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BACKWARD STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS RELATED TO UTILITY MAXIMIZATION AND HEDGING

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Abstract. We study the utility maximization problem, the problem of minimization of the hedging error and the corresponding dual problems using dynamic programming approach. We consider an incomplete financial market model, where the dynamics of asset prices are described by an $\mathbb{R}^d$-valued continuous semimartingale. Under some regularity assumptions, we derive the backward stochastic PDEs for the value functions related to these problems, and for the primal problem, we show that the strategy is optimal if and only if the corresponding wealth process satisfies a certain forward SDE. As examples we consider the mean-variance hedging problem and the cases of power, exponential, logarithmic utilities, and corresponding dual problems.

CONTENTS

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PART 1

BACKWARD STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS RELATED TO THE UTILITY MAXIMIZATION PROBLEM AND HEDGING

1.1. Introduction

Portfolio optimization, hedging, and derivative pricing are fundamental problems in mathematical finance, which are closely related to each other. The basic optimization problem of mathematical finance, the optimal portfolio choice or hedging, is to optimize

$$E[U(X_T^{x,\pi})] \quad \text{over all } \pi \text{ in the class } \Pi \text{ of strategies,} \tag{1.1.1}$$

where

$$X_t^{x,\pi} = x + \int_0^t \pi_u dS_u$$

is the wealth process starting from the initial capital x, determined by the self-financing trading strategy π, and Π is some class of admissible strategies. U is the objective function, which can also depend on ω. It can be interpreted as the utility function or a function that measures a hedging error.

If $U(x) = (x - H)^2$, where H is a contingent claim at time T, then (1.1.1) corresponds to the well-known mean-variance hedging problem

$$\text{minimize } E(X_T^{x,\pi} - H)^2 \text{ over all } \pi \in \Pi \tag{1.1.2}$$

introduced by Föllmer and Sondermann [29] and then developed by numerous authors (see, e.g., [20, 33, 36, 37, 83–85] for further generalizations and related results).

If the objective function U is the utility function, then (1.1.2) is the utility maximization problem

$$\text{maximize } E[U(X_T^{x,\pi})] \text{ over all } \pi \in \Pi, \tag{1.1.3}$$

i.e., for a given initial capital $x > 0$, the goal is to maximize the expected value from the terminal wealth.

The utility maximization problem was first studied by Merton (1971) in a classical Black–Scholes model. Using the Markov structure of the model, he derived the Bellman equation for the value function of the problem and obtained a closed-form solution of this equation in the cases of power, logarithmic, and exponential utility functions.

For general complete market models, it was shown by Pliska (1986), Cox and Huang (1989), and Karatzas et al. (1987) that the optimal portfolio of the utility maximization problem is (up to a constant) equal to the density of the martingale measure unique for complete markets. As was shown by He and Pearson (1991) and Karatzas et al. (1991), for incomplete markets described by the Itô processes, this method gives a duality characterization of optimal portfolios provided by the set of martingale measures. Their idea was to solve the dual problem of finding the suitable optimal martingale measure and then to express the solution of the primal problem by using the convex duality. Extending the domain of the dual problem, the approach was generalized to semimartingale models under weaker conditions on the utility functions by Kramkov and Schachermayer (1999) and Cvitanic, Schachermayer, and Wong (2001).

These approaches mainly give a reduction of the basic primal problem to the solution of the dual problem, but the constructive solution of the dual problem for general models of incomplete markets is itself a demanding task.

Our goal is to derive a semimartingale Bellman equation (stochastic version of the Bellman equation) directly related to the basic (or primal) optimization problem, to study the well-posedness of such equations, and to give constructions of optimal strategies. The application of the dynamic programming approach directly to the primal optimization problem can represent a valuable alternative to the commonly used convex duality approach in many cases.

Let S be an $\mathbb{R}^d$-valued continuous semimartingale defined on a filtered probability space satisfying the usual conditions. The process S describes the discounted price evolution of d risky assets in a financial market also containing a riskless bond of constant price. To exclude arbitrage opportunities, we assume that the set $\mathcal{M}^e$ of equivalent martingale measures for S is nonempty. Since S is continuous, the existence of an equivalent martingale measure implies that the structure condition is satisfied, i.e., S admits the decomposition

$$S_t = M_t + \int_0^t d\langle M\rangle_s \lambda_s, \quad \langle \lambda \cdot M\rangle < \infty \quad \text{for all } t \text{ a.s.,} \tag{1.1.4}$$

where M is a continuous local martingale and λ is a predictable $\mathbb{R}^d$-valued process.

We consider the utility function U mapping $(0, \infty)$ into $\mathbb{R}$. It is assumed to be continuously differentiable, strictly increasing, and strictly concave, and it satisfies the Inada conditions

$$U'(0) = \lim_{x\to 0} U'(x) = \infty, \quad U'(\infty) = \lim_{x\to\infty} U'(x) = \infty.$$

Also, we set $U(0) = \lim\limits_{x\to 0} U(x)$ and $U(x) = -\infty$ for all $x < 0$.

For any $x \in \mathbb{R}_+$, denote by Π_x the class of predictable S-integrable processes π such that the corresponding wealth process is nonnegative at any instant, i.e.,

$$X_t^{x,\pi} = x + \int_0^t \pi_u dS_u \geq 0 \quad \forall t \in [0, T].$$

For simplicity, in the introduction, we consider the case with a single risky asset.

Let us introduce the dynamical value function of problem (1.1.3) defined as

$$V(t, x) = \operatorname*{ess\,sup}_{\pi\in\Pi_x} E\left(U\left(x + \int_t^T \pi_u dS_u\right) \Big/ \mathcal{F}_t\right). \tag{1.1.5}$$

The classical Itô formula (or its generalization given by Krylov in 1980) plays a crucial role in deriving the Bellman equation for the value function of controlled diffusion processes. For our purposes, the Itô formula is no longer sufficient, since the function V also depends on ω even if U is deterministic. Therefore, the Itô–Ventzell formula must be used.

Under some regularity assumptions on the value function (sufficient for the application of the Itô–Ventzell formula), in Theorem 1.3.1, we show that the value function defined by (1.1.5) satisfies the following backward stochastic partial differential equation (BSPDE):

$$V(t, x) = V(0, x) + \frac{1}{2}\int_0^t \frac{(\varphi_x(s, x) + \lambda(s)V_x(s, x))^2}{V_{xx}(s, x)} d\langle M\rangle_s + \int_0^t \varphi(s, x)dM_s + L(t, x) \tag{1.1.6}$$

with the boundary condition

$$V(T, x) = U(x),$$

where $L(t, x)$ is a local martingale orthogonal to M for all x and the subscripts φ_x, V_x, and V_{xx} denote the partial derivatives. Moreover, a strategy π^* is optimal if and only if the corresponding wealth process X^{π^*} is a solution of the forward SDE

$$X_t^{\pi^*} = X_0^{\pi^*} - \int_0^t \frac{\varphi_x(u, X_u^{\pi^*}) + \lambda(u)V_x(u, X_u^{\pi^*})}{V_{xx}(s, X_u^{\pi^*})} dS_u. \tag{1.1.7}$$

Thus, to construct the optimal strategy, we need:

(1) first, to solve the backward equation (1.1.6) (which determines V and φ simultaneously) and substitute the corresponding derivatives of V and φ in Eq. (1.1.7);
(2) then to solve the forward equation (1.1.7) with respect to X^{π^*};
(3) finally, to reproduce the optimal strategy π^* from the corresponding wealth process X^{π^*}.

Theorem 1.3.1 is a verification theorem, since we require conditions directly imposed on the value function V (but not only on the function U). Therefore, we cannot state that the solution of Eq. (1.1.6) exists, but for the standard utility functions, all the conditions of Theorem 1.3.1 are satisfied, and in these cases, the existence of a unique solutions of the corresponding backward equations follows from this theorem.

If $U(x) = x^p$, $p \in (0,1)$, then (1.1.3) corresponds to the power utility maximization problem:

$$\text{maximize } E(x + \int_0^T \pi_u dS_u)^p \text{ over all } \pi \in \Pi_x. \tag{1.1.8}$$

In this case, $V(t,x) = x^p V_t$, where V_t is a semimartingale and all condition of Theorem 1.3.1 are satisfied. This theorem implies that the process V_t satisfies the following backward stochastic differential equation (BSDE):

$$V_t = V_0 + \frac{q}{2}\int_0^t \frac{(\varphi_s + \lambda_s V_s)^2}{V_s} d\langle M\rangle_s + \int_0^t \varphi_s dM_s + L_t, \quad V_T = 1, \tag{1.1.9}$$

where $q = p/(p-1)$ and L is a local martingale strongly orthogonal to M. In addition, Eq. (1.1.7) is transformed into the linear equation

$$X_t^* = x + (1-q)\int_0^t \frac{\varphi_u + \lambda_u V_u}{V_u} X_u^* dS_u \tag{1.1.10}$$

for the optimal wealth process.

Therefore,

$$X_t^* = x\mathcal{E}_t\left((1-q)\left(\frac{\varphi}{V} + \lambda\right)\cdot S\right),$$

and the optimal strategy is of the form

$$\pi_t^* = x(1-q)\left(\frac{\varphi_t}{V_t} + \lambda_t\right)\mathcal{E}_t\left((1-q)\left(\frac{\varphi}{V} + \lambda\right)\cdot S\right).$$

If we assume that $U(x)$ is strictly convex (for each ω), then we can interpret U as a function that measures a hedging error and consider the problem

$$\text{minimize } E[U(X_T^{x,\pi})] \text{ over all } \pi \text{ from } \Pi, \tag{1.1.11}$$

where the class Π will be specified later.

Note that the corresponding value function

$$V(t,x) = \operatorname*{ess\,inf}_{\pi\in\Pi} E\left(U\left(x + \int_t^T \pi_u dS_u\right)/\mathcal{F}_t\right) \tag{1.1.12}$$

satisfies the same Eq. (1.1.6) as the value function of problem (1.1.3) (in this case, V_{xx} is negative, and hence $V(t,x)$ is now a submartingale for all $x \in \mathbb{R}^+$). Equations for optimal wealth processes are also the same, and the proof (see Theorem 1.5.1) is mainly similar to the proof of Theorem 1.3.1.

If $U(x) = (x-H)^2$, where H is a contingent claim at time T, as was mentioned above, (1.1.11) corresponds to the mean-variance hedging problem.

We show that in this case, $V(t,x)$ is a quadratic trinomial of the form $V(t,x) = V_0(t) - 2V_1(t)x + V_2(t)x^2$ and Eq. (1.1.6) gives a triangular system of backward equations for the coefficients V_i, $i = 0,1,2$, of the value function. Moreover, Eq. (1.1.7) transforms into the linear equation

$$X_t^* = x + \int_0^t \frac{\varphi_1(s) + \lambda(s)V_1(s)}{V_2(s)} dS_s - \int_0^t \frac{\varphi_2(s) + \lambda(s)V_2(s)}{V_2(s)} X_s^* dS_s \tag{1.1.13}$$

for the optimal wealth process. A similar result was obtained in [2] for Markov diffusion processes by using the dynamic programming approach.

Note that (1.1.13) gives an alternative equivalent form to the well-known feedback form solution of problem (1.1.2), which is usually derived using the density process of the variance-optimal martingale measure [36] (see also [37, 73, 79, 86]). At the end of Sec. 1.5, we also establish relations between Eqs. (1.1.13) and (1.5.21) derived in [36] and between the equations for V_2 and for the value process of the variance-optimal martingale measure (see [49, 61]).

The main tools of the work are backward stochastic differential equations, which were introduced by J. M. Bismut in [4] for the linear case as the equations for the adjoint process in the stochastic maximum principle; other works on the maximum principle in stochastic control were written by Yu. Kabanov [42] and V. Arkin and M. Saksonov [1]. In [7, 74], the well-posedness results for BSDEs with more general generators were obtained (see also [24] for references and related results). The semimartingale backward equation that is a stochastic version of the Bellman equation in an optimal control problem was first derived in [7] by R. Chitashvili.

The main results of this part are based on the authors' papers [62, 63, 66].

1.2. Basic Assumptions and Some Auxiliary Facts

We consider an incomplete financial market model, in which the dynamics of asset prices are described by an $\mathbb{R}^d$-valued continuous semimartingale S defined on a filtered probability space $(\Omega, F, \mathcal{F} = (\mathcal{F}_t,\ t \in [0,T]), P)$ satisfying the usual conditions, where $F = \mathcal{F}_T$ and $T < \infty$ is a fixed time horizon. For all unexplained notation from the martingale theory, we refer the reader to [19, 39, 52].

Denote by $\mathcal{M}^e$ the set of martingale measures, i.e., the set of measures Q equivalent to P on $\mathcal{F}_T$ such that S is a local martingale under Q. Let $Z_t(Q)$ be the density process of Q with respect to the basic measure P, which is a strictly positive uniformly integrable martingale. For any $Q \in \mathcal{M}^e$, there is a P-local martingale M^Q such that $Z(Q) = \mathcal{E}(M^Q) = (\mathcal{E}_t(M^Q), t \in [0,T])$, where $\mathcal{E}(M)$ is the Doleans-Dade exponential of M.

Recall the definition of BMO-martingales and the reverse Hölder condition.

A square integrable continuous martingale M belongs to the class BMO if there is a constant $C > 0$ such that

$$E^{1/2}(\langle M\rangle_T - \langle M\rangle_\tau | F_\tau) \le C \quad P\text{-a.s.}$$

for every stopping time τ. The smallest constant with this property (or $+\infty$ if this does not exist) is called the BMO norm of M and is denoted by $\|M\|_{\mathrm{BMO}}$.

A strictly positive adapted process Z satisfies the reverse Hölder inequality $R_p(P)$, where $1 < p < \infty$, if there is a constant C such that

$$E\left(\left(\frac{Z_T}{Z_\tau}\right)^p |F_\tau\right) \le C \quad P\text{-a.s.}$$

for every stopping time τ.

Proposition 1.2.1 (Kazamaki [41]). *If M is a continuous BMO-martingale, then $\mathcal{E}(M)$ is a uniformly integrable martingale.*

The following assertion relates BMO and the reverse Hölder condition.

Proposition 1.2.2 (Doleans-Dade and Meyer [21]). *Let M be a local martingale and $\mathcal{E}(M)$ be its Doleans exponential. The following assertions are equivalent*:

(i) *M belongs to the class BMO*;

(ii) *$\mathcal{E}(M)$ is a uniformly integrable martingale satisfying the reverse Hölder inequality $R_p(P)$ for some $p > 1$.*

Let Π_x be the space of all predictable S-integrable processes π such that the corresponding wealth process is nonnegative at any instant, i.e.,

$$x + \int_0^t \pi_u dS_u \geq 0 \quad \forall t \in [0, T].$$

In the sequel, sometimes, we use the notation $(\pi \cdot S)_t$ for the stochastic integral $\int_0^t \pi_u dS_u$.

Assume that the objective function $U(x) = U(\omega, x)$ satisfies the following conditions:

(B1) $V(0, x) < \infty$ for some x;

(B2) $U(x)$ is the utility function P-a.s.;

(B3) the optimization problem (1.1.3) admits a solution, i.e., for any t and x, there is a strategy $\pi^*(t, x)$ such that

$$V(t,x) = E\left(U\left(x + \int_t^T \pi_s^*(t,x) dS_s\right) / \mathcal{F}_t\right). \tag{1.2.1}$$

Remark 1.2.1. As was shown by Kramkov and Schachermayer (1999), a sufficient (and necessary) condition for (B3) is that the utility function $U(x)$ have an asymptotic elasticity strictly less than 1, i.e.,

$$AE(U) = \limsup_{x \to \infty} \frac{xU_x(x)}{U(x)} < 1.$$

Remark 1.2.2. The strict concavity of U implies that the optimal strategy is unique, if it exists. Indeed, if there exist two optimal strategies π^1 and π^2, then by the concavity of U, the strategy $\bar{\pi} = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$ is also optimal. Therefore,

$$\frac{1}{2}E\left[U\left(x + \int_t^T \pi_s^1 dS_s\right) | \mathcal{F}_t\right] + \frac{1}{2}E\left[U\left(x + \int_t^T \pi_s^2 dS_s\right) | \mathcal{F}_t\right] = E\left[U\left(x + \int_t^T \bar{\pi}_s dS_s\right) | \mathcal{F}_t\right]$$

and

$$\frac{1}{2}U\left(x + \int_t^T \pi_s^1 dS_s\right) + \frac{1}{2}U\left(x + \int_t^T \pi_s^2 dS_s\right) = U\left(x + \int_t^T \bar{\pi}_s dS_s\right) \quad P\text{-a.s.}$$

Now the strict concavity of U leads to the relation

$$\int_t^T \pi_s^1 dS_s = \int_t^T \pi_s^2 dS_s.$$

For convenience, we give the proof of the following well-known assertion.

Lemma 1.2.1. *Under conditions* (B1)–(B3), *the value function $V(t, x)$ is a strictly concave function with respect to x.*

Proof. The concavity of $V(t,x)$ follows from (B2) and (B3), since for any $\alpha, \beta \in [0,1]$ with $\alpha+\beta=1$ and any $x_1, x_2 \in \mathbb{R}$, we have

$$\alpha V(t,x_1)+\beta V(t,x_2) = \alpha E\left[U\left(x_1+\int_t^T \pi_u^*(t,x_1)dS_u\right)|\mathcal{F}_t\right] + \beta E\left[U\left(x_2+\int_t^T \pi_u^*(t,x_2)dS_u\right)|\mathcal{F}_t\right]$$

$$\geq E\left[U\left(\alpha x_1+\beta x_2+\int_t^T (\alpha\pi_u^*(t,x_1)+\beta\pi_u^*(t,x_2))dS_u\right)|\mathcal{F}_t\right] \geq V(t,\alpha x_1+\beta x_2). \quad (1.2.2)$$

To show that $V(t,x)$ is strictly concave, we need to verify that if

$$\alpha V(t,x_1)+\beta V(t,x_2) = V(t,\alpha x_1+\beta x_2) \quad (1.2.3)$$

holds for some $\alpha,\beta \in (0,1)$ with $\alpha+\beta=1$, then $x_1=x_2$.

Indeed, if Eq. (1.2.3) holds, then from (1.2.2) and the strict convexity of U, it follows that

$$x_1+\int_t^T \pi_u^*(t,x_1)dS_u = x_2+\int_t^T \pi_u^*(t,x_2)dS_u \quad P\text{-a.s.},$$

which implies $x_1=x_2$. □

Remark 1.2.3. The concavity of $V(0,x)$ and condition (B1) imply that $V(0,x)<\infty$ for all $x \in \mathbb{R}$.

The Itô–Ventzell formula. Let $(Y(t,x), t \in [0,T], x \in \mathbb{R})$ be a family of special semimartingales with the decomposition

$$Y(t,x) = Y(0,x)+B(t,x)+N(t,x), \quad (1.2.4)$$

where $B(\cdot,x) \in \mathcal{A}_{\text{loc}}$ and $N(\cdot,x) \in \mathcal{M}_{\text{loc}}$ for any $x \in \mathbb{R}$. By the Galtchouk–Kunita–Watanabe (GKW) decomposition of $N(\cdot,x)$ with respect to M, a parametrized family of semimartingales Y admits the representation

$$Y(t,x) = Y(0,x)+B(t,x)+\int_0^t \psi(s,x)dM_s+L(t,x), \quad (1.2.5)$$

where $L(\cdot,x)$ is a local martingale strongly orthogonal to M for all $x \in \mathbb{R}$.

Assume that the following conditions hold.

(C1) There exists a predictable increasing process $(K_t, t \in [0,T])$ such that $B(\cdot,x)$ and $\langle M\rangle$ are absolutely continuous with respect to K, i.e., there is a measurable function $b(t,x)$ predictable for every x and a matrix-valued predictable process ν_t such that

$$B(t,x) = \int_0^t b(s,x)dK_s, \quad \langle M\rangle_t = \int_0^t \nu_s dK_s.$$

Note that, by continuity of M, the square characteristic $\langle M\rangle$ is absolutely continuous with respect to the continuous part K^c of the process K and

$$\langle M\rangle_t = \int_0^t \nu_s dK_s^c = \int_0^t \nu_s dK_s.$$

Without loss of generality, we can assume that ν is bounded, and in the sequel, the inner product $u'\nu_t v$ for $u,v \in \mathbb{R}^d$ is denoted by $(u,v)_{\nu_t}$;

(C2) the mapping $x \to Y(t,x)$ is twice continuously differentiable for all (ω,t);

(C3) the first derivative $Y_x(t,x)$ is a special semimartingale admitting the decomposition

$$Y_x(t,x) = Y_x(0,x) + B_{(x)}(t,x) + \int_0^t \psi_x(s,x)dM_s + L_{(x)}(t,x), \tag{1.2.6}$$

where $B_{(x)}(\cdot,x) \in \mathcal{A}_{\mathrm{loc}}$, $L_{(x)}(\cdot,x)$ is a local martingale orthogonal to M for all $x \in \mathbb{R}$, and ψ_x is the partial derivative of ψ at x (note that $A_{(x)}$ and $L_{(x)}$ are not assumed to be the derivatives of A and L, respectively, whose existence does not necessarily follow from condition (C2));

(C4) $Y_{xx}(t,x)$ is RCLL process for every $x \in \mathbb{R}$;

(C5) the functions $b(s,\cdot)$, $\psi(s,\cdot)$, and $\psi_x(s,\cdot)$ are continuous at x μ^K-a.e.;

(C6) for any $c > 0$,

$$E\int_0^T \sup_{|x|\le c} g(s,x)dK_s < \infty$$

for g equal to $|b|$, $|\psi|^2$, and $|\psi|_x^2$.

In what follows, we need the following version of the Itô–Ventzell formula.

Proposition 1.2.3. *Let $(Y(\cdot,x), x \in \mathbb{R})$ be a family of special semimartingales satisfying conditions* (C1)–(C6) *and $X^\pi = x + \pi \cdot S$. Then the transformed process $Y(t, X_t^\pi)$, $t \in [0,T]$ is a special semimartingale with the decomposition*

$$Y(t,X_t^\pi) = Y(0,c) + B_t + N_t,$$

where

$$B_t = \int_0^t \left[Y_x(s,X_s^\pi)\lambda_s' d\langle M\rangle_s \pi_s + \psi_x(s,X_s^\pi)' d\langle M\rangle_s \pi_s + \frac{1}{2} Y_{xx}(s,X_s^\pi)\pi_s' d\langle M\rangle_s \pi_s\right] + \int_0^t b(s,X_s^\pi)dK_s \tag{1.2.7}$$

and N is a continuous local martingale.

One can derive this assertion from of [50, Theorem 1.1] or [10, Theorem 2]. Here we do not require any conditions on $L(t,x)$ imposed in [10, 50], since the martingale part of the substituted process X^π is orthogonal to $L(\cdot,x)$ and since we do not give an explicit expression of the martingale part N, because this is not necessary for our purposes.

