth process.

Proof. By the lattice property (see Appendix A) we can choose a sequence $\tilde\pi^n\in\Pi$ such that $E(U(x+\int\limits_t^T\tilde\pi_s^n\,dS_s)/\mathcal{F}_t)\downarrow V(t,x)$ P-a.s. By condition B1) one can choose the sequence $\tilde\pi_n$ so that $E(U(x+\int_t^T\tilde\pi_s^n dS_s)/\mathcal{F}_t)\le E(U(x)/\mathcal{F}_t)$ for all $n\ge1$. Thus $E(U(x+\int\limits_t^T\tilde\pi_s^n\,dS_s)\to EV(t,x)$ as $n\to\infty$. By condition B3') there exists $R>0$

$$\gamma E\bigg|x+\int\limits_t^T\tilde\pi_s^n\,dS_s\bigg|^p\le EU\bigg(x+\int\limits_t^T\tilde\pi_s^n\,dS_s\bigg)+E\xi\le R.$$

Hence $x+\int_t^T \tilde{\pi}_s^n dS_s$ is bounded sequence in the space L^p and we can assume that it converges weakly. By Masur's lemma (see, e.g., [6]) there exists a sequence of strategies

$$\pi^n = \sum_{k=n}^{q(n)} \alpha_{kn} \tilde{\pi}^{kn}, \quad \text{where } \ q(n) > n, \quad \sum_{k=n}^{q(n)} \alpha_{kn} = 1, \ \ \alpha_{kn} \geq 0$$

such that $\int\limits_t^T \pi_s^n \, dS_s \to \int\limits_t^T \pi_s^* \, dS_s$ in L^p for some $\pi^* \in \Pi$. We can assume also that $\int\limits_t^T \pi_s^n \, dS_s \to \int\limits_t^T \pi_s^* \, dS_s$ P- a.s..

By convexity of U we have

$$E\Big[U\Big(x + \int\limits_t^T \pi_s^n dS_s\Big)/\mathcal{F}_t\Big] \leq E\Big[U\Big(x + \int\limits_t^T \tilde{\pi}_s^n dS_s\Big)/\mathcal{F}_t\Big].$$

Therefore

$$\overline{\lim_{n\to\infty}} E\Big[U\Big(x+\int\limits_t^T \pi_s^n \, dS_s\Big)\big|\mathcal{F}_t\Big] \leq \lim_{n\to\infty} E\Big[U\Big(x+\int\limits_t^T \tilde{\pi}_s^n \, dS_s\Big)\big|\mathcal{F}_t\Big] = V(t,x).$$

On the other hand the Fatou's lemma implies that

$$E\Big[U\Big(x + \int\limits_t^T \pi_s^* \, dS_s\Big)\big|\mathcal{F}_t\Big] \leq \underline{\lim_{n\to\infty}} E\Big[U\Big(x + \int\limits_t^T \pi_s^n \, dS_s\Big)\big|\mathcal{F}_t\Big] \ \text{ P-a.s..}$$

Therefore, π^* is optimal and π^* is unique by Remark 2.4. □

Acknowledgement

Research supported by Grant INTAS 99 00559.

We would like to thank N. Lazrieva for useful remarks and comments.

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(Received 18.05.2002)

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