Remark 1.2.4. Since the semimartingale S is assumed to be continuous and is of the form (1.1.4), only the latter term of (1.2.7) may have the jumps, i.e., the process K is not continuous in general.

1.3. Backward Stochastic Partial Differential Equation for the Value Function

In this section, we derive the backward stochastic partial differential equation for the value function related to the utility maximization problem.

Denote by $\mathcal{V}^{1,2}$ the class of functions $Y : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfying conditions (C1)–(C6).

Let us consider the following backward stochastic partial differential equation (BSPDE):

$$Y(t,x) = Y(0,x) + \frac{1}{2}\int_0^t \frac{(\psi_x(s,x) + \lambda(s)Y_x(s,x))'}{Y_{xx}(s,x)} d\langle M\rangle_s \big(\psi_x(s,x) + \lambda(s)Y_x(s,x)\big) + \int_0^t \psi(s,x)dM_s + L(t,x), \quad L(\cdot,x)\perp M, \tag{1.3.1}$$

with the boundary condition

$$Y(T, x) = U(x). \tag{1.3.2}$$

We say that Y solves problem (1.3.1), (1.3.2) if:

(i) $Y(\omega, t, x)$ is twice continuously differentiable for each (ω, t) and satisfies the boundary condition (1.3.2);

(i) $Y(t, x)$ and $Y_x(t, x)$ are special semimartingales admitting decompositions (1.2.5) and (1.2.6), respectively, where ψ_x is the partial derivative of ψ at x;

(iii) P-a.s. for all $x \in \mathbb{R}$,

$$B(t, x) = \frac{1}{2} \int_0^t \frac{(\psi_x(s, x) + \lambda(s) Y_x(s, x))'}{Y_{xx}(s, x)} d\langle M \rangle_s \big(\psi_x(s, x) + \lambda(s) Y_x(s, x)\big). \tag{1.3.3}$$

Remark 1.3.1. If we substitute the expression of $B(t, x)$ given by Eq. (1.3.3) in the canonical decomposition (1.2.5) for Y, then we obtain Eq. (1.3.1).

According to Proposition 1.7.1, the value process $V(t, x)$ is a supermartingale for any $x \in \mathbb{R}$ that admits the canonical decomposition

$$V(t, x) = V(0, x) + A(t, x) + \int_0^t \varphi(s, x) dM_s + m(t, x), \tag{1.3.4}$$

where $-A(\cdot, x) \in \mathcal{A}^+$ and $m(\cdot, x)$ is a local martingale strongly orthogonal to M for all $x \in \mathbb{R}_+$.

Assume that $V \in \mathcal{V}^{1,2}$. This implies that $V_x(t, x)$ is a special semimartingale with the decomposition

$$V_x(t, x) = V_x(0, x) + A_{(x)}(t, x) + \int_0^t \varphi_x(s, x) dM_s + m_{(x)}(t, x), \tag{1.3.5}$$

where $A_{(x)}(\cdot, x) \in \mathcal{A}_{\mathrm{loc}}$, $m_{(x)}(\cdot, x)$ is a local martingale orthogonal to M for all $x \in \mathbb{R}_+$, and φ_x coincides with the partial derivative of φ (μ^K-a.e.). Moreover,

$$A(t, x) = \int_0^t a(s, x) dK_s$$

for a measurable function $a(t, x)$.

Proposition 1.3.1. *Assume that conditions* (B1) *and* (B2) *are satisfied and the value function* $V(t, x)$ *belongs to the class* $\mathcal{V}^{1,2}$. *Then the inequality*

$$a(s, x) \le \frac{1}{2} \frac{|\varphi_x(s, x) + \lambda(s) V_x(s-, x)|^2_{\nu_s}}{V_{xx}(s-, x)} \tag{1.3.6}$$

holds for all $x \in \mathbb{R}$ μ^K*-a.e. Moreover, if the strategy* π^* *is optimal, then the corresponding wealth process* X^{π^*} *is a solution of the forward SDE*

$$X_t^{\pi^*} = X_0^{\pi^*} - \int_0^t \frac{\varphi_x(s, X_s^{\pi^*}) + \lambda(s) V_x(s, X_s^{\pi^*})}{V_{xx}(s, X_s^{\pi^*})} dS_s. \tag{1.3.7}$$

Proof. Using the Itô–Ventzell formula (Proposition 1.2.3) for the function $V(t, x, \omega) \in \mathcal{V}^{1,2}$ and for the process

$$\left(x + \int_s^t \pi_u dS_u, \ s \le t \le T \right)$$

we have

$$V\left(t, x+\int_s^t \pi_u dS_u\right) = V(s,x) + \int_s^t a\left(u, x+\int_s^u \pi_v dS_v\right) dK_u$$
$$+\int_s^t G\left(u, \pi_u, x+\int_s^u \pi_v dS_v\right) dK_u + N_t - N_s, \quad (1.3.8)$$

where

$$G(t,p,x,\omega) = V_x(t-,x)p'\nu_t\lambda(t) + p'\nu_t\varphi_x(t,x) + \frac{1}{2}V_{xx}(t-,x)p'\nu_t p \quad (1.3.9)$$

and N is a martingale. Since by Proposition 1.7.1 of the Appendix, the process

$$\left(V(t, x+\int_s^t \pi_u dS_u),\ t\in[s,T]\right)$$

is a supermartingale for all $s \geq 0$ and $\pi \in \Pi$, the process

$$-\int_s^t \left[G\left(u,\pi_u, x+\int_s^u \pi_v dS_v\right) + a\left(u, x+\int_s^u \pi_v dS_v\right)\right] dK_u$$

is increasing for any $s \geq 0$. Hence the process

$$-\int_s^t \left[G\left(u,\pi_u, x+\int_s^u \pi_v dS_v\right) + a\left(u, x+\int_s^u \pi_v dS_v\right)\right] dK_u^c$$

is also increasing for any $s \geq 0$, where $K = K^c + K^d$ is a decomposition of K into continuous and purely discontinuous increasing processes. Therefore, taking $\tau_s(\varepsilon) = \inf\{t \geq s : K_t^c - K_s^c \geq \varepsilon\}$ instead of t, we have that for any $\varepsilon > 0$ and $s \geq 0$,

$$\frac{1}{\varepsilon}\int_s^{\tau_s(\varepsilon)} a\left(u, x+\int_s^u \pi_v dS_v\right) dK_u^c \leq -\frac{1}{\varepsilon}\int_s^{\tau_s(\varepsilon)} G\left(u,\pi_u, x+\int_s^u \pi_v dS_v\right) dK_u^c. \quad (1.3.10)$$

Passing to the limit in (1.3.10) as $\varepsilon \to 0$, from Proposition 1.7.2 of the Appendix we obtain

$$a(s,x) \leq -G(s,\pi_s,x) \quad \mu^{K^c}\text{-a.e.}$$

for all $\pi \in \Pi$. Thus,

$$a(t,x) \leq \operatorname*{ess\,inf}_{\pi\in\Pi}\big(-G(t,\pi_t,x)\big) \quad \mu^{K^c}\text{-a.e.}$$

On the other hand,

$$\operatorname*{ess\,sup}_{\pi\in\Pi}\big(-G(t,\pi_t,x)\big) = \frac{|V_x(t-,x)\lambda(t)+\varphi_x(t,x)|^2_{\nu_t}}{2V_{xx}(t-,x)}$$
$$+\operatorname*{ess\,sup}_{\pi\in\Pi}\left(-\frac{1}{2}V_{xx}(t-,x)\Big|_{\pi_t} + \frac{V_x(t-,x)\lambda(t)+\varphi_x(t,x)}{V_{xx}(t-,x)}\Big|^2_{\nu_t}\right) = \frac{|V_x(t-,x)\lambda(t)+\varphi_x(t,x)|^2_{\nu_t}}{2V_{xx}(t-,x)}. \quad (1.3.11)$$

Indeed, since $V_{xx} < 0$, by Lemma 1.7.2 of Appendix 1.7, (1.3.11) holds.

Thus, for every $x \in \mathbb{R}_+$, we have

$$a(t,x) \leq \frac{|V_x(t-,x)\lambda(t)+\varphi_x(t,x)|^2_{\nu_t}}{2V_{xx}(t-,x)} \quad \mu^{K^c}\text{-a.e.}$$

Since μ^K-a.e. $a(t,x) \geq 0$ and $\mu^{K^d}\{\nu \neq 0\} = 0$, we obtain

$$a(t,x) \leq \frac{|V_x(t-,x)\lambda(t) + \varphi_x(t,x)|^2_{\nu_t}}{2V_{xx}(t-,x)} \quad \mu^K\text{-a.e.} \tag{1.3.12}$$

Conditions (C2) and (C5) imply that inequality (1.3.12) holds μ^K-a.e. for all $x \in \mathbb{R}$.

Let us show now that if a strategy π^* is optimal, then the corresponding wealth process X^{π^*} is a solution of Eq. (1.3.7). Let $\pi^*(s,x)$ be the optimal strategy; denote by

$$X_t^*(s,x) = x + \int_s^t \pi_u^*(s,x) dS_u$$

the corresponding wealth process.

By the optimality principle, the process

$$V(t, x + \int_s^t \pi_u^*(s,x) dS_u)$$

is a martingale on the interval $[s,T]$ and the Itô–Ventzell formula implies that μ^K-a.s.,

$$a(t, X_t^*(s,x)) + (\lambda_t, \pi_t(s,x))_{\nu_t} V_x(t-, X_t^*(s,x)) \\ + (\varphi_x(t, X_t^*(s,x)), \pi_t^*(s,x))_{\nu_t} + \frac{1}{2}|\pi_t^*(s,x)|^2_{\nu_t} V_{xx}(t-, X_t^*(s,x)) = 0. \tag{1.3.13}$$

It follows from (1.3.12) and (1.3.13) that μ^K-a.e.,

$$V_{xx}(t-, X_t^*(s,x)) \Big| \pi_t^*(s,x) + \frac{\varphi_x(t, X_t^*(s,x)) + \lambda(t) V_x(t-, X_t^*(s,x))}{V_{xx}(t-, X_t^*(s,x))} \Big|_{\nu_t} \leq 0.$$

Since $V_{xx} < 0$, integrating the latter relation by dK_u, we obtain

$$\int_s^t \left(\pi_u^*(s,x) + \frac{\varphi_x(u, X_u^*(s,x)) + \lambda(u) V_x(u, X_u^*(s,x))}{V_{xx}(u, X_u^*(s,x))} \right)' d\langle M \rangle_u \\ \times \left(\pi_u^*(s,x) + \frac{\varphi_x(u, X_u^*(s,x)) + \lambda(u) V_x(u, X_u^*(s,x))}{V_{xx}(u, X_u^*(s,x))} \right) = 0. \tag{1.3.14}$$

The Kunita–Watanabe inequality and (1.3.14) imply that the semimartingale

$$\int_s^t \left(\pi_u^*(s,x) + \frac{\varphi_x(u, X_u^*(s,x)) + \lambda(u) V_x(u, X_u^*(s,x))}{V_{xx}(u, X_u^*(s,x))} \right) dS_u$$

is indistinguishable from zero (since its $\mathcal{S}^2$-norm is zero), and we obtain that the wealth process of π^* satisfies the equation

$$X_t^*(s,x) = x - \int_s^t \frac{\varphi_x(u, X_u^*(s,x)) + \lambda(u) V_x(u, X_u^*(s,x))}{V_{xx}(u, X_u^*(s,x))} dS_u, \tag{1.3.15}$$

which gives Eq. (1.3.7) for $s = 0$. □

Recall that the process Z belongs to the class D if the family of random variables $Z_\tau I_{(\tau \leq T)}$ is uniformly integrable for all stopping times τ.

Under the additional condition

(C*) $(X_t^*(s,x),\ t \geq s)$ is a continuous function of (s,x) P-a.s. for each $t \in [s,T]$, we show that the value function V satisfies Eqs. (1.3.1)–(1.3.2).

This condition is satisfied, e.g., if the optimal wealth process $(X_t^*(s,x),\ t \geq s)$ is independent of s and x; this is the case for power, logarithmic, and exponential utility functions.

Theorem 1.3.1. *Let $V \in \mathcal{V}^{1,2}$. Assume that conditions* (B1)–(B3) *and* (C*) *hold. Then the value function is a solution of the BSPDE* (1.3.1)–(1.3.2), *i.e.,*

$$V(t,x) = V(0,x) + \frac{1}{2}\int_0^t \frac{(\varphi_x(s,x) + \lambda(s)V_x(s,x))'}{V_{xx}(s,x)} d\langle M\rangle_s \big(\varphi_x(s,x) + \lambda(s)V_x(s,x)\big)$$

$$+ \int_0^t \varphi(s,x)dM_s + m(t,x), \quad V(T,x) = U(x). \quad (1.3.16)$$

Moreover, the strategy π^* is optimal if and only if the corresponding wealth process X^{π^*} is a solution of the forward SDE (1.3.7) *such that the process $V(t, X^{\pi^*})$ is from the class D.*

Proof. Let $\pi^*(s,x)$ be the optimal strategy. By the optimality principle, $(V(t, X_t^*(s,x)), t \geq s)$ is a martingale. Therefore, using the Itô-Ventzell formula and taking (1.3.14) into account, we have

$$\int_s^t \left[a(u, X_u^*(s,x)) - g(u, X_u^*(s,x)) + \left| \pi_u^*(s,x) + \frac{V_x(u, X_u^*(s,x))\lambda(u) + \varphi_x(u, X_u^*(s,x))}{V_{xx}(u, X_u^*(s,x))}\right|_{\nu_u}^2 \right] dK_u = 0$$

for all $t \geq s$ P-a.s., where

$$g(s,x) = \frac{1}{2}\frac{|\varphi_x(s,x) + \lambda(s)V_x(s,x)|_{\nu_s}^2}{V_{xx}(s,x)}.$$

It follows from (1.3.14) that μ^K-a.e.,

$$\left| \pi_u^*(s,x) + \frac{V_x(u, X_u^*(s,x))\lambda(u) + \varphi_x(u, X_u^*(s,x))}{V_{xx}(u, X_u^*(s,x))}\right|_{\nu_u}^2 = 0,$$

and by (1.3.6),

$$a(s,x) \leq g(s,x) \quad \mu^K\text{-a.e.} \quad (1.3.17)$$

Thus,

$$\int_s^t \big[a(u, X_u^*(s,x)) - g(u, X_u^*(s,x))\big] dK_u = 0, \quad t \geq s \quad P\text{-a.s.}$$

This implies $(a(s,x) - g(s,x))(K_s - K_{s-}) = 0$ for any $s \in [0,T]$. Therefore,

$$a(s,x) = g(s,x) \quad \mu^{K^d}\text{-a.e.} \quad (1.3.18)$$

On the other hand,

$$\int_0^T \frac{1}{\varepsilon}\int_s^{\tau_s^\varepsilon} \big[a(u, X_u^*(s,x)) - g(u, X_u^*(s,x))\big] dK_u^c dK_s^c = 0 \quad P\text{-a.s.},$$

and by Proposition 1.7.2, we obtain that

$$\int_0^T [a(s,x) - g(s,x)] dK_s^c = 0 \quad P\text{-a.s.}$$

Now (1.3.17), (1.3.18), and the latter relation imply $a(s,x) = g(s,x)$ μ^K-a.e., and hence

$$A(t,x) = \frac{1}{2}\int_0^t \frac{(\varphi_x(s,x) + \lambda(s)V_x(s,x))'}{V_{xx}(s,x)} d\langle M\rangle_s \big(\varphi_x(s,x) + \lambda(s)V_x(s,x)\big),$$

and $V(t,x)$ satisfies (1.3.1)–(1.3.2).

If $\hat{\pi}$ is a strategy such that the corresponding wealth process $X^{\hat{\pi}}$ satisfies Eq. (1.3.7) and $V(t, X_t^{\hat{\pi}})$ is from the class D, then $\hat{\pi}$ is optimal. Indeed, using the Itô–Ventzell formula and Eqs. (1.3.7) and (1.3.16), we obtain that $V(t, X_t^{\hat{\pi}})$ is a local martingale, and hence it is a martingale, since it belongs to the class D. Therefore, $\hat{\pi}$ is optimal by the optimality principle. $\square$

Definition 1.3.1. We say that Y belongs to the class $D(\Pi)$ if for any $x \in \mathbb{R}$ and $\pi \in \Pi_x$, the process

$$Y\left(t, x + \int_0^t \pi_u dS_u\right)$$

is from the class D.

Theorem 1.3.2. *Let conditions* (B1)–(B3) *be satisfied. If the pair* $(Y, \mathcal{X})$ *is a solution of the forward-backward equation*

$$Y(t,x) = U(x) - \frac{1}{2}\int_t^T \frac{(\psi_x(s,x) + \lambda(s)Y_x(s,x))'}{Y_{xx}(s,x)} d\langle M\rangle_s \big(\varphi_x(s,x) + \lambda(s)V_x(s,x)\big)$$
$$- \int_t^T \psi(s,x)dM_s + L(T,x) - L(t,x), \quad (1.3.19)$$

$$\mathcal{X}_t = x - \int_0^t \frac{\psi_x'(s,\mathcal{X}_s) + Y_x(s,\mathcal{X}_s)\lambda(s)}{Y_{xx}(s,\mathcal{X}_s)} dS_s, \quad (1.3.20)$$

$\mathcal{X} \geq 0$ *and* Y *belongs to the class* $D(\Pi)$, *then such a solution is unique.*

Proof. Using the Itô–Ventzell formula for

$$Y\left(t, x + \int_s^t \pi_u dS_u\right),$$

we have

$$Y\left(t, x + \int_s^t \pi_u dS_u\right) = Y(s,x) + \int_s^t b\left(u, x + \int_s^u \pi_v dS_v\right) dK_u$$
$$+ \int_s^t G\left(u, \pi_u, c + \int_s^u \pi_v dS_v\right) dK_u + N_t - N_s, \quad (1.3.21)$$

where

$$G(t,p,x,\omega) = Y_x(t-,x)p'\nu_t\lambda(t) + p'\nu_t\psi_x(t,x) + \frac{1}{2}Y_{xx}(t-,x)p'\nu_t p.$$

Since Y solves (1.3.19), then Eq. (1.3.3) holds, which implies that

$$Y\left(t, x + \int_s^t \pi_u dS_u\right)$$

is a local supermartingale for each $\pi \in \Pi$.

Since Y is from the class $D(\Pi)$, the process

$$Y\left(t, x + \int_s^t \pi_u dS_u\right)$$

is a supermartingale of class D for any $\pi \in \Pi_x$, and using the boundary condition, we have

$$Y(s,x) \geq E\left[U\left(x + \int_s^T \pi_u dS_u\right)/\mathcal{F}_s\right],$$

which implies that

$$Y(s,x) \geq V(s,x). \tag{1.3.22}$$

Now using the Itô–Ventzell formula for $Y(t, \mathcal{X}_u)$ and taking into account that Y satisfies (1.3.19) and $\mathcal{X}$ solves (1.3.20), we obtain that $Y(t, \mathcal{X}_t)$ is a local martingale, and hence it is a martingale, since $Y(t, \mathcal{X}_t)$ is from the class D.

Therefore, since $\mathcal{X}_0 = x$ and $Y(T,x) = U(x)$, we have

$$Y(t,x) = E\left(U\left(x - \int_t^T \frac{Y_x(u,\mathcal{X}_u)\lambda_u + \psi_x(u,\mathcal{X}_u)}{Y_{xx}(u,\mathcal{X}_u)} dS_u\right)/\mathcal{F}_t\right). \tag{1.3.23}$$

Since

$$-\frac{\lambda(u)Y_x(u,\mathcal{X}_u) + \psi_x(u,\mathcal{X}_u)}{Y_{xx}(u,\mathcal{X}_u)} \in \Pi_x,$$

from (1.3.22) and (1.3.23) we obtain

$$Y(t,x) = V(t,x); \tag{1.3.24}$$

hence the solution of (1.3.19) is unique, if it exists.

Relations (1.3.19), (1.3.20), and (1.3.20) imply that $\mathcal{X}$ satisfies Eq. (1.3.7). Moreover, according to Proposition 1.3.1, the solution of (1.3.7) is the optimal wealth process, and hence $\mathcal{X} = X^{\pi*}$ by the uniqueness of the optimal strategy for problem (1.1.3) (see Remark 1.2.2). □

1.4. Utility Maximization Problem for Power, Logarithmic, and Exponential Utility Functions

In this section, we calculate the value function and give constructions of optimal strategies for the utility maximization problem corresponding to the cases of power, logarithmic, and exponential utility functions.

1.4.1. Power utility. Let $U(x) = x^p$, $p \in (0,1)$. Then (1.1.3) corresponds to the power utility maximization problem

$$\text{maximize } E\left(x + \int_0^T \pi_u dS_u\right)^p \text{ over all } \pi \in \Pi_x, \tag{1.4.1}$$

where Π_x is the class of admissible strategies.

In this case, the value function $V(t,x)$ is of the form $x^p V_t$, where V_t is a special semimartingale.

Indeed, since Π_x is a cone (for any $x > 0$, the strategy π belongs to Π_x if and only if $\pi/x \in \Pi_1$), we have

$$V(t,x) = \operatorname*{ess\,sup}_{\pi\in\Pi_x} E\left(\left(x + \int_t^T \pi_u dS_u\right)^p /\mathcal{F}_t\right) = x^p \operatorname*{ess\,sup}_{\pi\in\Pi_x} E\left(\left(1 + \int_t^T \frac{\pi_u}{x} dS_u\right)^p /\mathcal{F}_t\right) = x^p V_t,$$

where

$$V_t = \operatorname*{ess\,sup}_{\pi\in\Pi_1} E\left(\left(1 + \int_t^T \pi_u dS_u\right)^p /\mathcal{F}_t\right)$$

is a supermartingale by the optimality principle.

Let $V_t = V_0 + A_t + N_t$ be the canonical decomposition of V_t, where A is a decreasing process and N is a local martingale. Using the GKW decomposition, we have

$$V_t = V_0 + A_t + \int_0^t \varphi_s dM_s + L_t, \tag{1.4.2}$$

where L is a local martingale with $\langle L, M\rangle = 0$.

It is obvious that all the conditions of Theorem 1.3.1 are satisfied. Note that one can take $-A+\langle M\rangle$ as a dominated process K and that $V_t > 0$ for all t, since $\mathcal{M}^e \neq \emptyset$.

Therefore, we have the following consequence of Theorem 1.3.1.

Theorem 1.4.1. *If $U(x) = x^p, p \in (0,1)$, then the value function $V(t,x)$ has the form $x^p V_t$, where V_t satisfies the following backward stochastic differential equation* (BSDE):

$$V_t = V_0 + \frac{q}{2}\int_0^t \frac{(\varphi_s + \lambda_s V_s)'}{V_s} d\langle M\rangle_s(\varphi_s + \lambda_s V_s) + \int_0^t \varphi_s dM_s + L_t, \quad V_T = 1, \tag{1.4.3}$$

where $q = p/(p-1)$ and L is a local martingale strongly orthogonal to M. Moreover, the optimal wealth process is a solution of the linear equation

$$X_t^* = x - (q-1)\int_0^t \frac{\varphi_u + \lambda_u V_u}{V_u} X_u^* dS_u. \tag{1.4.4}$$

Therefore,

$$X_t^* = x\mathcal{E}_t\left(-(q-1)\left(\frac{\varphi}{V} + \lambda\right)\cdot S\right)$$

and the optimal strategy is of the form

$$\pi_t^* = -x(q-1)\mathcal{E}_t\left(-(q-1)\left(\frac{\varphi}{V} + \lambda\right)\cdot S\right)\left(\frac{\varphi_t}{V_t} + \lambda_t\right).$$

Now, we consider two cases where Eq. (1.4.3) admits an explicit solution.

Case 1. Let

$$S_t(q) = M_t + q\int_0^t d\langle M\rangle_s \lambda_s$$

and $Q(q)$ be a measure defined by $dQ(q) = \mathcal{E}_T(-q\lambda\cdot M)dP$. Note that $S(q)$ is a local martingale under $Q(q)$ by the Girsanov theorem.

Assume that

$$e^{\frac{q(q-1)}{2}\langle\lambda\cdot M\rangle_T} = c + \int_0^T h_u dS_u(q), \tag{1.4.5}$$

where c is a constant and h is a predictable $S(q)$-integrable process such that $h \cdot S(q)$ is a $Q(q)$-martingale.

This condition is satisfied if and only if the q-optimal martingale measure coincides with the minimal martingale measure. For diffusion market models, this condition holds for the so-called "almost complete" models, where the market price of risk is measurable with respect to the filtration generated by the price processes of basic securities.

Let condition (1.4.5) hold. Consider the process

$$Y_t = \left(E(\mathcal{E}^q_{t,T}(-\lambda\cdot M)/F_t)\right)^{\frac{1}{1-q}}. \tag{1.4.6}$$

Since

$$\mathcal{E}^q_t(-\lambda\cdot M) = \mathcal{E}_t(-q\lambda\cdot M)e^{\frac{q(q-1)}{2}\langle\lambda\cdot M\rangle_t},$$

condition (1.4.5) implies

$$Y_t = \left(E^{Q(q)}(e^{\frac{q(q-1)}{2}(\langle\lambda\cdot M\rangle_T - \langle\lambda\cdot M\rangle_t}/F_t)\right)^{\frac{1}{1-q}} = e^{\frac{q}{2}\langle\lambda\cdot M\rangle_t}\left(c + \int_0^t h_u dS_u(q)\right)^{\frac{1}{1-q}}.$$

By the Itô formula,

$$\begin{aligned} Y_t = Y_0 &+ \frac{q}{2}\int_0^t Y_s\lambda'_s d\langle M\rangle_s\lambda_s + \frac{q}{1-q}\int_0^t \frac{Y_s\lambda'_s}{c+(h\cdot S(q))_s}d\langle M\rangle_s h_s \\ &+ \frac{q}{2}\frac{1}{(1-q)^2}\int_0^t \frac{Y_s h'_s}{(c+(h\cdot S(q))_s)^2}d\langle M\rangle_s h_s + \frac{1}{1-q}\int_0^t \frac{Y_s h_s}{c+(h\cdot S(q))_s}dM_s, \end{aligned} \tag{1.4.7}$$

and denoting $\frac{1}{q-1}\frac{Y_s h_s}{c+(h\cdot S(q))_s}$ by ψ_s, we obtain

$$Y_t = Y_0 + \frac{q}{2}\int_0^t \frac{(\psi_s + \lambda_s Y_s)'}{Y_s}d\langle M\rangle_s(\psi_s+\lambda_s Y_s) + \int_0^t \psi_s dM_s.$$

It is obvious from (1.4.6) that $Y_T = 1$. Thus, the triple (Y, ψ, L), where $\psi = \frac{1}{q-1}\frac{Yh}{c+h\cdot S(q)}$, $L = 0$, and Y is defined by (1.4.6), satisfies Eq. (1.4.3).

Case 2. Assume that

$$e^{-\frac{q}{2}\langle\lambda\cdot M\rangle_T} = c + m_T, \tag{1.4.8}$$

where c is a constant and m is a martingale strongly orthogonal to M.

For diffusion market models, this condition is satisfied when the market price of risk is measurable with respect to the filtration independent of the asset price process.

Let us consider the process

$$Y_t = E(e^{-\frac{q}{2}(\langle\lambda\cdot M\rangle_T - \langle\lambda\cdot M\rangle_t)}/F_t). \tag{1.4.9}$$

Condition (1.4.7) implies

$$Y_t = e^{\frac{q}{2}\langle\lambda\cdot M\rangle_t}(c + m_t),$$

and by the Itô formula,

$$Y_t = Y_0 + \frac{q}{2}\int_0^t Y_s d\langle\lambda\cdot M\rangle_s + \int_0^t e^{\frac{q}{2}\langle\lambda\cdot M\rangle_s} dm_s.$$

It follows from here that the triple (Y, ψ, L), where $\psi = 0$, $L_t = \int_0^t e^{\frac{q}{2}\langle\lambda\cdot M\rangle_s} dm_s$ and Y is defined by (1.4.8), satisfies Eq. (1.4.3).

1.4.2. Exponential utility. Let us consider the case of the exponential utility function

$$U(x) = -e^{-\gamma(x-H)}$$

with risk aversion parameter $\gamma \in (0, \infty)$, where H is a contingent claim describing a random payoff at time T. We assume that H is a bounded F_T-measurable random variable.

Consider the maximization problem

$$\max_{\pi\in\Pi} E\left(-\exp\left(-\gamma\left(x + \int_0^T \pi_u dS_u - H\right)\right)\right), \tag{1.4.10}$$

which is the maximal expected utility that can be attained starting from the initial capital x, using some strategy $\pi \in \Pi$, and paying out H at time T.

The corresponding value function

$$V(t,x) = \operatorname*{ess\,sup}_{\pi\in\Pi_x} E\left(-\exp\left(-\gamma\left(x + \int_t^T \pi_u dS_u - H\right)\right)/\mathcal{F}_t\right) \tag{1.4.11}$$

is of the form $V(t,x) = -e^{-\gamma x}V_t$, where

$$V_t = \operatorname*{ess\,inf}_{\pi\in\Pi_x} E\left(\exp\left(-\gamma\left(\int_t^T \pi_u dS_u - H\right)\right)|\mathcal{F}_t\right) \tag{1.4.12}$$

is a special semimartingale.

Let $V_t = V_0 + A_t + N_t$ be the canonical decomposition of V_t, where A is a decreasing process and N is a local matingale. Using the GKW decomposition, we have

$$V_t = V_0 + A_t + \int_0^t \varphi_s dM_s + L_t, \tag{1.4.13}$$

where L is a local martingale with $\langle L, M\rangle = 0$.

It is obvious that all the conditions of Theorem 1.3.1 hold. Note that one can take $-A + \langle M\rangle$ as a dominated process K and that $V_t > 0$ for all t, since $\mathcal{M}^e \neq \emptyset$.

Therefore, we have the following consequence of Theorem 1.3.1.

Theorem 1.4.2. *The value function* (1.4.11) *has the form* $-e^{-\gamma x}V_t$, *where* V_t *satisfies the following BSDE*:

$$V_t = V_0 + \frac{1}{2}\int_0^t \frac{(\varphi_s + \lambda_s V_s)^2}{V_s} d\langle M\rangle_s + \int_0^t \varphi_s dM_s + L_t \tag{1.4.14}$$

with the boundary condition

$$V_T = e^{\gamma H},$$

where L is a local martingale strongly orthogonal to M. Moreover, the optimal wealth process is expressed as

$$X_t^* = x + \int_0^t \frac{\varphi_u + \lambda_u V_u}{\gamma V_u} dS_u \tag{1.4.15}$$

and the optimal strategy is of the form

$$\pi_t^* = \frac{\varphi_t + \lambda_t V_t}{\gamma V_t}.$$

Now we give explicit solutions of Eq. (1.4.10) in two extreme cases.

Case 1. Assume that

$$\gamma H - \frac{1}{2}\langle \lambda \cdot M\rangle_T = c + \int_0^T h_u dS_u, \tag{1.4.16}$$

where c is a constant and h is a predictable S-integrable process such that $h \cdot S$ is a martingale with respect to the minimal martingale measure.

This condition is satisfied if and only if the minimal martingale measure coincides with the minimal martingale measure and H is attainable. For diffusion market models, this condition holds for the so-called "almost complete" models, i.e., when the market price of risk is measurable with respect to the filtration generated by the price processes of basic securities.

Similarly to the case of power utility, we can show that the triple (Y, ψ, L), where

$$Y_t = e^{E^{Q_{\min}}(\gamma H - \frac{1}{2}\langle \lambda\cdot M\rangle_{tT}/F_t)}, \quad \psi_t = Y_t h_t, \quad L_t = 0,$$

satisfies Eq. (1.4.14).

Case 2. Assume that

$$e^{\gamma H - \frac{1}{2}\langle \lambda\cdot M\rangle_T} = c + m_T, \tag{1.4.17}$$

where c is a constant and m is a martingale strongly orthogonal to M.

For diffusion market models, this condition holds when the market price of risk is measurable with respect to the filtration independent of the asset price process.

We can show that the triple (Y, ψ, L), where

$$Y_t = e^{E(\gamma H - \frac{1}{2}\langle \lambda\cdot M\rangle_{tT}/F_t)}, \quad \psi_t = 0, \quad L_t = \int_0^t e^{\frac{1}{2}\langle \lambda\cdot M\rangle_s} dm_s,$$

satisfies Eq. (1.4.14).

1.4.3. Logarithmic utility. For the logarithmic utility

$$U(x) = \log x, \quad x > 0,$$

the value function of the corresponding utility maximization problem takes the form

$$V(t, x) = \log x + V_t,$$

where V_t is a special semimartingale.

Indeed, since for any $x > 0$, the strategy π belongs to Π_x if and only if $\pi/x \in \Pi_1$, we have

$$V(t,x) = \operatorname*{ess\,sup}_{\pi \in \Pi_x} E\left(\log\left(x + \int_t^T \pi_u dS_u\right)/\mathcal{F}_t\right) = \operatorname*{ess\,sup}_{\pi \in \Pi_x} E\left(\log x\left(1 + \int_t^T \frac{\pi_u}{x} dS_u\right)/\mathcal{F}_t\right)$$

$$\log x + \operatorname*{ess\,sup}_{\pi \in \Pi_x} E\left(\log\left(1 + \int_t^T \frac{\pi_u}{x} dS_u\right)/\mathcal{F}_t\right) = \log x + V_t,$$

where

$$V_t = \operatorname*{ess\,sup}_{\pi \in \Pi_1} E\left(\log\left(1 + \int_t^T \pi_u dS_u\right)/\mathcal{F}_t\right)$$

is a supermartingale by the optimality principle.

It is also obvious that all the conditions of Theorem 1.3.1 hold. In this case, $\varphi_x(t,x) = 0$, $V_x(t,x) = 1/x$, $V_{xx}(t,x) = -1/x^2$, and Eq. (1.3.7) gives the following linear BSDE for V_t:

$$V_t = V_0 + \frac{1}{2}\int_0^t V_s d\langle \lambda \cdot M\rangle_s + \int_0^t \varphi_s dM_s + L_t, \quad V_T = 0, \tag{1.4.18}$$

which admits the explicit solution

$$V_t = \frac{1}{2} E(\langle \lambda \cdot M\rangle_T - \langle \lambda \cdot M\rangle_t / F_t).$$

Thus, we have the following consequence of Theorem 1.3.1.

Theorem 1.4.3. *If $U(x) = \log x$, then the value function of the problem is represented as*

$$V(t,x) = \log x + \frac{1}{2} E(\langle \lambda \cdot M\rangle_T - \langle \lambda \cdot M\rangle_t / F_t).$$

Moreover, the optimal wealth process is a solution of the linear equation

$$X_t^* = x + \int_0^t \lambda_u X_u^* dS_u. \tag{1.4.19}$$

Thus,

$$X_t^* = x\mathcal{E}_t(\lambda \cdot S),$$

and the optimal strategy is of the form

$$\pi_t^* = \lambda_t X_t^* = x\lambda_t \mathcal{E}_t(\lambda \cdot S).$$

1.5. Minimization of the Hedging Error. Mean-Variance Hedging

In this section, we consider the minimization problem

$$\min_{\pi \in \Pi} E[U(X_T^{x,\pi})], \tag{1.5.1}$$

where

$$X_t^{x,\pi} = x + \int_0^t \pi_u dS_u$$

is the wealth process starting from the initial capital x and U is the objective function, which can also depend on ω. It can be interpreted as a function that measures a hedging error.

Let Π be the class of predictable S-integrable self-financing trading strategies closed with respect to bifurcation, i.e., such that for any $t \geq 0$, $B \in F_t$, and $\pi^1, \pi^2 \in \Pi$, the strategy

$$\pi_s = \tilde{\pi}_s I_{(0 \leq s < t)} + \pi_s^1 I_B I_{(s \geq t)} + \pi_s^2 I_{B^c} I_{(s \geq t)}$$

belongs to the class Π.

Note that for all known classes of admissible strategies, this condition holds. This condition guarantees the fulfillment of the optimality principle.

Let

$$V(t,x) = \operatorname*{ess\,inf}_{\pi \in \Pi} E\left(U\left(x + \int_t^T \pi_u dS_u\right) / \mathcal{F}_t\right) \tag{1.5.2}$$

be the value function of problem (1.5.1).

Let Π^p, $p > 1$, be the space of all predictable S-integrable processes π such that the stochastic integral

$$(\pi \cdot S)_t = \int_0^t \pi_u dS_u, \quad t \in [0,T],$$

lies in the $\mathcal{S}^p$ space of semimartingales, i.e.,

$$E\left(\int_0^T \pi_s' d\langle M \rangle_s \pi_s\right)^{p/2} + E\left(\int_0^T |\pi_s dA_s|\right)^p < \infty.$$

Define G_T^p as the space of terminal values of stochastic integrals, i.e.,

$$G_T^p(\Pi) = \{(\pi \cdot S)_T : \pi \in \Pi^p\}.$$

For convenience, we give some assertions from [34, Theorem 4.1] (previously proved in [17] for the case $p = 2$), which establishes necessary and sufficient conditions for the closedness of the space G_T^p in L^p.

Proposition 1.5.1. *Let S be a continuous semimartingale. Let $p > 1$ and q be conjugate to p. Then the following assertions are equivalent*:

(1) *there is a martingale measure $Q \in \mathcal{M}^e$, and G_T^p is closed in L^p*;
(2) *there is a martingale measure Q that satisfies the reverse Hölder condition $R_q(P)$*;
(3) *there is a constant C such that for all $\pi \in \Pi^p$, we have*

$$\left\| \sup_{t \leq T} (\pi \cdot S)_t \right\|_{L^p(P)} \leq C \|(\pi \cdot S)_T\|_{L^p(P)}.$$

(4) *there is a constant c such that for every stopping time τ, every $A \in \mathcal{F}_\tau$, and for every $\pi \in \Pi^p$ with $\pi = \pi I_{]\tau,T]}$, we have*

$$\|I_A - (\pi \cdot S)_T\|_{L^p(P)} \geq c P(A)^{1/p}.$$

Remark 1.5.1. Assertion (4) implies that for every stopping time τ and for every $\pi \in \Pi^p$, we have

$$E\left(\left|1 - \int_\tau^T \pi_u dS_u\right|^p / \mathcal{F}_\tau\right) \geq c^p.$$

We assume that the function $U(x) = U(\omega, x)$ satisfies the following conditions:

(D1) $U(x)$ is nonnegative and $EU(x) < \infty$ for all x;
(D2) $U(x)$ is strictly convex function P-a.s.;

(D3) the optimization problem (1.5.2) admits a solution, i.e., for any t and x, there is a strategy $\pi^*(t,x)$ such that

$$V(t,x) = E\left(U\left(x + \int_t^T \pi_s^*(t,x)dS_s\right)|\mathcal{F}_t\right). \tag{1.5.3}$$

Remark 1.5.2. For (D3), the following condition is sufficient:

(D3′) there exist $\gamma > 0$, a positive integrable random variable ξ, and $p > 1$ such that $U(x) \geq \gamma|x|^p - \xi$.

In Proposition 1.7.3 of the Appendix, it is proved that if conditions (D1), (D2), and (D3′), and the reverse Hölder condition $R_q(P)$, $q = p/(p-1)$ hold, then the optimal strategy exists in the class Π^p.

Note that the function $U(x) = |H - x|^p$ for $H \in L^p$ satisfies (D3′) as well as conditions (D1)–(D2).

Now we give the formulations of the main statements, which are similar to the corresponding assertions of Sec. 1.3.

The optimality principle in this case is of the same form (as Proposition 1.7.1) except that the notion "supermartingale" must be replaced by the notion "submartingale." Moreover, in the proof of the existence of an RCLL modification of submartingale

$$\tilde{V}\left(t, x + \int_0^t \pi_u dS_u\right),$$

we need the following additional condition.

For any real number α, let constants C_α and B_α and an integrable random variable η exist such that

$$U(\alpha x) \leq C_\alpha U(x) + B_\alpha \eta \quad \text{for all } x \in \mathbb{R}. \tag{1.5.4}$$

This condition is needed to use the Fatou lemma (see the proof of Proposition 1.7.1 here or in [61]). Note that the function $U(x) = |H - x|^p$ for $H \in L^p$ also satisfies this condition.

Theorem 1.5.1. *Let $V \in \mathcal{V}^{1,2}$. Assume that conditions* (D1)–(D3) *and* (C*) *hold. Then the value function is a solution of the BSPDE*

$$V(t,x) = V(0,x) + \frac{1}{2}\int_0^t \frac{(\varphi_x(s,x) + \lambda(s)V_x(s,x))'}{V_{xx}(s,x)} d\langle M\rangle_s \big(\varphi_x(s,x) + \lambda(s)V_x(s,x)\big)$$
$$+ \int_0^t \varphi(s,x)dM_s + m(t,x), \quad V(T,x) = U(x). \tag{1.5.5}$$

Moreover, a strategy π^* is optimal if and only if the corresponding wealth process X^{π^*} is a solution of the forward SDE (1.3.7).

The proof is similar to the proof of Theorem 1.3.1. Note that in this case the process $V(t, X^{\pi^*})$ is from the class D as a positive submartingale.

Definition 1.5.1. We say that Y belongs to the class $D^p(\Pi)$ if:

(i) there is a positive process c_t from the class D such that

$$Y(t,x) \geq -c_t \quad \text{for all } x \in \mathbb{R},$$

(ii) the process

$$Y\left(t, x + \int_0^t \pi_u dS_u\right)$$

is of class D for every $\pi \in \Pi^p$, where the class Π^p is defined in Sec. 1.2.

Remark 1.5.3. Note that the value function $V(t,x)$ belongs to the class $D^p(\Pi)$, since for any $\pi \in \Pi^p$,

$$0 \le V\left(t, x + \int_0^t \pi_u dS_u\right) \le E\left(U\left(x + \int_0^T \pi_u dS_u\right) | \mathcal{F}_t\right) \tag{1.5.6}$$

and the right-hand-side of (1.5.6) is a uniformly integrable martingale.

Theorem 1.5.2. *Let conditions* (D1), (D2), (D3′), *and the reverse Hölder condition* $R_q(P), q = p/(p-1)$ *hold. If the pair* $(Y, \mathcal{X})$ *is a solution of the forward-backward equation*

$$Y(t,x) = U(x) - \frac{1}{2}\int_t^T \frac{((\psi_x(s,x) + \lambda(s)Y_x(s,x))'}{Y_{xx}(s,x)} d\langle M\rangle_s \big(\varphi_x(s,x) + \lambda(s)V_x(s,x)\big)$$
$$- \int_t^T \psi(s,x)dM_s + L(T,x) - L(t,x), \tag{1.5.7}$$

$$\mathcal{X}_t = x - \int_0^t \frac{\psi_x'(s,\mathcal{X}_s) + Y_x(s,\mathcal{X}_s)\lambda(s)}{Y_{xx}(s,\mathcal{X}_s)} dS_s, \tag{1.5.8}$$

and Y *belongs to the class* $\mathcal{V}^{1,2} \cap D^p(\Pi)$, *then such a solution is unique. Moreover,* Y *coincides with the value function and* $\mathcal{X}$ *with the optimal wealth process.*

Proof. The inequality

$$Y(s,x) \le V(s,x) \tag{1.5.9}$$

is proved similarly to (1.3.22). Let us show the converse inequality.

Now using the Itô–Ventzell formula for $Y(t, \mathcal{X}_u)$ and taking into account the fact that Y satisfies (1.5.7) and $\mathcal{X}$ solves (1.5.8), we obtain that $Y(t, \mathcal{X}_u)$ is a local martingale, and hence it is a supermartingale, since Y is bounded from below by the process of class D. Therefore, since $\mathcal{X}_0 = x$ and $Y(T,x) = U(x)$, we have

$$Y(t,x) \ge E(Y(T,\mathcal{X}_T)/\mathcal{F}_t) = E\left(U\left(x + \int_t^T \frac{Y_x(u,\mathcal{X}_u)\lambda_u + \psi_x(u,\mathcal{X}_u)}{Y_{xx}(u,\mathcal{X}_u)} dS_u\right)/\mathcal{F}_t\right). \tag{1.5.10}$$

Applying inequalities (1.5.9) and (1.5.10) for $s = 0$, we obtain

$$EU\left(x + \int_0^T \frac{Y_x(u,\mathcal{X}_u)\lambda_u + \psi_x(u,\mathcal{X}_u)}{Y_{xx}(u,\mathcal{X}_u)} dS_u\right) \le Y(0,x) \le V(0,x) \le EU(x) < \infty. \tag{1.5.11}$$

Condition (D3′) implies that

$$E\left(x + \int_0^T \frac{Y_x(u,\mathcal{X}_u)\lambda_u + \psi_x(u,\mathcal{X}_u)}{Y_{xx}(u,\mathcal{X}_u)} dS_u\right)^p$$
$$\le E\left(U\left(x + \int_0^T \frac{Y_x(u,\mathcal{X}_u)\lambda_u + \psi_x(u,\mathcal{X}_u)}{Y_{xx}(u,\mathcal{X}_u)} dS_u\right)\right) + E\xi < \infty.$$

Therefore, by Proposition 1.5.1,

$$\frac{\lambda(u)Y_x(u,\mathcal{X}_u) + \psi_x(u,\mathcal{X}_u)}{Y_{xx}(u,\mathcal{X}_u)} \in \Pi^p$$

and it follows from (1.5.9) and (1.5.10) that

$$Y(t,x) = V(t,x), \tag{1.5.12}$$

hence the solution of (1.5.5) is unique and coincides with the value function.

The relations (1.5.8) and (1.5.12) imply that $\mathcal{X}$ satisfies Eq. (1.3.7). Since $\mathcal{X} = X^{\pi*}$ satisfies (1.3.7), $V(t,\mathcal{X})$ is a local martingale and hence it is a martingale, because $V = Y \in D(\Pi)$. By the optimality principle, $\mathcal{X}$ is optimal; moreover, it coincides with optimal wealth process by the uniqueness of the optimal strategy for problem (1.1.2) (see Remark 1.5.2). □

Now let us consider the case where $U(x) = (x - H)^2$, which corresponds to the mean-variance hedging problem (1.1.2), where H is a F_T-measurable random variable describing the net payoff at time T of some financial instrument.

Assume that

(A*) there exists a martingale measure that satisfies the reverse Hölder condition $R_2(P)$.

Theorem 1.5.3. *Let H be a square integrable $\mathcal{F}_T$-measurable random variable, and let the objective function be of the form $U(x) = |H - x|^2$. Then the value function of problem* (1.1.2) *admits the representation*

$$V(t,x) = V_0(t) - 2V_1(t)x + V_2(t)x^2, \tag{1.5.13}$$

where the processes $V_0(t)$, $V_1(t)$, and $V_2(t)$ satisfy the following system of backward equations:

$$\begin{aligned} V_2(t) = V_2(0) &+ \int_0^t \frac{(\varphi_2(s) + \lambda(s)V_2(s))'}{V_2(s)} d\langle M\rangle_s(\varphi_2(s) + \lambda(s)V_2(s)) \\ &+ \int_0^t \varphi_2(s)dM_s + L_2(t), \quad V_2(T) = 1, \end{aligned} \tag{1.5.14}$$

$$\begin{aligned} V_1(t) = V_1(0) &+ \int_0^t \frac{(\varphi_2(s) + \lambda(s)V_2(s))'}{V_2(s)} d\langle M\rangle_s(\varphi_1(s) + \lambda(s)V_1(s)) \\ &+ \int_0^t \varphi_1(s)dM_s + L_1(t), \quad V_1(T) = H, \end{aligned} \tag{1.5.15}$$

$$\begin{aligned} V_0(t) = V_0(0) &+ \int_0^t \frac{(\varphi_1(s) + \lambda(s)V_1(s))'}{V_2(s)} d\langle M\rangle_s(\varphi_1(s) + \lambda(s)V_1(s)) \\ &+ \int_0^t \varphi_0(s)dM_s + L_0(t), \quad V_0(T) = H^2, \end{aligned} \tag{1.5.16}$$

where L_0, L_1, and L_2 are local martingales orthogonal to M.

If a triple (Y_0, Y_1, Y_2), where $Y_0 \in D$, $Y_1^2 \in D$, and $c \le Y_2 \le C$ for some constants $0 < c < C$, satisfies system (1.5.14)–(1.5.16), *then such solution is unique and coincides with the triple* (V_0, V_1, V_2).

Moreover, the optimal wealth process X^{π^*} satisfies the linear equation

$$X_t^{\pi^*} = x + \int_0^t \frac{\varphi_1(s) + \lambda(s)V_1(s)}{V_2(s)} dS_s - \int_0^t \frac{\varphi_2(s) + \lambda(s)V_2(s)}{V_2(s)} X_s^{\pi^*} dS_s. \tag{1.5.17}$$

Proof. It is obvious that $U(x) = |H-x|^2$ satisfies conditions (D1) and (D2) and condition (D3) follows from Proposition 1.7.3 of the Appendix, since the function $U(x) = |H-x|^2$ satisfies condition (D3′) for $p=2$ and the space G_T^2 of stochastic integrals is closed by Proposition 1.5.1. Hence there exists an optimal strategy $\pi^*(t,x)$ and

$$V(t,x) = E\left[\left|H - x - \int_t^T \pi_u^*(t,x)dS_u\right|^2 |\mathcal{F}_t\right].$$

Since $\int_t^T \pi_u^*(t,x)dS_u$ coincides with the orthogonal projection of $H-x \in L^2$ on the closed subspace of stochastic integrals, the optimal strategy is linear with respect to x, i.e., $\pi_u^*(t,x) = \pi_u^0(t) + x\pi_u^1(t)$. This implies that the value function $V(t,x)$ is of the form (1.5.13), where

$$\begin{aligned} V_0(t) &= E\left[\left|\int_t^T \pi_u^0(t)dS_u - H\right|^2 |\mathcal{F}_t\right], \\ V_1(t) &= E\left[\left(1+\int_t^T \pi_u^1(t)dS_u\right)\left(\int_t^T \pi_u^0(t)dS_u - H\right)|\mathcal{F}_t\right], \\ V_2(t) &= E\left[\left|\int_t^T \pi_u^1(t)dS_u + 1\right|^2 |\mathcal{F}_t\right]. \end{aligned} \tag{1.5.18}$$

Obviously, the function $U(x) = |x-H|^2$ satisfies all the conditions of Proposition 2.9.1 and assertion (3) of Proposition 1.5.1 implies $\tilde{\Pi} = \Pi^2$, where the class $\tilde{\Pi}$ is defined in the Appendix 1.7. Therefore, according to Proposition 2.9.1 of the Appendix, $V(t,x)$ is an RCLL submartingale for each $x \in \mathbb{R}$. Thus, $V_0(t) = V(t,0)$ is an RCLL submaringale. On the other hand, for any $s \geq t$,

$$E[V_2(t)|\mathcal{F}_s] = \lim_{x\to\infty} \frac{1}{x^2}E[V(t,x)|\mathcal{F}_s] \geq \lim_{x\to\infty}\frac{1}{x^2}V(s,x) = V_2(s) \quad P\text{-a.s.},$$

and $V_2(t)$ is also a submartingale with RCLL trajectories as the uniform limit of RCLL processes. Hence $V_1(t) = \frac{1}{2}(V_0(t) + V_2(t) - V(t,1))$ is a special semimartingale.

Since V_0 and V_2 are submartingales,

$$V_2(t) \leq E(V_2(T)/\mathcal{F}_t) \leq 1, \quad V_0(t) \leq E(H^2/\mathcal{F}_t),$$

and $V(t,x) = V_0(t) - 2V_1(t)x + V_2(t)x^2 \geq 0$ for all $x \in \mathbb{R}$, we have $V_1^2(t) \leq V_0(t)V_2(t)$; hence

$$V_1^2(t) \leq E(H^2/\mathcal{F}_t).$$

Since $V(t,x)$ is strictly convex and $V_{xx}(t,x) = 2V_2(t)$, the process V_2 is strictly positive. Moreover, from Proposition 1.5.1 (see Remark 1.5.1), it follows that there is a constant $c>0$ such that $V_2(t) \geq c$.

Thus, V_0 and V_1^2 belong to the class D and the process V_2 satisfies the two-sided inequality

$$c \leq V_2(t) \leq 1.$$

Let

$$V_i(t) = V_0(0) + A_i(t) + \int_0^t \varphi_i(u)dM_u + m_i(t)$$

be the canonical decomposition of V_i for $i = 0,1,2$, where m_i is a local martingale strongly orthogonal to M and $A_i \in \mathcal{A}_{\text{loc}}$ (moreover, A_0 and A_2 are increasing processes). Taking

$$K(t) = A_0(t) + A_2(t) + \mathrm{Var}(A_1)(t) + \langle M\rangle_t + t,$$

we obviously see that condition (C1) is satisfied. It is easy to see that conditions (C2)–(C6) also hold. By Proposition 1.5.1, $X_t^*(s,x)$ is a solution of the forward equation (1.3.7), which coincides with the linear equation (1.5.17) in this case and can be explicitly solved in terms of $V_i, i=1,2$. Therefore condition (C*) also holds, and we may apply Theorem 1.3.1. Equating the coefficients of the quadratic trinomial (1.5.14) in Eq. (1.3.8), we obtain that V_2, V_1, and V_0 satisfy Eqs. (1.5.14), (1.5.15), and (1.5.16), respectively. The boundary conditions for these equations follow from Eq. (1.5.18).

Proof of uniqueness. If a triple (Y_0, Y_1, Y_2) is a solution of system (1.5.14)–(1.5.16), then the function $Y(t,x) = Y_0(t) - 2Y_1(t)x + Y_2(t)x^2$ is a solution of (1.3.1), (1.3.2). By assertion (3) of Proposition 1.5.1, the process $\left(\int_0^t \pi_u dS_u\right)^2$ is of class D. Since $Y_1^2(t) \in D$, the Hölder inequality implies that the process $Y_1(t)\left(\int_0^t \pi_u dS_u\right)$ is of class D. Therefore, $Y\left(t, x + \int_0^t \pi_u dS_u\right)$ belongs to the class D for every $\pi \in \Pi_x$.

It is easy to see that $Y_2(t) > c$ implies

$$Y(t,x) = Y_0(t) - 2Y_1(t)x + Y_2(t)x^2 \geq -\frac{1}{c}Y_1^2$$

for all $x \in \mathbb{R}$. Thus, Y belongs to the class $D(\Pi)$, and $Y(t,x) = V(t,x)$ by Theorem 1.5.2, which implies $Y_i = V_i$ for $i = 0,1,2$. □

Remark 1.5.4. In a similar way, one can show that for $U(x) = |H-x|^p$, the optimal strategy is also linear with respect to x. Moreover if p is even, i.e., $p = 2n$, then the value function is a polynomial in x, i.e., $V(t,x) = \sum_{j=0}^{p} V_j(t)x^j$ and (1.3.1) and (1.3.2) are transformed into a system of backward SDEs of order $2n+1$ for the processes $V_j(t)$.

Remark 1.5.5. Equation (1.5.15) is linear with respect to (V_1, φ_1), and V_1 is explicitly expressed in terms of (V_2, φ_2) as follows:

$$V_1(t) = E\left(H\mathcal{E}_{tT}\left(-\left(\frac{\varphi_2}{V_2} + \lambda\right)\cdot S\right)/\mathcal{F}_t\right). \tag{1.5.19}$$

1.5.1. Comparison of the direct and dual approaches. Now we give the relationship between Eq. (1.5.14) and the known feedback form solution of problem (1.1.2) expressed in terms of the variance-optimal martingale measure (see, e.g., [36]). For simplicity, we do this under the assumption of continuity of the filtration F. To this end, recall the notion of the variance-optimal martingale measure.

The variance-optimal martingale measure is a signed measure such that its density with respect to the reference measure P is of minimal L^2-norm (see [17, 86] for the precise definition and related results). According to [17, 86], the variance-optimal martingale measure Q^* always exists, and it is a probability measure equivalent to P if S is continuous and if the subset $\mathcal{M}_2^e$ of equivalent martingale measures with square integrable densities is nonempty. Moreover, as was shown in [17], if Q^* is the variance-optimal martingale measure, then the density Z_T^* of Q^* with respect to the basic measure P can be written as a constant plus a stochastic integral of S, and the density process Z_t^* defined by $E^*(Z_T/\mathcal{F}_t)$ admits the same representation

$$Z_t^* = E^* Z_T + \int_0^t h_u^* dS_u$$

for a predictable S-integrable process h^*.

Let $V_t^H = E^*(H/\mathcal{F}_t)$ and

$$V_t^H = E^*H + \int_0^t \xi_u^H dS_u + L_t^H, \quad \langle L^H, X\rangle = 0, \tag{1.5.20}$$

be the Galtchouk–Kunita–Watanabe decomposition of V_t^H with respect to the variance-optimal martingale measure Q^*.

It was shown in [36] (see also [37, 73, 79, 86]) that the optimal mean-variance hedging strategy is expressed in the feedback form

$$\pi_t^* = \xi_t^H - \frac{h_t^*}{Z_t^*}\left(V_{t-}^H - c - \int_0^t \pi_u^* dS_u\right).$$

Integrating both sides with respect to dS_u, we obtain the following linear equation for the optimal wealth process:

$$X_t^{\pi^*} = x + \int_0^t \left[\xi_s^H - \frac{h_s^*}{Z_s^*}V_s^H\right] dS_s + \int_0^t \frac{h_s^*}{Z_s^*} X_s^{\pi^*} dS_s. \tag{1.5.21}$$

To show that Eqs. (1.5.21) and (1.5.17) are equivalent, we need the following assertion proved in [57, 61]. Under the above assumptions, the variance-optimal martingale measure is a solution of the optimization problem

$$\inf_{Q\in\mathcal{M}_2^e} EZ_T^2(Q).$$

Let

$$V_t = \operatorname*{ess\,inf}_{Q\in\mathcal{M}_2^e} E\left(\frac{Z_T^2(Q)}{Z_t^2(Q)}/\mathcal{F}_t\right)$$

be the value process of the problem.

The following proposition is proved in Part 2. It is a consequnce of Theorem 2.3.1.

Proposition 1.5.2. *Assume that the filtration F is continuous and condition* (A*) *holds. Then the value process V is a unique solution of the semimartingale backward equation*

$$V_t = V_0 - \int_0^t \left(V_s\lambda_s' d\langle M\rangle_s \lambda_s - 2\lambda_s' d\langle M\rangle_s \varphi_s\right) + \frac{1}{V_s} d\langle M\rangle_s + \int_0^t \varphi_s dM_s + m_t, \quad V_T = 1, \tag{1.5.22}$$

in the class of semimartingales Y satisfying the two-sided inequality

$$c \le Y_t \le C. \tag{1.5.23}$$

Moreover, the martingale measure Q^* is variance-optimal if and only if the corresponding density is represented as

$$Z_T^* = \mathcal{E}_T\left(-\int_0^{\cdot} \lambda_s d\, M_s - \int_0^{\cdot} \frac{1}{V_s} dm_s\right). \tag{1.5.24}$$

or, equivalently, if and only if

$$Z_T^* = c\mathcal{E}_T\left(\left(\frac{\varphi}{V} - \lambda\right)\cdot S\right). \tag{1.5.25}$$

The following proposition shows that Eqs. (1.5.21) and (1.5.17) are equivalent.

Proposition 1.5.3. *Assume that the filtration F is continuous and condition* (A*) *holds. Then*

$$V(t) = \frac{1}{V_2(t)}, \quad \frac{h_t^*}{Z_t^*} = \frac{\varphi_2(t)}{V_2(t)} - \lambda_t, \quad V^H(t) = \frac{V_1(t)}{V_2(t)},$$

and the optimal wealth process X^* satisfies Eq. (1.4.9).

Proof. If we write the Itô formula for $1/V_2(t)$, taking into account the fact that $V_2(t)$ satisfies (1.5.14), we obtain that the semimartingale $1/V_2(t)$ satisfies Eq. (1.5.22) with $\varphi = -\varphi_2/V_2^2$, $m = -1/V_2^2 \cdot L$, and by the uniqueness of solution (since $c \le V_2(t) \le 1$), we have

$$V(t) = \frac{1}{V_2(t)}, \quad \frac{\varphi(t)}{V(t)} = -\frac{\varphi_2(t)}{V_2(t)}. \tag{1.5.26}$$

It follows from (1.5.25) that

$$Z_t^* = E^*(Z_T^*/\mathcal{F}_t) = V_0 \mathcal{E}_t \left(\left(\frac{\varphi}{V} - \lambda \right) \cdot S \right)$$

and

$$h_t^* = V_0 \left(\frac{\varphi_t}{V_t} - \lambda_t \right) \mathcal{E}_t \left(\left(\frac{\varphi}{V} - \lambda \right) \cdot S \right).$$

Therefore, (1.5.22) and (1.5.23) imply

$$\frac{h_t^*}{Z_t^*} = \frac{\varphi_t}{V_t} - \lambda_t = \frac{\varphi_2(t) + \lambda(t) V_2(t)}{V_2(t)}. \tag{1.5.27}$$

Now let us show that

$$\frac{\varphi_1(t) + \lambda(t) V_1(t)}{V_2(t)} = \xi^H(t) - \frac{h_t^*}{Z_t^*} V_t^H.$$

From (1.5.21), we have

$$V^H(t) = E\left(H \mathcal{E}_{tT} \left(-\lambda \cdot M - \frac{\varphi}{V} \cdot m \right) / \mathcal{F}_t \right).$$

Therefore, (1.5.19), (1.5.22), and the relation

$$\mathcal{E}_T \left(-\lambda \cdot M - \frac{\varphi}{V} \cdot m \right) = c \mathcal{E}_T \left(\left(\frac{\varphi}{V} - \lambda \right) \cdot S \right)$$

imply

$$V^H(t) = c V_1(t) \frac{\mathcal{E}_t((\frac{\varphi}{V} - \lambda) \cdot S)}{\mathcal{E}_t(-\lambda \cdot M - \frac{\varphi}{V} \cdot m)} = c V_1(t) \frac{E^*(\mathcal{E}_T((\frac{\varphi}{V} - \lambda) \cdot S)/\mathcal{F}_t)}{\mathcal{E}_t(-\lambda \cdot M - \frac{\varphi}{V} \cdot m)} = V_1(t) V(t) = \frac{V_1(t)}{V_2(t)},$$

and hence $V_1(t) = V^H(t) V_2(t)$.

Using the formula of integration by parts and equating the martingale parts of $V_1(t)$ and $V^H(t)V_2(t)$, we obtain that μ^K-a.e.,

$$\varphi_1(t) = \varphi_2(t) V^H(t) + \xi^H(t) V_2(t).$$

Therefore, (1.5.26), (1.5.27), and the latter equality imply

$$\begin{aligned} \frac{\varphi_1(t) + \lambda(t) V_1(t)}{V_2(t)} &= \frac{\varphi_2(t) V^H(t) + \xi^H(t) V_2(t) + \lambda(t) V_1(t)}{V_2(t)} \\ &= \xi^H(t) - V^H(t) \frac{\varphi(t)}{V(t)} + \lambda(t) V^H(t) = \xi^H(t) - V_t^H \frac{h_t^*}{Z_t^*}, \end{aligned}$$

and hence (1.5.21) and (1.5.17) are equivalent. □

Remark 1.5.6. Proposition 1.5.2 holds without assumption on the continuity of the filtration. To this end, one needs to apply [60, Theorem 1] instead of Proposition 1.5.1.

Remark 1.5.7. The condition $V \in \mathcal{V}^{1,2}$ also holds in several other particular cases (e.g., in the case of exponential hedging where $U(x) = \exp(H - x)$), but it is important to derive the required properties of the value function from the assumptions on the basic objects U and X, which we intend to do in the future.

Now let us consider the optimization problem

$$\text{minimize } E\left(c+\int_0^T \pi_s dS_s - H\right)^2 \tag{1.5.28}$$

over all $c \in \mathbb{R}$ and $\pi \in \Pi$. Then for any $c \in \mathbb{R}$,

$$E\left(c+\int_0^T \pi_s dS_s - H\right)^2 \geq E\left(c+\int_0^T \pi_s^*(c) dS_s - H\right)^2 = V(0,c) = V_0(0) - 2V_1(0) + c^2 V_2(0). \tag{1.5.29}$$

The infinum on the right-hand side of (1.5.29) is attained for $c = V_1(0)/V_2(0)$. It follows from Proposition 1.5.2 that

$$\frac{V_1(0)}{V_2(0)} = V_0^H = E^* H,$$

where E^* is the expectation with respect to the variance-optimal martingale measure.

Therefore,

$$E\left(c+\int_0^T \pi_s dS_s - H\right)^2 \geq E\left(E^*H+\int_0^T \pi_s dS_s - H\right)^2$$

for all c and π. Thus, if (c^*, π^*) is a solution of (1.5.28), then $c^* = E^* H$, which was proved by Schweizer in [86].

1.6. Stochastic Volatility Models

The main goal of this section is to establish the connection between the semimartingale backward equation for the value process and the classical Bellman equation for the value function related to the utility maximization problem in the case of Markov diffusion processes. For Markov diffusion models, the value process can be represented as a space-transformation of an asset price process by the value function. The problem is to establish the differentiability properties of the value function from the fact that the value process satisfies the corresponding BSDE. The role of the bridge between these equations is played by the statements describing all invariant space-transformations of diffusion processes studied by Chitashvili and Mania [8] and formulated here in the Appendix in a suitable case adapted to financial market models. This approach allows us to prove that there exists a solution (in a certain sense) of the Bellman equation and that this solution is differentiable (in a generalized sense) under mild assumptions on the model coefficients. Although, in our case, the generalized derivative in t and the second-order generalized derivatives in x do not separately exist in general (we prove the existence of a generalized L-operator), these derivatives do not enter the construction of the optimal strategy explicitly given in terms of the first-order derivatives of the value function. It should be noted that the theory of viscosity solutions is usually applied to such problems (see, e.g., El Karoui et al. (1997)), but the differentiability of the value function is in general beyond the framework of this method.

We assume that the dynamics of the asset price process is determined by the following system of stochastic differential equations:

$$dS_t = \mathrm{diag}(S_t)(\mu(t,S_t,R_t)dt + \sigma^l(t,S_t,R_t)dW_t^l), \tag{1.6.1}$$

$$dR_t = b(t,S_t,R_t)dt + \delta(t,S_t,R_t)dW_t^l + \sigma^\perp(t,S_t,R_t)dW_t^\perp, \tag{1.6.2}$$

where $W = (W^1, \ldots, W^n)$ is the n-dimensional standard Brownian motion defined on a complete probability space (Ω, F, P) equipped with the P- augmented filtration generated by W, $F = (F_t, t \in [0,T])$. The d- and $(n-d)$-dimensional Brownian motions are denoted by $W^l = (W^1, \ldots, W^d)$ and $W^\perp = (W^{d+1}, \ldots, W^n)$, respectively.

Assume that the following conditions hold.

(S1) the coefficients μ, b, δ, σ^l, and $\sigma^\perp$ are measurable and bounded;

(S2) the $(n \times n)$-matrix function $\sigma\sigma'$ is uniformly elliptic, i.e., there is a constant $c > 0$ such that

$$(\sigma(t,s,r)\lambda, \sigma(t,s,r)\lambda) \geq c|\lambda|^2$$

for all $t \in [0,T]$, $s \in \mathbb{R}^d_+$, $r \in \mathbb{R}^{n-d}$, and $\lambda \in \mathbb{R}^n$, where σ is defined by

$$\sigma(t,s,r) = \begin{pmatrix} \sigma^l(t,s,r) & 0 \\ \delta(t,s,r) & \sigma^\perp(t,s,r), \end{pmatrix};$$

(S3) system (1.6.1), (1.6.2) admits a unique strong solution.

Straightforward calculations yield that in this case

$$\lambda = \operatorname{diag}(S)^{-1}(\sigma^l\sigma^{l'})^{-1}\mu,$$

where $\sigma^{l'}$ denotes the transposition of σ^l,

$$\frac{d\langle M\rangle_t}{dt} = \operatorname{diag}(S_t)(\sigma^l\sigma^{l'})(t,S_t,R_t)\operatorname{diag}(S_t)$$

is the ν_t process, $\theta = (\sigma^l)^{-1}\mu$ is the market price of risk, and

$$\langle\lambda \cdot M\rangle_t = \int_0^t \|\theta_s\|^2 ds$$

is the mean variance tradeoff.

By the results of Krylov (1980), for sufficiently smooth coefficients μ, σ, b, and δ, the value function $V(t,x)$ can be represented as $v(t,x,S_t,R_t)$ with a sufficiently smooth function $v(t,x,s,r)$, $t \in [0,T]$, $x \in \mathbb{R}_+$, $s \in \mathbb{R}^d_+$, $r \in \mathbb{R}^{n-d}$. Hence, by Eq. (1.3.1) and the Itô formula, we obtain that $v(t,x,s,r)$ satisfies the partial differential equation

$$\begin{aligned}\mathcal{L}v(t,x,s,r) &+ v_s(t,x,s,r)'\operatorname{diag}(s)\mu(t,s,r) + v_r(t,x,s,r)'b(t,s,r) \\ &= \frac{1}{2}\frac{|v_{sx}(t,x,s,r) + \operatorname{diag}(s)^{-1}\sigma^{l'}(t,s,r)^{-1}\delta'(t,s,r)v_{rx}(t,x,s,r) + \lambda'(t,s,r)v_x(t,x,s,r)|^2_{\nu_t}}{v_{xx}(t,x,s,r)}, \end{aligned}\tag{1.6.3}$$

$$v(T,x,s,r) = U(x), \tag{1.6.4}$$

which coincides with the Bellman equation of the optimization problem (1.1.3), (1.6.1), (1.6.2) for a controlled Markov process. Moreover, the optimal strategy is

$$\pi^*(t,x,s,r) = \frac{v_{sx}(t,x,s,r) + \operatorname{diag}(s)^{-1}\sigma^{l'}(t,s,r)^{-1}\delta'(t,s,r)v_{rx}(t,x,s,r) + \lambda'(t,s,r)v_x(t,x,s,r)}{v_{xx}(t,x,s,r)}.$$

In this section, we study the solvability of (1.6.3), (1.6.4) in the particular cases of utility functions but with weaker conditions on the coefficients.

First, we consider the case of a power utility.

Theorem 1.6.1. *Let conditions* (S1)–(S3) *be satisfied. Then the value function $v(t,s,r)$ admits all first-order generalized derivatives v_s and v_r, and the generalized L-operator*

$$\begin{aligned}\mathcal{L}v = v_t + \frac{1}{2}\operatorname{tr}(\operatorname{diag}(s)\sigma^l\sigma^{l'}(t,s,r)\operatorname{diag}(s)v_{ss} &+ \operatorname{tr}(\delta\sigma^{l'}(t,r,s)\operatorname{diag}(s)v_{sr}) \\ &+ \frac{1}{2}\operatorname{tr}((\delta\delta'(t,s,r) + \sigma^\perp\sigma^{\perp'}(t,s,r))v_{rr})\end{aligned}$$

(in the sense of Definition 1.7.1 of the Appendix) is a unique bounded solution of the equation

$$\begin{aligned}&\mathcal{L}v(t,s,r) + v_s(t,s,r)' \operatorname{diag}(s)\mu(t,s,r) + v_r(t,s,r)'b(t,s,r)\\&= \frac{q}{2}\frac{|v_s(t,s,r) + \operatorname{diag}(s)^{-1}\sigma^{l'}(t,s,r)^{-1}\delta'(t,s,r)v_r(t,s,r) + \lambda(t,s,r)v(t,s,r)|^2_{\nu_t}}{v(t,s,r)} \quad dt\,ds\,dr\text{-a.e.}\end{aligned} \tag{1.6.5}$$

with the boundary condition

$$v(T,s,r) = 1. \tag{1.6.6}$$

Moreover, the optimal strategy is defined as

$$\pi^*(t,x,s,r) = (1-q)\left(\lambda(t,s,r) + \frac{\varphi(t,s,r)}{v(t,s,r)}\right)x$$

and the optimal wealth process is of the form

$$X_t^* = x\mathcal{E}_t\left((1-q)\left(\frac{\varphi}{v}+\lambda\right)\cdot S\right),$$

where

$$\varphi(t,s,r) = v_s(t,s,r) + \operatorname{diag}(s)^{-1}\sigma^{l'}(t,s,r)^{-1}\delta'(t,s,r)v_r(t,s,r).$$

Proof. Existence. Since (S,R) is a Markov process, the feedback controls are sufficient, and the value process is expressed by

$$V_t = v(t,S_t,R_t) \quad \text{a.s.,} \tag{1.6.7}$$

where

$$v(t,s,r) = \sup_{\pi\in\Pi_1} E\left(\left(1+\int_t^T \pi_u ds_u\right)^p |S_t = s, R_t = r\right)$$

(one can show this fact, e.g., similarly to [8]).

Since the value process satisfies Eq. (1.4.3), it is an Itô process. From

$$\mathcal{E}(-\lambda\cdot M) = \mathcal{E}\left(-\int_0^\cdot \theta_u dw_u^l\right)$$

and the boundedness of θ it follows that $\mathcal{E}(-\lambda\cdot M)$ satisfies the reverse Hölder inequality. Thus, by the Hölder inequality,

$$V_t = \operatorname*{ess\,sup}_{\pi\in\Pi_1} E\left(\left(1+\int_t^T \pi_u dS_u\right)^p /\mathcal{F}_t\right)$$

is bounded, and its martingale part is in BMO by [56]. Hence the finite variation part of V_t is of integrable variation, and from (1.6.7) we see that $v(t,S_t,R_t)$ is an Itô process of the form (1.7.14) (Appendix). Therefore, Proposition 1.7.4 of the Appendix implies that the function $v(t,s,r)$ admits a generalized L-operator and all first-order generalized derivatives, and it can be represented as follows:

$$
\begin{aligned}
v(t,S_t,R_t) = v_0 + \int_0^t \big(v_s(u,S_u,R_u)' \operatorname{diag}(S_u)\sigma^l(u,S_u,R_u) \\
+ v_r(u,S_u,R_u)'\delta(u,S_u,R_u)\big)dW_s^l + \int_0^t v_r(u,S_u,R_u)'\sigma^\perp(u,S_u,R_u)dW_s^\perp \\
+ \int_0^t \mathcal{L}v(u,S_u,R_u)ds + \int_0^t \big(v_s(u,S_u,R_u)' \operatorname{diag}(X_s)\mu(u,S_u,R_u) \\
+ v_r(u,S_u,R_u)b(u,S_u,R_u)\big)du, \quad (1.6.8)
\end{aligned}
$$

where $\mathcal{L}V$ is the generalized $\mathcal{L}$-operator.

On the other hand, the value process is a solution of (1.4.3), and by the uniqueness of the canonical decomposition of semimartingales, comparing the martingale parts of (1.6.8) and (1.4.3), we have that $dt \times dP$-a.e.

$$\varphi_t = v_s(t,S_t,R_t) + \operatorname{diag}(S_t)^{-1}\sigma^{l'}(t,S_t,R_t)^{-1}\delta'(t,S_t,R_t)v_r(t,S_t,R_t), \quad (1.6.9)$$

$$\varphi_t^\perp = \sigma^{\perp'}(t,S_t,R_t)v_r(t,S_t,R_t). \quad (1.6.10)$$

Then, equating the processes of bounded variation of the equations and taking into account (1.6.8) and (1.6.9), we derive that

$$
\begin{aligned}
\int_0^t \big(\mathcal{L}v(u,S_u,R_u) + v_s(u,S_u,R_u)' \operatorname{diag}(S_u)\mu(u,S_u,R_u) + v_r(u,S_u,R_u)b(u,S_u,R_u)\big)du \\
= \frac{q}{2}\int_0^t \frac{|\varphi_u + \lambda(u,S_u,R_u)v(u,S_u,R_u)|^2_{\nu_u}}{v(u,S_u,R_u)}du, \quad (1.6.11)
\end{aligned}
$$

which gives that $v(t,s,r)$ solves the Bellman equation (1.6.5).

Uniqueness. Let $\tilde{v}(t,s,r)$ be a bounded positive solution of (1.6.5), (1.6.6) from the class V^L. Then using the generalized Itô formula (Proposition 1.7.1 of the Appendix) and Eq. (1.6.5), we obtain that $\tilde{v}(t,S_t,R_t)$ is a solution of (1.4.3) and hence $\tilde{v}(t,S_t,R_t)$ coincides with the value process v by Theorem 1.4.1. Therefore, $\tilde{v}(t,S_t,R_t) = v(t,S_t,R_t)$-a.s. and $\tilde{v} = v$, $dt\,dx\,dy$-a.e. □

Now we consider extreme cases for the stochastic volatility models. In the first extreme case, we assume that the coefficients μ and σ^l do not contain the variable r. Hence θ and λ are also independent of the variable r and Eq. (2.4.19) takes the form

$$dS_t = \operatorname{diag}(S_t)(\mu(t,S_t)dt + \sigma^l(t,S_t)dW_t^l). \quad (1.6.12)$$

Let $S(q)$ be the Itô process governed by SDE

$$dS_t(q) = \operatorname{diag}(S_t(q))\sigma^l(t,S_t(q))(dW_t^l + q\theta(t,S_t(q))dt), \quad (1.6.13)$$

where $dW_t^l + q\theta(t,S_t)dt$ is the Brownian motion with respect to measure

$$dQ(q) = \mathcal{E}_T(-q\int_0^\cdot \theta_u dw_u^l)dP.$$

Thus, by Theorem 1.4.1, the value process is represented as

$$V_t = v(t, S_t(q)) = (\tilde{v}(t, S_t(q))^{\frac{1}{1-q}},$$

where

$$\tilde{v}(t,s) = E^{Q(q)}\left(\exp\left(\frac{q(q-1)}{2}\int_t^T |\theta_u|^2 du\right) | S_t(q) = s\right).$$

Therefore, we have the following assertion.

Corollary 1.6.1. *Let conditions* (S1)–(S3) *hold for the coefficients of system* (1.6.13)*. Then the value process can be represented as* $(\tilde{v}(t, S_t(q))^{\frac{1}{1-q}}$*, where* $\tilde{v}(t,s)$ *is the classical solution of the linear partial differential equation*

$$\tilde{v}_t(t,s) + \frac{1}{2}\operatorname{tr}(\operatorname{diag}(s)\sigma^l\sigma^{l'}(t,s)\operatorname{diag}(s)\tilde{v}_{ss}(t,s)) + \frac{q(q-1)}{2}|\theta(t,s)|^2\tilde{v}(t,s) = 0, \tag{1.6.14}$$

$$\tilde{v}(T,s) = 1. \tag{1.6.15}$$

The second extreme case corresponds to the stochastic volatility model of the form

$$\begin{aligned} dS_t &= \operatorname{diag}(S_t)(\mu(t,S_t,R_t)dt + \sigma^l(t,S_t,R_t)dW_t^l), \\ dR_t &= b(t,R_t)dt + \sigma^{\perp}(t,R_t)dW_t^{\perp}. \end{aligned} \tag{1.6.16}$$

Corollary 1.6.2. *Let conditions* (S1)–(S3) *hold for the coefficients of system* (1.6.16) *and* θ *be independent of the variable* s*. Then the value process of the optimization problem* (1.4.1) *has the form* $V_t = v(t, R_t)$*, where*

$$v(t,r) = E\left(\exp\left(-\frac{q}{2}\int_t^T |\theta(u,R_u)|^2 du\right) | R_t = r\right)$$

satisfies the linear partial differential equation

$$v_t(t,r) + \frac{1}{2}\operatorname{tr}(\sigma^{\perp}\sigma^{\perp'}(t,r)v_{rr}(t,r)) + v_r(t,r)'b(t,r) - \frac{q}{2}|\theta(t,r)|^2 v(t,r) = 0, \tag{1.6.17}$$

$$v(T,r) = 1. \tag{1.6.18}$$

A similar result can be obtained for the exponential utility function.

Proposition 1.6.1. *Let conditions* (S1)–(S3) *hold and* $H = g(S_T, R_T)$ *for a continuous bounded function* $g(s,r)$*. Then the value function* $v(t,s,r)$ *for problem* (1.4.10) *admits all first-order generalized derivatives* v_s *and* v_r *and the generalized L-operator is a unique bounded solution of the equation*

$$\begin{aligned} &\mathcal{L}v(t,s,r) + v_s(t,s,r)'\operatorname{diag}(s)\mu(t,s,r) + v_r(t,s,r)'b(t,s,r) \\ &\quad = \frac{1}{2}\frac{|v_s(t,s,r) + diag(s)^{-1}\sigma^{l'}(t,s,r)^{-1}\delta'(t,s,r)v_r(t,s,r) + \lambda(t,s,r)v(t,s,r)|^2_{\nu_t}}{v(t,s,r)} \quad dt\,ds\,dr\text{-}a.e. \end{aligned} \tag{1.6.19}$$

with the boundary condition

$$v(T,s,r) = e^{-\gamma g(s,r)}. \tag{1.6.20}$$

Moreover, the optimal strategy is defined as

$$\pi^*(t,x,s,r) = \frac{1}{\gamma}\left(\lambda(t,s,r) + \frac{\varphi(t,s,r)}{v(t,s,r)}\right)x,$$

where

$$\varphi(t,s,r) = v_s(t,s,r) + \operatorname{diag}(s)^{-1}\sigma^{l'}(t,s,r)^{-1}\delta'(t,s,r)v_r(t,s,r)$$

and the optimal wealth process is defined by (1.4.15).

In the case of logarithmic utility, from Theorem 1.4.2 and the Feynmann–Kac formula, we immediately obtain the following assertion.

Proposition 1.6.2. *Let conditions* (S1)–(S3) *hold and* $U(x) = \ln x$. *Then the value function can be represented as* $v(t, S_t, R_t)$, *where* $v(t, s, r)$ *is a unique solution of the linear partial differential equation*

$$\mathcal{L}v(t,s,r) + v_s(t,s,r)' \operatorname{diag}(s)\mu(t,s,r) + v_r(t,s,r)'b(t,s,r) + |\theta(t,s,r)|^2 v(t,s,r) = 0, \tag{1.6.21}$$

$$v(T,s,r) = 1 \tag{1.6.22}$$

and the optimal strategy is $\pi^*(t,x,s,r) = \lambda(t,s,r)x$.

Now we specify the result presented in Theorem 1.5.1 in the case of the stochastic volatility model given by (2.4.19) and (2.4.20). For simplicity, we consider the case where $\delta = 0$ and assume that $H = g(S_T, R_T)$ for some continuous bounded g. In this case, the value process has the form

$$V(t) = v_2(t, S_t, R_t)x^2 - 2v_1(t, S_t, R_t)x + v_0(t, S_t, R_t).$$

The following assertion can be proved similarly to Theorem 1.4.3.

Theorem 1.6.2. *Let conditions* (S1)–(S3) *hold. Then the value function* $v(t,s,r)$ *of problem* (1.1.2) *admits all first-order generalized derivatives* v_s *and* v_r, *and the generalized* $\mathcal{L}$*-operator* (*in the sense of Definition* 1.7.1 *of the Appendix*) *is a unique solution of the system of partial differential equations*

$$\mathcal{L}v_2(t,s,r) + v_{2s}(t,s,r)' \operatorname{diag}(s)\mu(t,s,r) + v_{2r}(t,s,r)'b(t,s,r)$$
$$= \frac{|v_{2s}(t,s,r) + \lambda(t,s,r)v_2(t,s,r)|^2_{\nu(t,s,r)}}{v_2(t,s,r)} \quad dt\,ds\,dr\text{-a.e.}, \tag{1.6.23}$$

$$v_2(T,s,r) = 1, \tag{1.6.24}$$

$$\mathcal{L}v_1(t,s,r) + v_{1s}(t,s,r)' \operatorname{diag}(s)\mu(t,s,r) + v_{1r}(t,s,r)'b(t,s,r)$$
$$= \frac{\big(v_{1s}(t,s,r) + \lambda(t,s,r)v_1(t,s,r), v_{2s}(t,s,r) + \lambda(t,s,r)v_2(t,s,r)\big)_{\nu(t,s,r)}}{v_2(t,s,r)}, \tag{1.6.25}$$

$$v_1(T,s,r) = g(s,r), \tag{1.6.26}$$

$$\mathcal{L}v_0(t,s,r) + v_{0s}(t,s,r)' \operatorname{diag}(s)\mu(t,s,r) + v_{0r}(t,s,r)'b(t,s,r)$$
$$= \frac{|v_{1s}(t,s,r) + \lambda(t,s,r)v_1(t,s,r)|^2_{\nu(t,s,r)}}{v_2(t,s,r)}, \tag{1.6.27}$$

$$v_0(T,s,r) = g(s,r)^2. \tag{1.6.28}$$

Moreover, the optimal strategy has the form

$$\pi(t,x,s,r) = \frac{v_{1s}(t,s,r) + \lambda(t,s,r)v_1(t,s,r)}{v_2(t,s,r)} - \frac{v_{2s}(t,s,r) + \lambda(t,s,r)v_2(t,s,r)}{v_2(t,s,r)}x.$$

1.7. Appendix

A. Let us show that the family

$$\Lambda_t^\pi = E\left(U\left(x + \int_0^T \pi_u dS_u\right)|\mathcal{F}_t\right), \quad \pi \in \Pi_x(\tilde{\pi}, t, T) \tag{1.7.1}$$

satisfies the ε-lattice property (with $\varepsilon = 0$) for any $t \in [0,T]$ and $\tilde{\pi}$. $\Pi(\tilde{\pi}, t, T)$ is the set of predictable S-integrable processes π from Π_x such that

$$\pi_s = \tilde{\pi}_s I_{(0 \le s < t)}.$$

We write $\Pi(t,T)$ instead of $\Pi(0,t,T)$ for the class of strategies corresponding to $\tilde{\pi} = 0$ up to time t.

We need to show that for any $\pi^1, \pi^2 \in \Pi(\tilde{\pi}, t, T)$, there exists a strategy $\pi \in \Pi(\tilde{\pi}, t, T)$ such that

$$\Lambda_t^{\pi} = \max(\Lambda_t^{\pi^1}, \Lambda_t^{\pi^2}). \tag{1.7.2}$$

For any π^1 and π^2, define the set

$$B = \{\omega : \Lambda_t^{\pi^1} \leq \Lambda_t^{\pi^2}\},$$

and let

$$\pi_s = \tilde{\pi}_s I_{(0 \leq s < t)} + \pi_s^1 I_B I_{(s \geq t)} + \pi_s^2 I_{B^c} I_{(s \geq t)}.$$

Obviously, $\pi \in \Pi_x$ (respectively, $\pi \in \Pi^p$) if $\tilde{\pi}, \pi^1, \pi^2 \in \Pi_x$ (respectively, Π^p).

Since B is $\mathcal{F}_t-$measurable, we have

$$\begin{aligned}
\Lambda_t^{\pi} &= E\left(U\left(x + \int_0^T \pi_u dS_u\right) |\mathcal{F}_t\right) = E\left(U\left(x + \int_0^t \tilde{\pi}_u dS_u + I_B \int_t^T \pi_u^1 dS_u + I_{B^c} \int_t^T \pi_u^2 dS_u\right) |\mathcal{F}_t\right) \\
&= I_B E\left(U\left(x + \int_0^t \tilde{\pi}_u dS_u + \int_t^T \pi_u^1 dS_u\right) |\mathcal{F}_t\right) + I_{B^c} E\left(U\left(x + \int_\tau^t \tilde{\pi}_u dS_u + \int_t^T \pi_u^2 dS_u\right) |\mathcal{F}_t\right) \\
&= I_B E\left(U\left(x + \int_0^T \pi_u^1 dS_u\right) |\mathcal{F}_t\right) + I_{B^c} E\left(U\left(x + \int_0^T \pi_u^2 dS_u\right) |\mathcal{F}_t\right) \\
&= E\left(U\left(x + \int_0^T \pi_u^1 dS_u\right) |\mathcal{F}_t\right) \vee E\left(U\left(x + \int_0^T \pi_u^2 dS_u\right) |\mathcal{F}_t\right),
\end{aligned}$$

and hence (1.7.2) is satisfied.

Proposition 1.7.1 (optimality principle). *Let condition* (B1) *hold.*

(a) *For all $x \in \mathbb{R}$, $\pi \in \Pi$, and $s \in [0, T]$, the process*

$$\left(V\left(t, x + \int_s^t \pi_u dS_u\right),\ t \geq s\right)$$

is a supermartingale, admitting an RCLL modification.

(b) *$\pi^*(s, x)$ is optimal if and only if*

$$\left(V\left(t, x + \int_s^t \pi_u^* dS_u\right),\ t \geq s\right)$$

is a martingale.

(c) *For all $s < t$,*

$$V(s, x) = \operatorname*{ess\,sup}_{\pi \in \Pi(s,T)} E\left(V\left(t, x + \int_s^t \pi_u dS_u\right) |\mathcal{F}_s\right). \tag{1.7.3}$$

Proof. (a) For simplicity, we take s to be equal to zero. Let us show that

$$Y_t = V\left(t, x + \int_0^t \tilde{\pi}_u dS_u\right)$$

is a supermartingale for all x and $\tilde{\pi}$. Since

$$Y_t = \operatorname*{ess\,sup}_{\pi\in\Pi(t,T)} E\left(U\left(x + \int\limits_0^t \tilde{\pi}_u dS_u + \int\limits_t^T \pi_u dS_u\right)|\mathcal{F}_t\right)$$

using the lattice property of family (1.7.1) from [26, Lemma 16.A.5], we have

$$\begin{aligned}E(Y_t|\mathcal{F}_s) &= E\left(\operatorname*{ess\,sup}_{\pi\in\Pi(t,T)} E\left(U\left(x + \int\limits_0^t \tilde{\pi}_u dS_u + \int\limits_t^T \pi_u dS_u\right)|\mathcal{F}_t\right)|\mathcal{F}_s\right)\\ &= E\left(\operatorname*{ess\,sup}_{\pi\in\Pi(\tilde{\pi},t,T)} E\left(U\left(x + \int\limits_0^T \pi_u dS_u\right)|\mathcal{F}_t\right)|\mathcal{F}_s\right) = \operatorname*{ess\,sup}_{\pi\in\Pi(\tilde{\pi},t,T)} E\left(U\left(x + \int\limits_0^T \pi_u dS_u\right)|\mathcal{F}_s\right).\end{aligned} \tag{1.7.4}$$

Obviously, $\Pi(\tilde{\pi},t,T) \subseteq \Pi(\tilde{\pi},s,T)$ for $s \le t$, which implies the inequality

$$\begin{aligned}&\operatorname*{ess\,sup}_{\pi\in\Pi(\tilde{\pi},t,T)} E\left(U\left(x + \int\limits_0^T \pi_u dS_u\right)|\mathcal{F}_s\right)\\ &\qquad\le \operatorname*{ess\,sup}_{\pi\in\Pi(\tilde{\pi},s,T)} E\left(U\left(x + \int\limits_0^T \pi_u dS_u\right)|\mathcal{F}_t\right) = V\left(s, x + \int\limits_0^s \tilde{\pi}_u dS_u\right).\end{aligned} \tag{1.7.5}$$

Thus, (1.7.4) and (1.7.5) imply $E(Y_t/F_s) \le Y_s$.

(b) If

$$V\left(t, x + \int\limits_0^t \pi^*_u dS_u\right)$$

is a martingale, then

$$\inf_{\pi\in\Pi} EU\left(x + \int\limits_0^T \pi_u dS_u\right) = V(0,x) = EV(0,x) = EV\left(T, x + \int\limits_0^T \pi^*_u dS_u\right) = EU\left(x + \int\limits_0^T \pi^*_u dS_u\right),$$

and hence π^* is optimal.

Conversely, if π^* is optimal, then

$$EV(0,x) = \sup_{\pi\in\Pi} EU\left(x + \int\limits_0^T \pi_u dS_u\right) = EU\left(x + \int\limits_0^T \pi^*_u dS_u\right) = EV\left(T, x + \int\limits_0^T \pi^*_u dS_u\right).$$

Since

$$V\left(t, x + \int\limits_0^t \pi^*_u dS_u\right)$$

is a supermartingale, the latter equality implies that this process is a martingale (this follows from [51, Lemma 6.6]).

(c) Since

$$Y_t = V\left(t, x + \int\limits_s^t \tilde{\pi}_u dS_u\right)$$

is a supermartingale, for any $\tilde{\pi} \in \Pi(s,T)$, $x \in \mathbb{R}$, and $t \geq s$, we have

$$V(s,x) \geq E\left(V\left(t, x + \int_s^t \tilde{\pi}_u dS_u\right) | \mathcal{F}_s\right),$$

and hence

$$V(s,x) \leq \operatorname*{ess\,sup}_{\tilde{\pi} \in \Pi(s,T)} E\left(V\left(t, x + \int_s^t \tilde{\pi}_u dS_u\right) | \mathcal{F}_s\right). \tag{1.7.6}$$

On the other hand, for any $\tilde{\pi}$,

$$\begin{aligned} E\left(V\left(t, x + \int_s^t \tilde{\pi}_u dS_u\right) | \mathcal{F}_s\right) &= E\left(\operatorname*{ess\,sup}_{\pi \in \Pi(t,T)} E\left(U\left(x + \int_s^t \tilde{\pi}_u dS_u + \int_t^T \pi_u dS_u\right) | \mathcal{F}_t\right) \mathcal{F}_s\right) \\ &\geq E\left(E\left(U\left(x + \int_s^T \tilde{\pi}_u dS_u\right) | \mathcal{F}_t\right) \mathcal{F}_s\right) = E\left(U\left(x + \int_s^T \tilde{\pi}_u dS_u\right) | \mathcal{F}_s\right). \end{aligned}$$

Taking esssup of both parts, we obtain

$$\operatorname*{ess\,sup}_{\tilde{\pi} \in \Pi(s,T)} E\left(V\left(t, x + \int_s^t \tilde{\pi}_u dS_u\right) | \mathcal{F}_s\right) \geq \operatorname*{ess\,sup}_{\tilde{\pi} \in \Pi(s,T)} E\left(U\left(x + \int_s^T \tilde{\pi}_u dS_u\right) | \mathcal{F}_s\right) = V(s,x). \tag{1.7.7}$$

Thus (1.7.3) follows from (1.7.6) and (1.7.7).

Let us show now that the process

$$\tilde{V}\left(t, x + \int_0^t \tilde{\pi}_u dS_u\right)$$

admits an RCLL modification for all $x \in \mathbb{R}$ and $\pi \in \tilde{\Pi}$. According to [51, Theorem 3.1], it suffices to prove that the function

$$E\tilde{V}\left(t, x + \int_0^t \tilde{\pi}_u dS_u\right), \quad t \in [0,T],$$

is right-continuous for every $x \in \mathbb{R}$.

Let $(t_n, n \geq 1)$ be a sequence of positive numbers such that $t_n \downarrow t$ as $n \to \infty$. Since

$$\tilde{V}\left(t, x + \int_0^t \tilde{\pi}_u dS_u\right)$$

is a supermartingale, we have

$$E\tilde{V}\left(t, x + \int_0^t \tilde{\pi}_u dS_u\right) \geq \lim_{n \to \infty} E\tilde{V}\left(t_n, x + \int_0^{t_n} \tilde{\pi}_u dS_u\right). \tag{1.7.8}$$

Let us show the converse inequality. For $s = 0$, (1.7.4) takes the form

$$E\tilde{V}\left(t, x + \int_0^t \tilde{\pi}_u dS_u\right) = \max_{\pi \in \tilde{\Pi}(\tilde{\pi},t,T)} EU\left(x + \int_0^T \pi_u dS_u\right). \tag{1.7.9}$$

Therefore, for any $\varepsilon > 0$, there exists a strategy π^ε such that

$$E\tilde{V}\left(t, x + \int_0^t \tilde{\pi}_u dS_u\right) \le EU\left(x + \int_0^t \tilde{\pi}_u dS_u + \int_t^T \pi_u^\varepsilon dS_u\right) + \varepsilon. \tag{1.7.10}$$

Let us define the sequence $(\pi^n, n \ge 1)$ of strategies

$$\pi_s^n = \tilde{\pi}_s I_{(s<t_n)} + \pi_s^\varepsilon I_{(s\ge t_n)}.$$

Using inequality (1.7.10), the continuity of U (it follows from (B1) and (B2)), the convergence of the stochastic integrals, and the Fatou lemma, we have

$$\begin{aligned}
E\tilde{V}\left(t, x + \int_0^t \tilde{\pi}_u dS_u\right) &\le EU\left(x + \int_0^t \tilde{\pi}_u dS_u + \int_t^T \pi_u^\varepsilon dS_u\right) + \varepsilon \\
&= E\left(\lim_n U\left(x + \int_0^{t_n} \tilde{\pi}_u dS_u + \int_{t_n}^T \pi_u^\varepsilon dS_u\right)\right) + \varepsilon \\
&\ge \varliminf_{n\to\infty} E\left(E\left(U\left(x + \int_0^{t_n} \tilde{\pi}_u dS_u + \int_{t_n}^T \pi_u^\varepsilon dS_u\right) / \mathcal{F}_{t_n}\right)\right) + \varepsilon \\
&\ge \varliminf_{n\to\infty} E\left(\operatorname*{ess\,sup}_{\pi\in\tilde{\Pi}(\tilde{\pi},t_n,T)} E\left(U\left(x + \int_0^{t_n} \tilde{\pi}_u dS_u + \int_{t_n}^T \pi_u dS_u\right) / \mathcal{F}_{t_n}\right)\right) + \varepsilon \\
&= \varliminf_{n\to\infty} E\tilde{V}\left(t_n, x + \int_0^{t_n} \tilde{\pi}_u dS_u\right) + \varepsilon.
\end{aligned} \tag{1.7.11}$$

Since ε is an arbitrary positive number, from (1.7.11) we obtain

$$E\tilde{V}\left(t, x + \int_0^t \tilde{\pi}_u dS_u\right) \le \varliminf_{n\to\infty} E\tilde{V}\left(t_n, x + \int_0^{t_n} \tilde{\pi}_u dS_u\right), \tag{1.7.12}$$

which, together with (1.7.9), implies that the function

$$E\tilde{V}\left(t, x + \int_0^t \tilde{\pi}_u dS_u\right), \quad t \in [0, T],$$

is right-continuous. □

B. Let $(K(t), t \in \mathbb{R})$ be a strictly increasing continuous function. Define

$$\tau_s(\varepsilon) = \inf\{t \ge s : K_t - K_s \ge \varepsilon\}, \quad \sigma_s(\varepsilon) = \inf\{t \ge 0 : K_t - K_s \ge -\varepsilon\}.$$

Obviously, $K_{\tau_s(\varepsilon)} = K_s + \varepsilon$ and $K_{\sigma_s(\varepsilon)} = K_s - \varepsilon$.

Lemma 1.7.1. *For any K-integrable function F,*

$$\int_{\mathbb{R}} \frac{1}{\varepsilon} \int_s^{\tau_s^\varepsilon} \left|F(t) - F(s)\right| dK(t) dK(s) \to 0 \quad \textit{as } \varepsilon \to 0.$$

Proof. First, assume that F is continuous and $F(t)=0$ if $|t|>T$ for some $T>0$. Then

$$\int_{\mathbb{R}} \frac{1}{\varepsilon} \int_s^{\tau_s^\varepsilon} \big|F(s)-F(t)\big| dK_t dK_s \le \int_{\mathbb{R}} \max_{t\le s\le \tau_t^\varepsilon} \big|F(s)-F(t)\big| dK_t$$

$$\le \max_{0\le s-t\le \tau_t^\varepsilon - t} \big|F(s)-F(t)\big| \quad \text{as } \varepsilon\to 0$$

since F is uniformly continuous on $[-T,T]$ and $\tau_t^\varepsilon - t\to 0$ as $\varepsilon\to 0$.

On the other hand,

$$\int_{\mathbb{R}} \tfrac{1}{\varepsilon} \int_s^{\tau_s^\varepsilon} \big|F(s)-F(t)\big| dK_t dK_s \le |F|_{L^1(\mathbb{R},dK)} + \int_{\mathbb{R}} \tfrac{1}{\varepsilon} \int_s^{\tau_s^\varepsilon} |F(t)| dK_t dK_s$$

$$|F|_{L^1(\mathbb{R},dK)} + \int_{\mathbb{R}} \frac{1}{\varepsilon} \int_{\sigma_t^\varepsilon}^{t} |F(t)| dK_s dK_t \le 2|F|_{L^1(\mathbb{R},dK)}, \quad (1.7.13)$$

since by the Fubini theorem

$$\int_{\mathbb{R}} \int_s^{\tau_s^\varepsilon} |F(t)| dK_t dK_s = \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{(s\le t\le \tau_s^\varepsilon)} |F(t)| dK_s dK_t$$

$$= \int_{\mathbb{R}} \int_{\sigma_t^\varepsilon}^{t} |F(t)| dK_s dK_t \le \int_{\mathbb{R}} |F(t)| (K_t - K_{\sigma_t^\varepsilon}) dK_t \le \varepsilon |F|_{L^1(\mathbb{R},dK)}.$$

Using inequality (1.7.13), we can approximate each function $F\in L^1(\mathbb{R},dK)$ by compactly supported continuous functions. □

Corollary 1.7.1. *For $F\in L^1(\mathbb{R},dK)$,*

$$\int_{\mathbb{R}} \left| \frac{1}{\varepsilon} \int_s^{\tau_s^\varepsilon} F(t) dK_t - F(s) \right| dK_s \to 0 \quad \text{as } \varepsilon\to 0.$$

If

$$\int_t^{\tau_t^\varepsilon} F(s) dK_s = 0 \quad dK\text{-a.s.},$$

then $F_t = 0$ dK-a.s.

Proposition 1.7.2. *Let $(f(t,x),(t,x)\in\mathbb{R}^2)$ and $(X(t,s),t\ge s)$ be measurable functions such that the family $x\to f(\cdot,x)$ is continuous in $L^1(\mathbb{R},dK)$, and let $X(s,t)$ be a continuous function on $\{(t,s);t\ge s\}$ with $X(s,s)=x$ for all $s\in\mathbb{R}$ and some $x\in\mathbb{R}$. Then*

$$\int_{\mathbb{R}} \left| \frac{1}{\varepsilon} \int_s^{\tau_s^\varepsilon} f(t,X(t,s)) dK_t - f(s,x) \right| dK_s \to 0 \quad \text{as } \varepsilon\to 0.$$

Proof. Denote by b_t^ε the expression $\max\limits_{\sigma_t^\varepsilon \le s \le t} |X(t,s) - x|$. Then

$$\int\limits_{\mathbb{R}} \frac{1}{\varepsilon} \int\limits_{s}^{\tau_s^\varepsilon} \big|f(t, X(t,s)) - f(s,x)\big| dK_t dK_s \le \int\limits_{\mathbb{R}} \frac{1}{\varepsilon} \int\limits_{\sigma_t^\varepsilon}^{t} \big|f(t, X(t,s)) - f(t,x)\big| dK_s dK_t$$

$$+ \int\limits_{\mathbb{R}} \frac{1}{\varepsilon} \int\limits_{s}^{\tau_s^\varepsilon} \big|f(t,x) - f(s,x)\big| dK_t dK_s.$$

The first term in the latter expression can be estimated by

$$\int\limits_{\mathbb{R}} \max_{|x-y| \le b_t^\varepsilon} \big|f(t,x) - f(t,y)\big| dK_t.$$

Since $X(\cdot,\cdot)$ is continuous, $b_t^\varepsilon \to 0$ uniformly on each $[-T,T]$ as $\varepsilon \to 0$, and by the continuity of the family $f(\cdot, x) \in L^1$, we obtain that the first summand tends to zero. The second summand tends to zero by Lemma 1.7.1. $\square$

Remark 1.7.1. If the functions f and K are defined on the subsets $[0,T] \times \mathbb{R}$ and $[0,T]$, respectively, then we can consider the functions

$$\tilde{f}(t,x) = \begin{cases} f(t,x), & (t,x) \in [0, T \times \mathbb{R}, \\ 0, & (t,x) \in [0,T] \times \mathbb{R}, \end{cases} \qquad \tilde{K}(t) = \begin{cases} K(t), & t \in [0,T], \\ t + K(0), & t < 0, \\ K(T) + t - T, & t > T \end{cases}$$

and further we can use Proposition 1.7.2.

C. Assume that the following condition holds:

(D3′) there exist $\gamma > 0$, a positive integrable random variable ξ, and $p > 1$ such that $U(x) \ge \gamma |x|^p - \xi$.

Note that the function $U(x) = |H - x|^p$ for $H \in L^p$ satisfies (D3′), as well as conditions (D1)–(D2) of Sec. 1.5.

Proposition 1.7.3. *Assume that one of the assertions of Proposition* 1.2.1 *and conditions* (D1), (D2), *and* (D3′) *are satisfied. Then for any* t *and* x, *the problem*

$$\operatorname*{ess\,inf}_{\pi \in \Pi} E\left(U\left(x + \int\limits_{t}^{T} \pi_s dS_s \right) / \mathcal{F}_t \right)$$

admits a unique solution with a p-integrable wealth process.

Proof. By the lattice property (see Sec. 1.7), we can choose a sequence $\tilde{\pi}^n \in \Pi$ such that

$$E\left(U\left(x + \int\limits_{t}^{T} \tilde{\pi}_s^n dS_s \right) / \mathcal{F}_t \right) \downarrow V(t,x) \quad P\text{-a.s.}$$

By condition (B1), one can choose a sequence $\tilde{\pi}_n$ such that

$$E\left(U\left(x + \int\limits_{t}^{T} \tilde{\pi}_s^n dS_s \right) / \mathcal{F}_t \right) \le E(U(x)/\mathcal{F}_t)$$

for all $n \ge 1$. Thus

$$EU\left(x + \int\limits_{t}^{T} \tilde{\pi}_s^n dS_s \right) \to EV(t,x) \quad \text{as } n \to \infty.$$

By condition (B3′), there exists $R > 0$ such that

$$\gamma E\left|x+\int_t^T \tilde{\pi}_s^n dS_s\right|^p \le EU\left(x+\int_t^T \tilde{\pi}_s^n dS_s\right)+E\xi \le R.$$

Hence $x+\int_t^T \tilde{\pi}_s^n dS_s$ is a bounded sequence in the space L^p, and we can assume that it weakly converges. By the Masure lemma (see, e.g., [25]), there exists a sequence of strategies

$$\pi^n=\sum_{k=n}^{q(n)} \alpha_{kn}\tilde{\pi}^{kn},$$

where

$$q(n)>n, \quad \sum_{k=n}^{q(n)} \alpha_{kn}=1, \quad \alpha_{kn}\ge 0,$$

such that

$$\int_t^T \pi_s^n\, dS_s \to \int_t^T \pi_s^*\, dS_s$$

in L^p for some $\pi^*\in\Pi$. We can assume also that

$$\int_t^T \pi_s^n\, dS_s \to \int_t^T \pi_s^*\, dS_s \quad P\text{-a.s.}$$

By the convexity of U, we have

$$E\left[U\left(x+\int_t^T \pi_s^n dS_s\right)/\mathcal{F}_t\right] \le E\left[U\left(x+\int_t^T \tilde{\pi}_s^n dS_s\right)/\mathcal{F}_t\right].$$

Therefore,

$$\overline{\lim_{n\to\infty}}\, E\left[U\left(x+\int_t^T \pi_s^n dS_s\right)|\mathcal{F}_t\right] \le \lim_{n\to\infty} E\left[U\left(x+\int_t^T \tilde{\pi}_s^n dS_s\right)|\mathcal{F}_t\right]=V(t,x).$$

On the other hand, the Fatou lemma implies that

$$E\left[U\left(x+\int_t^T \pi_s^* dS_s\right)|\mathcal{F}_t\right] \le \underline{\lim}_{n\to\infty}\, E\left[U\left(x+\int_t^T \pi_s^n dS_s\right)|\mathcal{F}_t\right] \quad P\text{-a.s.}$$

Therefore, π^* is optimal and π^* is unique by Remark 1.2.3. □

Finally, we prove the following lemma used in the proof of Proposition 1.3.1.

Lemma 1.7.2. *Let b_t be a predictable process and S be a continuous semimartingale. Denote by Π_x the space of all predictable S-integrable processes π such that for all $t\in[0,T]$,*

$$x+\int_0^t \pi_u dS_u \ge 0.$$

Then P-a.s. for all $t\in[0,T]$,

$$\operatorname*{ess\,inf}_{\pi\in\Pi_x} |\pi_t-b_t|=0.$$

Proof. Taking $b_t^n = b_t I_{(|b_t|\le n)}$, we have

$$\operatorname*{ess\,inf}_{\pi\in\Pi_x} |\pi_t - b_t| \le \operatorname*{ess\,inf}_{\pi\in\Pi_x} |\pi_t - b_t^n| + |b_t^n - b_t|.$$

Therefore, without loss of generality, we may assume that b is S-integrable. Let τ be a predictable stopping time. Denote by $(\tau_n, n\ge 1)$ the predicted sequence of stopping times.

For each $n\ge 1$, let us define the strategy

$$\pi_t^n = b_t I_{(\tau_n,\tau]}(t)\mathcal{E}_t\left(\frac{1}{x} b I_{(\tau_n,\tau]}\cdot S\right).$$

Obviously, π^n belongs to Π_x for all $n\ge 1$.

Indeed,

$$x + \int_0^t \pi_u dS_u = x + x\int_0^t \mathcal{E}_u\left(\frac{1}{x} b I_{(\tau_n,\tau]}\cdot S\right)\frac{1}{x} b_u I_{(\tau_n,\tau]}(u) dS_u = x\mathcal{E}_t\left(\frac{1}{x} b I_{(\tau_n,\tau]}\cdot S\right)\ge 0.$$

Since $\pi_\tau^n = b_\tau \mathcal{E}_\tau(\frac{1}{x} b I_{(\tau_n,\tau]}\cdot S)$ and S is continuous, we have that P-a.s.,

$$\pi_\tau^n \to b_\tau \quad \text{as } n\to\infty.$$

Denote by γ_t the expression $\operatorname*{ess\,inf}_{\pi\in\Pi_x} |\pi_t - b_t|$. Then

$$\gamma_\tau = \big(\operatorname*{ess\,inf}_{\pi\in\Pi_x} |\pi_t - b_t|\big)_\tau \le \big(|\pi_t^n - b_t|\big)_\tau = |\pi_\tau^n - b_\tau| \to 0$$

as $n\to\infty$. By the arbitrariness of n and τ, we obtain that P-a.s., $\gamma_\tau = 0$ for any predictable stopping time τ. Therefore γ_t is indistinguishable from zero. □

Corollary 1.7.2. *Let K be an increasing process. Then*

$$\operatorname*{ess\,inf}_{\pi\in\Pi_x} |\pi_t - b_t| = 0 \quad \mu^K\text{-a.e.}$$

D. Now we introduce some notions, which allow us to present an application of Theorem 1.3.1 to the Markov case.

Consider the system of stochastic differential equations (2.4.19), (2.4.20) and assume that conditions (B1) and (B2) are satisfied. Under these conditions, there exists a unique weak solution of (2.4.19), (2.4.20), which is a Markov process, and its transition probability function admits a density $p(s,\ (x_0,y_0),\ t,\ (x,y))$ with respect to the Lebesgue measure. We use the notation $p(t,x,y) = p(0,(x_0,y_0),t,(x,y))$ for the fixed initial condition $S_0 = x_0, R_0 = y_0$.

Introduce the measure μ on the space $([0,T]\times\mathbb{R}_+^d\times\mathbb{R}^{n-d}, \mathcal{B}([0,T]\times\mathbb{R}_+^d\times\mathbb{R}^{n-d}))$:

$$\mu(dt,dx,dy) = p(t,x,y)dt\,dx\,dy.$$

Let $C^{1,2}$ be the class of functions f continuously differentiable at t and twice differentiable at x, y on $[0,T]\times\mathbb{R}_+^d\times\mathbb{R}^{n-d}$. For functions $f\in C^{1,2}$, the L operator is defined as

$$Lf = f_t + \operatorname{tr}\left(\frac{1}{2}\operatorname{diag}(x)\sigma^l\sigma^{l'}\operatorname{diag}(x) f_{xx}\right) + \operatorname{tr}(\delta\sigma^{l'}\operatorname{diag}(x) f_{xy}) + \operatorname{tr}\left(\frac{1}{2}(\delta\delta' + \sigma^\perp\sigma^{\perp'}) f_{yy}\right),$$

where f_t, f_{xx}, f_{xy}, and f_{yy} are partial derivatives of the function f, for which we use the matrix notation.

Definition 1.7.1. We say that a function $f = (f(t,x,y),\ t\ge 0,\ x\in\mathbb{R}_+^d,\ y\in\mathbb{R}^{n-d})$ belongs to the class V_μ^L if there exists a sequence of functions $(f^n, n\ge 1)$ from $C^{1,2}$ and measurable μ-integrable functions $f_{x_i}(i \le d)$, $f_{y_j}(d < j \le n)$ and (Lf) such that

$$E \sup_{s \le T} \left| f^n(s, S_s, R_s) - f(u, S_u, R_u) \right| \to 0 \quad \text{as } n \to \infty,$$

$$\iint\limits_{[0,T]\times\mathbb{R}_+^d\times\mathbb{R}^{n-d}} \left(f_{x_i}^n(s,x,y) - f_{x_i}(s,x,y)\right)^2 x_i^2 \mu(ds, dx, dy) \to 0, \quad i \le d,$$

$$\iint\limits_{[0,T]\times\mathbb{R}_+^d\times\mathbb{R}^{n-d}} \left(f_{y_j}^n(s,x,y) - f_{y_j}(s,x,y)\right)^2 \mu(ds, dx, dy) \to 0, \quad d < j \le n,$$

$$\iint\limits_{[0,T]\times\mathbb{R}_+^d\times\mathbb{R}^{n-d}} \left| Lf^n(s,x,y) - (Lf)(s,x,y) \right| \mu(ds, dx, dy) \to 0$$

as $n \to \infty$.

Now we formulate the statement proved by Chitashvili and Mania [8] in the case convenient for our purposes.

Proposition 1.7.4. *Let conditions* (B1)–(B2) *hold and $f(t, S_t, R_t)$ be a bounded process. Then the process $(f(t, S_t, R_t), t \in [0,T])$ is an Itô process of the form*

$$f(t, S_t, R_t) = f(0, S_0, R_0) + \int_0^t g(s,\omega) dW_s + \int_0^t a(s,\omega) ds \quad a.s.$$

with

$$E \int_0^t g^2(s,\omega) ds < \infty, \quad E \int_0^t |a(s,\omega)| ds < \infty \tag{1.7.14}$$

if and only if f belongs to V_μ^L. Moreover the process $f(t, S_t, R_t)$ admits the decomposition

$$f(t, S_t, R_t) = f(0, S_0, R_0) + \sum_{i=1}^d \int_0^t f_{x_i}(s, S_s, R_s) dS_s^i$$
$$+ \sum_{j=d+1}^n \int_0^t f_{y_j}(s, S_s, R_s) dR_s^j + \int_0^t (Lf)(s, S_s, R_s) ds. \tag{1.7.15}$$

Remark 1.7.2. For continuous functions $f \in V_\mu^L$, the condition

$$\sup_{(t,x,y)\in D} \left| f^n(t,x,y) - f(t,x,y) \right| \to 0 \quad \text{as } n \to \infty \tag{1.7.16}$$

for every compact set $D \in [0,T]\times\mathbb{R}_+^d\times\mathbb{R}^{n-d}$ can be used instead of the first relation in Definition 1.7.1.

Part 2

SEMIMARTINGALE BACKWARD EQUATION RELATED TO DUAL PROBLEMS

2.1. Introduction

The well-known tool in studying the optimization problem (1.1.3) is the use of the duality relationships between the optimal strategies and the optimal martingale measures (see, e.g., [46]).

Let us denote by $\tilde{U} : (0, \infty) \to \mathbb{R}$ the conjugate function of the utility $U(x)$, i.e.,

$$\tilde{U}(y) = \sup_{x>0} [U(x) - xy].$$

It is well known that if U is a utility function, then $\tilde{U}$ is a continuously differentiable, decreasing, and strictly convex function satisfying $\tilde{U}'(0) = -\infty$, $\tilde{U}'(\infty) = 0$, $\tilde{U}(0) = U(\infty)$, $\tilde{U}(\infty) = U(0)$, and

$$U(x) = \inf_{y>0} \left[\tilde{U}(y) + xy\right], \quad x > 0.$$

Moreover, the derivative of U is the inverse function of the derivative of $\tilde{U}$.

Note that for $\log x$, x^p/p, and $-e^{-\gamma x}$, the corresponding convex conjugate functions are $-\log y - 1$, $-\dfrac{p-1}{p} y^{\frac{p}{p-1}}$, and $\dfrac{y}{\gamma}\left(\log \dfrac{y}{\gamma} - 1\right)$, respectively.

The function $\tilde{U}(y)$ is the Legendre transform of $-U(-x)$, which is a useful tool in solving the utility maximization problems (see, e.g., [4] for the application in finance).

The dual problem to (1.1.3) is given by

$$\tilde{V}(0, y) = \inf_{Q \in \mathcal{M}^e} E\tilde{U}(yZ_T^Q), \tag{2.1.1}$$

where Z_T^Q is the Radon–Nikodym density of Q relative to the measure P.

It was shown in [46] that if $\tilde{V}(0, y) < \infty$ for each $y > 0$ and the dual minimizer $Q^*(y) \in \mathcal{M}^e$ (called the minimax martingale measure) exists, then the optimal solution $\pi^*(x)$ to (1.1.3) also exists, and letting $y = V_x(0, x)$, we have the following duality relation between $\pi^*(x)$ and the dual minimizer $Q^*(y)$:

$$x + (\pi^*(x) \cdot S)_T = -\tilde{U}_y(yZ_T^{Q^*}), \quad yZ_T^{Q^*} = U_x(x + (\pi^*(x) \cdot S)_T). \tag{2.1.2}$$

Thus, the solution of the primal problem (1.1.3) of utility maximization reduces to the solution of the dual problem (1.1.3), but the dual problem needs to be solved constructively. If the market considered is complete (or "almost complete"), then the martingale measure is unique (respectively, the minimax martingale measure coincides with the minimal martingale measure), and the dual problem is easier to solve than the corresponding primal problem. The solution of the dual problem for more general incomplete market models is quite complicated.

Let us introduce the value function of the dual problem defined by

$$\tilde{V}(t, y) = \operatorname*{ess\,inf}_{Q \in \mathcal{M}^e} E[\tilde{U}(y\mathcal{E}_{tT}(M^Q))|\mathcal{F}_t].$$

Similarly to (2.1.2), the optimal wealth process

$$X_t^* = x + \int_0^t \pi_s^* dS_u$$

and the optimal martingale measure $Y_t^* = y\mathcal{E}_t(M^{Q*})$ satisfy the following duality relations (see [46]):

$$X_t^* = -\tilde{V}_y(t, Z_t^*), \quad Z_t^* = V_x(t, X_t^*). \tag{2.1.3}$$

Using the same approach as in Part 1, we can also derive the backward stochastic PDE for the value function of the dual problem. This equation is more complicated than Eq. (1.1.6). Therefore, in addition, we assume the continuity of the filtration, since without this assumption, the equation for $\tilde{V}$ is very complicated (whereas the form of Eq. (1.1.6) is the same with and without assumption of the continuity of the filtration). Since the dual optimizer contains an orthogonal (to M) martingale part in general, we need stronger regularity assumptions on $\tilde{V}(t,y)$ for the application of the Itô-Venzell formula. The BSPDE for the function $\tilde{V}(t,y)$ is

$$\tilde{V}(t,y) = \tilde{V}(0,y) - \frac{y^2}{2}\int_0^t \tilde{V}_{yy}(s,y)\lambda_s^2 d\langle M\rangle_s + y\int_0^t \tilde{\varphi}_y(s,y)\lambda_s d\langle M\rangle_s$$

$$+ \int_0^t \frac{1}{\tilde{V}_y(s,y)} d\langle L_y(\cdot,y)\rangle_s + \int_0^t \tilde{\varphi}(s,y)dM_s + \tilde{L}(t,y) \quad (2.1.4)$$

with the boundary condition

$$\tilde{V}(T,y) = \tilde{U}(y),$$

where $\tilde{L}(t,y)$ is a local martingale orthogonal to M for all y. Moreover, the density of the optimal martingale measure Z^* is a unique solution of the forward semimartingale equation

$$Z_t^* = y - \int_0^t \lambda_s Z_s^* dM_s + \int_0^t \frac{1}{\tilde{V}_{yy}(s,Z_s^*)}\tilde{L}_{(y)}(ds, Z_s^*), \quad (2.1.5)$$

where $\int_0^t \tilde{L}_{(y)}(ds,Z_s^*)$ is the stochastic line integral with respect to the family of local martingales $(L_y(t,y),\ y\in\mathbb{R}^+)$ (see [8, 31] for the definition of stochastic line integrals). Thus, we see that for conjugate functions of general utility functions, Eqs. (2.1.4) and (2.1.5) are complicated. Therefore, we do not give here the derivation of Eqs. (2.1.4) and (2.1.5) (and do not specify conditions sufficient to this end) and in this part, we study only dual problems of utility maximization and hedging for power and exponential functions, which are problems of finding the p-optimal and the minimal entropy martingale measures. The main results of the next sections were published in [57–59, 61].

2.2. p-Optimal Martingale Measures

Assume that the dynamics of the discounted prices of some traded assets is described by an $\mathbb{R}^d$-valued continuous semimartingale $X = (X_t, t\in[0,T])$ defined on a filtered probability space $(\Omega, \mathcal{F}, F = (F_t,\ t\in[0,T]),\ P)$ satisfying the usual conditions, where $\mathcal{F} = F_T$ and $T<\infty$ is a fixed time horizon. The process X is adapted to the filtration F and admits the decomposition

$$X_t = X_0 + \Lambda_t + M_t, \quad (2.2.1)$$

where M is a continuous local martingale and Λ is a continuous process of finite variation. For the absence of "arbitrage" in this market, it is necessary to assume that X satisfies the structure condition; this means that there exists a predictable $\mathbb{R}^d$-valued process $\lambda = (\lambda_t, t\in[0,T])$ such that

$$d\Lambda_t = d\langle M\rangle_t\lambda_t \quad \text{a.s. for } t\in[0,T], \qquad K_T = \int_0^T \lambda_s' d\langle M\rangle_s\lambda_s < \infty \quad \text{a.s.},$$

where $'$ denotes the transposition. The process K is called the mean-variance tradeoff process of X (see [84, 85] for the interpretation of the process K).

By $\mathcal{M}^{abs}$ we denote the set of measures Q absolutely continuous with respect to P on F_T such that X is a local martingale under Q. Let $\mathcal{M}^e$ be the set of equivalent martingale measures, i.e., a subset of $\mathcal{M}^{abs}$ containing probability measures that are equivalent to P. Let $Z_t(Q)$ be the density process of Q relative to the basic measure P. For any $Q \in \mathcal{M}^e$, there is a P-local martingale M^Q such that $Z^Q = \mathcal{E}(M^Q) = (\mathcal{E}_t(M^Q), t \in [0,T])$. If the local martingale $\hat{Z}_t = \mathcal{E}_t(-\lambda \cdot M), t \in [0,T])$ is a strictly positive martingale, then $d\hat{P}/dP = \hat{Z}_T$ defines an equivalent probability measure called the minimal martingale measure for X.

Let

$$\mathcal{M}_p^e = \left\{ Q \in \mathcal{M}^e : E\eta \left(\frac{dQ}{dP}\right)^p < \infty \right\},$$

where η is a nonnegative F_T-measurable random variable.

Throughout this section, we make the following assumptions:

(A) there is an equivalent martingale measure $\tilde{Q}$ such that

$$E\eta\mathcal{E}_T^p(M^{\tilde{Q}}) < \infty;$$

(B) all P-local martingales are continuous;

(C) there is a constant k_1 such that $\eta \geq k_1 > 0$.

Remark 2.2.1. Condition (A) is natural and is related to some kind of nonarbitrage condition if $\eta = 1$ (see [18] for the definition of "arbitrage" and related results). We note that since X is continuous, the existence of an equivalent martingale measure implies that the structure condition holds. In particular, Assumption (B) means the continuity of filtration F, and it is restrictive, but it is satisfied if the filtration F is generated by a Brownian motion, or, more generally, if F admits the integral representation property relative to some vector-valued continuous martingale. Also, we note that the main results are true if we replace condition (C) by $E\eta^{\frac{1}{1-p}} < \infty$, $p > 1$.

Sometimes, we replace condition (A) by the following stronger condition:

(A*) the random variable η is bounded, i.e.,

$$\eta \leq k_2 \tag{2.2.2}$$

for some constant $k_2 > k_1$, and there exists the minimal martingale measure satisfying the reverse Hölder inequality $R_p(P)$, i.e., there is a constant C such that

$$E(\mathcal{E}_{\tau,T}^p(-\lambda \cdot M)|F_\tau) \leq C$$

for any stopping time τ.

Here and in what follows, we use the notation

$$\mathcal{E}_{\tau,T}(N) = \frac{\mathcal{E}_T(N)}{\mathcal{E}_\tau(N)} = \mathcal{E}_T(N - N_{\cdot\wedge\tau})$$

for a continuous local martingale N.

We consider the following optimization problems:

$$\min_{Q \in \mathcal{M}_p^e} E\eta\mathcal{E}_T^p(M^Q), \quad p \geq 1, \tag{2.2.3}$$

$$\max_{Q \in \mathcal{M}^e} E\eta\mathcal{E}_T^p(M^Q), \quad 0 < p \leq 1. \tag{2.2.4}$$

Let

$$V_t(p) = \operatorname*{ess\,inf}_{Q \in \mathcal{M}_p^e} E(\eta\mathcal{E}_{tT}^p(M^Q)|F_t), \quad p \geq 1, \tag{2.2.5}$$

$$\overline{V}_t(p) = \operatorname*{ess\,sup}_{Q \in \mathcal{M}_p^e} E(\eta\mathcal{E}_{tT}^p(M^Q)|F_t), \quad 0 < p \leq 1, \tag{2.2.6}$$

be the value processes of problems (2.2.3) and (2.2.4), respectively.

For $p = 1$, the processes $V_t(p)$ and $\overline{V}_t(p)$ represent the lower and upper prices of a contingent claim η at the moment t.

For $\eta = 1$, (2.2.3) is the problem of finding the p-optimal martingale measure, in particular, for $p = 2$ the solution of problem (2.2.3) gives the variance optimal martingale measure, which plays an essential role in the mean variance hedging problem (see, e.g., [16, 33, 73, 78, 86]).

It is well known that the p-optimal martingale measure Q^* exists in the class $\mathcal{M}^{abs}$, and it was shown in [18] (in [34] for the case $p > 1$) that Q^* is equivalent to P if condition (A) is satisfied and X is continuous. It was proved in [18] (this fact was already observed in [20, 83, 86] to various extents of generality) that if X is a locally bounded semimartingale, and if the measure Q^* is variance optimal, then the corresponding density Z^* is represented as

$$Z_T^* = c + \int_0^T h_s' dX_s$$

for a constant c and an X-integrable process h, where the process

$$\int_0^t h_s' dX_s, \quad t \in [0, T],$$

is a Q-martingale for any $Q \in \mathcal{M}_2^e$.

We derive the corresponding fact for $p > 1$ (under assumptions (A) and (B)) using the semimartingale backward equation for the value process. Moreover, we obtain an explicit expression of the integrand h in terms of the value process $V_t(p)$ and show that $V_t(p)$ uniquely solves a suitable semimartingale backward equation.

Now we formulate the main statement of this part, which is a combination of Theorem 2.3.1 and Corollary 2.3.2 of Proposition 2.3.3.

Let Y be a semimartingale with the decomposition

$$Y_t = Y_0 + B_t + L_t, \quad B \in \mathcal{A}_{\text{loc}}, \quad L \in \mathcal{M}_{\text{loc}}^2, \tag{2.2.7}$$

and let

$$L_t = \int_0^t \psi_s' dM_s + \tilde{L}_t, \quad \langle \tilde{L}, M \rangle = 0, \tag{2.2.8}$$

be the Galtchouk–Kunita–Watanabe decomposition of L with respect to the martingale M.

If conditions (A*), (B), and (C) are satisfied, then the value process $V(p)$ is a unique solution of the semimartingale backward equation

$$\begin{aligned} Y_t = Y_0 - \frac{p(p-1)}{2} \int_0^t Y_s \lambda_s' d\langle M \rangle_s \lambda_s + p \int_0^t \lambda_s' d\langle M \rangle_s \psi_s \\ + \frac{p}{2(p-1)} \int_0^t \frac{1}{Y_s} d\langle \tilde{L} \rangle_s + \int_0^t \psi_s dM_s + \tilde{L}_t, \quad t < T, \end{aligned} \tag{2.2.9}$$

with the boundary condition

$$Y_T = \eta \tag{2.2.10}$$

in the class of processes Y satisfying the two-sided inequality

$$c \le Y_t \le C \quad \text{for all } t \in [0, T] \text{ a.s.}, \tag{2.2.11}$$

for some positive constants $c < C$.

Moreover, the martingale measure Q^* is p-optimal if and only if its density $Z^* = \mathcal{E}_T(M^{Q^*})$ is expressed as

$$\mathcal{E}_T^{p-1}(M^{Q^*}) = Y_0 + \int_0^T \mathcal{E}_s\left(\frac{\psi}{Y} + (1-p)\lambda \cdot X\right)\left(\frac{\psi_s}{Y_s} + (1-p)\lambda_s\right)' dX_s. \tag{2.2.12}$$

We also show that the value process satisfies (2.2.9), (2.2.10) if we replace (A*) by condition (A) (Theorem 2.3.1a), but in this case, the class of processes in which this solution is unique is not explicitly described.

The same problem was studied by Laurent and Pham [49], in the case $p = 2$ and $\eta = e^{-\int_0^T r_s ds}$, where the process r is the instantaneous interest rate. Using the dynamic programming approach, they obtain a characterization of the variance-optimal martingale measure in terms of the value function of a stochastic control problem (equivalent to (2.2.3)) in the case of Brownian filtration.

Note that one can use the processes $V_t(p)$ and $\overline{V}_t(p)$ to calculate upper and lower prices of contingent claims, since as proved in [57]

$$\lim_{p\downarrow 1} V_t(p) = \operatorname*{ess\,inf}_{Q\in\mathcal{M}^e} E(\eta\mathcal{E}_{tT}(M^Q)|F_t), \quad \lim_{p\uparrow 1} \overline{V}_t(p) = \operatorname*{ess\,sup}_{Q\in\mathcal{M}^e} E(\eta\mathcal{E}_{tT}(M^Q)|F_t).$$

2.3. Backward Semimartingale Equation for the Value Process Related to the p-Optimal Martingale Measure

We say that the process B strongly dominates the process A and write $A \prec B$ if the difference $B - A \in \mathcal{A}^+_{\text{loc}}$, i.e., is a locally integrable increasing process.

Let $(A^Q, Q \in \mathcal{Q})$ be a family of processes of finite variations, zero at time zero. Denote by $\operatorname{ess\,inf}_{Q\in\mathcal{Q}}(A^Q)$ the largest process of finite variation, zero at time zero, which is strongly dominated by the process $(A^Q_t, t \in [0,T])$ for every $Q \in \mathcal{Q}$, i.e., this is "ess inf of the family $(A^Q, Q \in \mathcal{Q})$ relative to the partial order $\prec$.

We will use the following assertion proved by Delbaen and Schachermayer [17] in the case $p = 2$.

Proposition 2.3.1. *If $U = (U_t, t \in [0,T])$ is a nonnegative p-integrable martingale ($p > 1$) with $U_0 > 0$ and if the stopping time $\tau = \inf\{t : U_t = 0\}$ is predictable and announced by a sequence of stopping times $(\tau_n, n \geq 1)$, then*

$$E\left(\frac{U_T^p}{U_{\tau_n}^p}|F_{\tau_n}\right) \to \infty, \quad n \to \infty,$$

on the F_{τ_-}-measurable set $\{U_\tau = 0\}$.

If $p < 1$ and U is a uniformly integrable martingale, then

$$E\left(\frac{U_T^p}{U_{\tau_n}^p}|F_{\tau_n}\right) \to 0, \quad n \to \infty,$$

on the set $\{U_\tau = 0\}$.

Proof. For $p > 1$, the proof is the same as in [17]. In the case $0 < p < 1$, one can prove this assertion using arguments similar to [17]. Using the Hölder inequality, we have

$$E\left(\frac{U_T^p}{U_{\tau_n}^p}|F_{\tau_n}\right) = E\left(\frac{U_T^p}{U_{\tau_n}^p} I_{(U_\tau\neq 0)}|F_{\tau_n}\right) \leq E^{1-p}(I_{(U_\tau\neq 0)}|F_{\tau_n})$$

and the Lévy theorem implies that $E^{1-p}(I_{(U_\tau=0)}|F_{\tau_n})$ tends to zero on the set $(U_\tau = 0)$. □

Since X is continuous, any element Q of $\mathcal{M}^e$ is given by the density $Z_t(Q)$, which is expressed as an exponential martingale of the form

$$\mathcal{E}_t(-\lambda \cdot M + N),$$

where N is a local martingale strongly orthogonal to M.

By $\mathcal{N}(X)$ we denote the class of local martingales N strongly orthogonal to M such that the process $(\mathcal{E}_t(-\lambda\cdot M+N), t\in[0,T])$ is a martingale under P.

Let $\mathcal{N}_p(X)$ be the subclass of $\mathcal{N}(X)$ of local martingales N such that the process $(\mathcal{E}_t(-\lambda\cdot M+N),$ $t\in[0,T])$ is a strictly positive P-martingale with $E\eta\mathcal{E}_T^p(-\lambda\cdot M+N)<\infty$. Then

$$\mathcal{M}_p^e=\left\{Q\sim P:\frac{dQ}{dP}\mid F_T=\mathcal{E}_T(-\lambda\cdot M+N), N\in\mathcal{N}_p(X)\right\}. \tag{2.3.1}$$

The following assertion can be proved in the standard manner (see, e.g., [26, 49]).

Proposition 2.3.2 (optimality principle). (a) *There exists an RCLL semimartingale, still denoted by $V_t(p)$, such that for each $t\in[0,T]$,*

$$V_t(p)=\operatorname*{ess\,inf}_{Q\in\mathcal{M}_p^e}E(\eta\mathcal{E}_{tT}^p(M^Q)|F_t)\quad a.s.$$

$V_t(p)$ is the largest RCLL process equal to η at time T such that $V_t(p)\mathcal{E}_t^p(M^Q)$ is a submartingale for every $Q\in\mathcal{M}_p^e$.

(b) *The following properties are equivalent*:

(i) *Q^* is p-optimal, i.e.,*

$$V_0(p)=\inf_{Q\in\mathcal{M}_p^e}E\eta\mathcal{E}_T^p(M^Q)=E\eta\mathcal{E}_T^p(M^{Q^*});$$

(ii) *Q^* is p-optimal for all conditional criteria, i.e., for all $t\in[0,T]$,*

$$V_t(p)=E(\eta\mathcal{E}_{tT}^p(M^{Q^*})|F_t)\quad a.s.;$$

(iii) *$V_t(p)\mathcal{E}_t^p(M^{Q^*})$ is a P-martingale.*

We recall that the process X belongs to the class D if the family of random variables $X_\tau I_{(\tau\le T)}$ for all stopping times τ is uniformly integrable.

Let $\mathcal{S}$ (respectively, $\mathcal{S}_+$) be the class of semimartingales (respectively, strictly positive semimartingales).

Definition 2.3.1. We say that Y belongs to the class D_p if Y is an RCLL process such that for every $Q\in\mathcal{M}_p^e$ the process $\mathcal{E}_t^p(M^Q)Y_t$ is in D.

Remark 2.3.1. Since for every $Q\in\mathcal{M}_p^e$, the process $\mathcal{E}_t^p(M^Q)$ belongs to the class D as a positive submartingale (see [19]), then any bounded positive process Y belongs to the class D_p.

Definition 2.3.2. By $S(X)$ we denote the class of strictly positive semimartingales Y such that $Y\in D_p$ and $-\frac{1}{(p-1)Y}\cdot\tilde{L}\in\mathcal{N}(X)$, i.e., such that

$$\left(\mathcal{E}_t\left(-\lambda\cdot M-\frac{1}{(p-1)Y}\cdot\tilde{L}\right),\ t\in[0,T]\right)$$

is a martingale, where $\tilde{L}$ is the local martingale introduced in (2.2.8).

Let us consider the optimization problem (2.2.3). One can rewrite the value process $V(p)$ of this problem in the form

$$V_t(p)=\operatorname*{ess\,inf}_{N\in\mathcal{N}_p(X)}E(\eta\mathcal{E}_{tT}^p(-\lambda\cdot M+N)|F_t), \tag{2.3.2}$$

Since $\mathcal{M}^e\ne\emptyset$, the process $V(p)$ is a semimartingale with respect to the measure P; let

$$V_t(p)=m_t+A_t,\quad m\in M_{\rm loc}^2,\quad A\in\mathcal{A}_{\rm loc}, \tag{2.3.3}$$

be the canonical decomposition of $V(p)$.

Let

$$m_t = \int_0^t \varphi_s dM_s + \tilde{m}_t, \quad \langle \tilde{m}, M \rangle = 0, \tag{2.3.4}$$

be the Galtchouk–Kunita–Watanabe decomposition of m with respect to M.

Theorem 2.3.1. *Let conditions* (A), (B), *and* (C) *be satisfied. Then the following assertions hold.*

(a) *The value process $V(p)$ is a solution of the semimartingale backward equation*

$$Y_t = Y_0 - \operatorname*{ess\,inf}_{N \in \mathcal{N}_p(X)} \left[\frac{1}{2} p(p-1) \int_0^t Y_s d\langle -\lambda \cdot M + N \rangle_s + p \langle \lambda \cdot M + N, L \rangle_t \right] + L_t, \quad t < T, \tag{2.3.5}$$

with the boundary condition

$$Y_T = \eta. \tag{2.3.6}$$

This solution is unique in the class $S(X)$ of semimartingales. Moreover, the martingale measure Q^* is p-optimal if and only if it is given by the density $dQ^* = \mathcal{E}_T(M^{Q^*})dP$, where

$$M_t^{Q^*} = -\int_0^t \lambda_s' dM_s - \frac{1}{p-1} \int_0^t \frac{1}{V_s(p)} d\tilde{m}_s. \tag{2.3.7}$$

(b) *If, in addition, condition* (A*) *is satisfied, then the value process V is a unique solution of the semimartingale backward equation* (2.3.5), (2.3.6) *in the class of semimartingales Y satisfying the two-sided inequality*

$$c \le Y_t \le C \quad \textit{for all } t \in [0,T] \ \textit{a.s.} \tag{2.3.8}$$

for some positive constants $c < C$.

Proof. (a) *Existence.* According to (2.3.2), $\mathcal{E}_t^p(M^Q)V_t(p)$ is a P-submartingale for every $Q \in \mathcal{M}_p^e$. Therefore, by assumption (A) (since there exists $Q \in \mathcal{M}_p^e$ with $\mathcal{E}(M^Q)$ strictly positive), $V(p)$ is a P-semimartingale with decomposition (2.3.3).

Using the relation

$$\mathcal{E}^p(M^Q) = \mathcal{E}\left(pM^Q + \frac{p(p-1)}{2} \langle M^Q \rangle \right)$$

and the Itô formula for $\mathcal{E}_t^p(M^Q)V_t(p)$, we have

$$\begin{aligned}
\mathcal{E}_t^p(M^Q)V_t(p) &= V_0(p) + \int_0^t \mathcal{E}_s^p(M^Q) dV_s(p) \\
&\quad + \int_0^t V_{s-}(p)\mathcal{E}_s^p(M^Q) d(pM_s^Q + \frac{p(p-1)}{2}[M^Q]_s) + p\int_0^t \mathcal{E}_s^p(M^Q) d[V(p), M^Q]_s \\
&= V_0(p) + \int_0^t \mathcal{E}_s^p(M^Q) d(A_s + \frac{p(p-1)}{2}(V_-(p)\cdot\langle M^Q\rangle)_s + p\langle M^Q, m\rangle_s) \\
&\quad + \int_0^t \mathcal{E}_s^p(M^Q) dm_s + p\int_0^t V_{s-}(p)\mathcal{E}_s^p(M^Q) dM_s^Q.
\end{aligned} \tag{2.3.9}$$

Since $\mathcal{E}_t^p(M^Q)V_t(p)$ is a P-submartingale for all $Q \in \mathcal{M}_p^e$ and $\mathcal{E}_t(M^Q)$ is strictly positive, we obtain from (2.3.9) that

$$A_t + \frac{p(p-1)}{2}\int_0^t V_s(p)d\langle M^Q\rangle_s + p\langle M^Q, m\rangle_t \in \mathcal{A}^+_{\text{loc}} \tag{2.3.10}$$

for every $Q \in \mathcal{M}_p^e$.

It is well known that for convex coercive continuous functions defined on a closed convex subset of a reflexive Banach space, the infimum is attained (see [25]). Since the set of densities $Z_T(Q)$ of absolutely continuous local martingale measures Q with $E\eta Z_T^p(Q) < \infty$ is a closed convex subset of $L^p(\eta \cdot P)$ and $\|\cdot\|^p_{L^p(\eta\cdot P)}$ is a convex coercive function, the optimal martingale measure Q^* exists. Note that the class of densities $(Z(Q), Q \in \mathcal{M}_p^e)$ is not closed in general. Therefore, we only have that $Q^* \in \mathcal{M}_p^{abs}$, where

$$\mathcal{M}_p^{abs} = \left\{Q \in \mathcal{M}^{abs} : E\eta\left(\frac{dQ}{dP}\right)^p < \infty\right\}.$$

Let us show that the existence of an equivalent martingale measure $\tilde{Q}$ with $E\eta\mathcal{E}_T^p(M^{\tilde{Q}}) < \infty$ implies that Q^* is equivalent to P. We prove this fact using the idea of Delbaen and Schachermayer [17].

Since Q^* is optimal, we have

$$E\eta Z_T^p(Q^*) \le E\eta\mathcal{E}_T^p(M^{\tilde{Q}}), \tag{2.3.11}$$

where we denote by $Z_t(Q^*)$ the density process of Q^* relative to the measure P.

Following [17], we define the stopping times

$$\tau_n = \inf\{t : Z_t(Q^*) \le 1/n\}, \quad \tau = \inf\{t : Z_t(Q^*) = 0\}. \tag{2.3.12}$$

Inequality (2.3.11) implies that for every $n \ge 1$,

$$E\left[\eta\frac{Z_T^p(Q^*)}{Z_{\tau_n}^p(Q^*)}|F_{\tau_n}\right] \le E\left[\eta\frac{\mathcal{E}_T^p(M^{\tilde{Q}})}{\mathcal{E}_{\tau_n}^p(M^{\tilde{Q}})}|F_{\tau_n}\right] \quad \text{a.s.} \tag{2.3.13}$$

Indeed, if the measure of the set B defined by

$$B = \left\{\omega : E\left[\eta\frac{Z_T^p(Q^*)}{Z_{\tau_n}^p(Q^*)}|F_{\tau_n}\right] > E\left[\eta\frac{\mathcal{E}_T^p(M^{\tilde{Q}})}{\mathcal{E}_{\tau_n}^p(M^{\tilde{Q}})}|F_{\tau_n}\right]\right\} \tag{2.3.14}$$

is strictly positive, then constructing a new (absolutely continuous) martingale measure $\hat{Q}$ by $d\hat{Q} = \hat{Z}_T dP$,

$$\hat{Z}_T = I_B Z_{\tau_n}(Q^*)\frac{\mathcal{E}_T(M^{\tilde{Q}})}{\mathcal{E}_{\tau_n}(M^{\tilde{Q}})} + I_{B^c}Z_T(Q^*)$$

we have

$$\begin{aligned} E\eta(\hat{Z}_T)^p &= E\eta Z_{\tau_n}^p(Q^*)\left[I_B\frac{\mathcal{E}_T(M^{\tilde{Q}})}{\mathcal{E}_{\tau_n}(M^{\tilde{Q}})} + I_{B^c}\frac{Z_T(Q^*)}{Z_{\tau_n}(Q^*)}\right]^p \\ &= EZ_{\tau_n}^p(Q^*)\left[I_B E\left(\eta\frac{\mathcal{E}_T^p(M^{\tilde{Q}})}{\mathcal{E}_{\tau_n}^p(M^{\tilde{Q}})}|F_{\tau_n}\right) + I_{B^c}E\left(\eta\frac{Z_T^p(Q^*)}{Z_{\tau_n}^p(Q^*)}\right)|F_{\tau_n})\right] < E\eta Z_T^p(Q^*), \end{aligned}$$

which contradicts the optimality of Q^*. Now by Proposition (2.3.1) and condition (C), the left-hand side of (2.3.13) tends to infinity on the set $Z_\tau(Q^*) = 0$ as $n \to \infty$. On the other hand, since the measure $\tilde{Q}$ is equivalent to P, the limit of the right-hand side of (2.3.13) is finite. Thus, $P(Z_\tau(Q^*) = 0) = 0$, and hence Q^* is an equivalent local martingale measure.

Therefore, by the optimality principle (see Proposition 2.3.2) the process $V_t(p)\mathcal{E}_t^p(M^{Q^*})$ is a martingale and using the Itô formula (2.3.9) for $V_t(p)\mathcal{E}_t^p(M^{Q^*})$, we obtain

$$A_t + \frac{p(p-1)}{2}\int_0^t V_s(p)d\langle M^{Q^*}\rangle_s + p\langle M^{Q^*}, m\rangle_t = 0. \tag{2.3.15}$$

The last equation, together with relation (2.3.10), implies

$$A_t = -\operatorname*{ess\,inf}_{Q\in\mathcal{M}_p^e}\left[\frac{p(p-1)}{2}\int_0^t V_s(p)d\langle M^Q\rangle_s + p\langle M^Q, m\rangle_t\right], \tag{2.3.16}$$

and hence the value process $V(p)$ satisfies Eq. (2.3.5) (obviously, $V(p)$ also satisfies the boundary condition $V_t(p) = \eta$). We note that (2.3.15) implies that the process A_t and hence $V_t(p)$ is continuous.

Now let us show that the optimal martingale measure Q^* is given by (2.3.7) and that the value process $V(p)$ belongs to the class $S(X)$ of semimartingales. From (2.3.16) and (2.3.1), we have

$$A_t = -\frac{p(p-1)}{2}\int_0^t V_s(p)d\langle\lambda\cdot M\rangle_s + p\int_0^t d\langle\lambda\cdot M, m\rangle_s - \operatorname*{ess\,inf}_{N\in N_p(X)}\left(\frac{p(p-1)}{2}\int_0^t V_s(p)d\langle N\rangle_s + p\langle N, m\rangle_t\right)$$

$$= -\frac{p(p-1)}{2}\int_0^t V_s(p)d\langle\lambda\cdot M\rangle_s + p\int_0^t d\langle\lambda\cdot M, m\rangle_s$$

$$-\operatorname*{ess\,inf}_{N\in N_p(X)}\left[\left\langle\sqrt{\frac{p(p-1)}{2}}\int_0^t\sqrt{V_s(p)}dN_s + \sqrt{\frac{p}{2(p-1)}}\int_0^t\frac{1}{\sqrt{V_s(p)}}d\tilde{m}_s\right\rangle_t - \frac{p}{2(p-1)}\int_0^t\frac{1}{V_s(p)}d\langle\tilde{m}\rangle_s\right]$$

$$= -\frac{p(p-1)}{2}\int_0^t V_s(p)d\langle\lambda\cdot M\rangle_s + p\int_0^t d\langle\lambda\cdot M, \varphi\cdot M\rangle_s + \frac{p}{2(p-1)}\int_0^t\frac{1}{V_s(p)}d\langle\tilde{m}\rangle_s, \tag{2.3.17}$$

since

$$\operatorname*{ess\,inf}_{N\in N_p(X)}\left\langle\sqrt{\frac{p(p-1)}{2}}\int_0^t\sqrt{V_s(p)}dN_s + \sqrt{\frac{p}{2(p-1)}}\int_0^t\frac{1}{\sqrt{V_s(p)}}d\tilde{m}_s\right\rangle_t = 0. \tag{2.3.18}$$

Indeed, it is obvious that for any stopping time τ_n defined by

$$\tau_n = \inf\{t : \mathcal{E}_t(\tilde{N}) \geq n\},$$

where

$$\tilde{N}_t = -\frac{1}{p-1}\int_0^t\frac{1}{V_s(p)}d\tilde{m}_s, \tag{2.3.19}$$

the stopped martingale $\tilde{N}_t^{\tau_n}$ belongs to the class $N_p(X)$ and $\tau_n \uparrow T$. Therefore,

$$\operatorname*{ess\,inf}_{N\in N_p(X)}\left[\left\langle\sqrt{\frac{p(p-1)}{2}}\int_0^t\sqrt{V_s(p)}dN_s + \sqrt{\frac{p}{2(p-1)}}\int_0^t\frac{1}{\sqrt{V_s(p)}}d\tilde{m}_s\right\rangle_t\right] \leq \frac{p}{2(p-1)}\int_{t\wedge\tau_n}^t\frac{1}{V_s(p)}d\langle\tilde{m}\rangle_s \tag{2.3.20}$$

for each $n \geq 1$, and (2.3.17) holds since the right-hand side of the latter relation tends to zero as $n \to \infty$.

We observe that by the Jensen inequality, from condition (C) we have the inequality $V_t(p) \geq k_1$, so that all integrals in (2.3.17) are well defined.

By the optimality principle, $V_t(p)\mathcal{E}_t^p(M^{Q^*})$ is a martingale. Since $V(p)$ solves Eq. (2.3.5), this implies that

$$\operatorname*{ess\,inf}_Q\left[\frac{p(p-1)}{2}\int_0^t V_s(p)d\langle M^Q\rangle_s + p\langle M^Q, m\rangle_t\right] = \frac{p(p-1)}{2}\int_0^t V_s(p)d\langle M^{Q^*}\rangle_s + p\langle M^{Q^*}, m\rangle_t. \quad (2.3.21)$$

Since M^{Q^*} is represented in the form $-\lambda\cdot M + N^*$ for some $N^*\in N^p(X)$, it follows from (2.3.17) and (2.3.19) that the processes N^* and $\tilde{N}$ and hence the processes

$$M^{Q^*}, \quad -\int_0^t \lambda_s dM_s - \frac{1}{p-1}\int_0^t \frac{1}{V_s(p)}d\tilde{m}_s$$

are indistinguishable.

Therefore,, the p-optimal martingale measure is unique and admits representation (2.3.7).

By definition of $V(p)$, we have that for any $Q\in\mathcal{M}_p^e$,

$$V_\tau(p)\mathcal{E}_\tau^p(M^Q) \le E(\eta\mathcal{E}_T^p(M^Q)|F_\tau). \quad (2.3.22)$$

Therefore, for any $Q\in\mathcal{M}_p^e$, the process $V_t(p)\mathcal{E}_t^p(M^Q)$ is a submartingale of class D as a positive process majorized by a uniformly integrable martingale (see [19]) and $V(p)\in D_p$ by Definition 2.3.1.

Finally, since $Q^*\in\mathcal{M}_p^e$ and the processes M^{Q^*} and $-\lambda\cdot M - \frac{1}{p-1}\frac{1}{V(p)}\cdot\tilde{m}$ are indistinguishable, we have that $\mathcal{E}_t(-\lambda\cdot M - \frac{1}{(p-1)V}\cdot\tilde{m})$ is a martingale and hence $V(p)\in S(X)$.

Uniqueness. Let Y be a solution of (2.3.5), (2.3.6) of class $S(X)$. This means that Y is a semimartingale with decomposition (2.2.7), (2.2.8) such that $Y_T=\eta$,

$$B_t = -\operatorname*{ess\,inf}_{Q\in\mathcal{M}_p^e}\left(\frac{p(p-1)}{2}\int_0^t Y_s d\langle M^Q\rangle_s + p\langle M^Q, L\rangle_t\right), \quad (2.3.23)$$

and $(\mathcal{E}_t(-\lambda\cdot M - \frac{1}{(p-1)Y}\cdot\tilde{L}), t\in[0,T])$ is a martingale.

Since (2.3.21) implies that

$$B_t + \frac{p(p-1)}{2}\int_0^t Y_s d\langle M^Q\rangle_s + p\langle M^Q, L\rangle_t \in \mathcal{A}^+_{\mathrm{loc}},$$

using decomposition (2.3.7) and the Itô formula for $\mathcal{E}_t^p(M^Q)Y_t$ we obtain that the process $\mathcal{E}_t^p(M^Q)Y_t$ is a local submartingale for all $Q\in\mathcal{M}_p^e$. Since $Y\in D_p$, we have that $\mathcal{E}_t^p(M^Q)Y_t$ is a submartingale of class D. Therefore, it follows from the boundary condition (2.3.6) that for every $Q\in\mathcal{M}_p^e$,

$$\mathcal{E}_t^p(M^Q)Y_t \le E[\mathcal{E}_T^p(M^Q)Y_T|F_t] = E[\eta\mathcal{E}_T^p(M^Q)|F_t].$$

Hence

$$Y_t \le E[\eta\mathcal{E}_{tT}^p(M^Q)|F_t]$$

for all $Q\in\mathcal{M}_p^e$ and

$$Y_t \le \operatorname*{ess\,inf}_Q E[\eta\mathcal{E}_{tT}^p(M^Q)|F_t] = V_t(p). \quad (2.3.24)$$

Let us show the converse inequality. Similarly to (2.3.17), we can show that

$$B_t = -\frac{p(p-1)}{2}\int_0^t Y_s\lambda_s' d\langle M\rangle_s\lambda_s + p\int_0^t \lambda_s' d\langle M\rangle_s\psi_s + \frac{p}{2(p-1)}\int_0^t \frac{1}{Y_s}d\langle\tilde{L}\rangle_s \quad (2.3.25)$$

and the infimum is attained for the martingale

$$N_t^0 = -\frac{1}{p-1}\int_0^t \frac{1}{Y_s} d\tilde{L}_s, \tag{2.3.26}$$

where $\tilde{L}$ is the orthogonal martingale part of L in the Kunita–Watanabe decomposition (2.2.8).

Therefore, using the Itô formula once again, one can show that $\mathcal{E}_t^p(-\lambda \cdot M + N^0)Y_t$ is a local martingale, since (2.3.23) and (2.2.8) imply

$$\mathcal{E}_t^p(-\lambda \cdot M + N^0)Y_t = Y_0 + \int_0^t \mathcal{E}_s^p(-\lambda \cdot M + N^0)(\psi_s - pY_s\lambda_s)'dM_s - \frac{1}{p-1}\int_0^t \mathcal{E}_s^p(-\lambda \cdot M + N^0)d\tilde{L}_s.$$

By the definition of the class $S(X)$, we have that the process $\mathcal{E}_t(M^{Q^0})$ is a martingale, where $M^{Q^0} = -\lambda \cdot M + N^0$. Hence $dQ^0 = \mathcal{E}_T(M^{Q^0})dP$ is an absolutely continuous local martingale measure. Let us show that $Q^0 \in \mathcal{M}_p^e$.

To show that $\mathcal{E}_T(M^{Q^0})$ is strictly positive, we use the Delbaen–Schachermayer lemma (see Proposition 2.3.1). Let τ_n and τ be stopping times defined by (2.3.12) for the process $\mathcal{E}_t(M^{Q^0})$.

From inequality (2.3.22), we have that for any stopping time σ,

$$Y_\sigma \le V_\sigma(p) = \operatorname*{ess\,inf}_{Q\in\mathcal{M}_p^e} E(\eta\mathcal{E}_{\tau,T}^p(M^Q)|F_\sigma) \le E(\eta\mathcal{E}_{\sigma,T}^p(M^{\tilde{Q}})|F_\sigma) \tag{2.3.27}$$

for any $\tilde{Q} \in \mathcal{M}_p^e$.

Since any positive local martingale is a supermartingale, we have

$$\mathcal{E}_\sigma^p(M^{Q^0})Y_\sigma \ge E(Y_T\mathcal{E}_T^p(M^{Q^0})|F_\sigma), \tag{2.3.28}$$

and from the boundary condition (2.3.6), replacing σ by τ_n, we obtain

$$Y_{\tau_n} \ge E\left[\eta\frac{\mathcal{E}_T^p(M^{Q^0})}{\mathcal{E}_{\tau_n}^p(M^{Q^0})}|F_{\tau_n}\right]. \tag{2.3.29}$$

Therefore, (2.3.25) and (2.3.27) imply the inequality

$$E\left[\eta\frac{\mathcal{E}_T^p(M^{Q^0})}{\mathcal{E}_{\tau_n}^p(M^{Q^0})}|F_{\tau_n}\right] \le E\left[\eta\frac{\mathcal{E}_T^p(M^{\tilde{Q}})}{\mathcal{E}_{\tau_n}^p(M^{\tilde{Q}})}|F_{\tau_n}\right]. \tag{2.3.30}$$

Now (2.3.28), (2.3.1), and condition (C) imply that Q^0 is an equivalent local martingale measure.

On the other hand, using inequalities (2.3.25) and (2.3.26) we have that for $\sigma = 0$

$$E\eta\mathcal{E}_T^p(M^{Q^0}) \le Y_0 \le V_0(p) \le E\eta\mathcal{E}_T^p(M^{\tilde{Q}}) < \infty$$

and $\eta\mathcal{E}_T^p(M^{Q^0})$ is integrable.

Thus, $Q^0 \in \mathcal{M}_p^e$, and since $Y \in D_p$, the process $Y_t\mathcal{E}_t^p(M^{Q^0})$ is from the class D and hence it is a uniformly integrable martingale. Now, the martingale property and the boundary condition imply

$$Y_t = E(\eta\mathcal{E}_{t,T}^p(-\lambda \cdot M + N^0)|F_t). \tag{2.3.31}$$

Therefore, (2.3.22) and (2.3.29) imply $Y_t = V_t(p)$ a.s. for all $t \in [0,T]$, and hence the solution of equation (2.3.5), (2.3.6) is unique in the class $S(X)$.

(b) It is easy to see that the value process satisfies the two-sided inequality

$$k_1 \le V_t(p) \le Ck_2 \quad \text{a.s.} \tag{2.3.32}$$

for all $t \in [0,T]$.

By the Jensen inequality,

$$V_t(p) = \operatorname*{ess\,inf}_{Q \in \mathcal{M}_p^e} E(\eta \mathcal{E}_{t,T}^p(M^Q)|F_t) \geq k_1 \operatorname*{ess\,inf}_{Q \in \mathcal{M}_p^e} E^p(\mathcal{E}_{t,T}(M^Q)|F_t) = k_1.$$

On the other hand, if there exists a martingale measure $\tilde{Q}$ satisfying the reverse Hölder inequality, we have that V is bounded from above, since

$$V_t(p) = \operatorname*{ess\,inf}_{Q \in \mathcal{M}_p^e} E(\eta \mathcal{E}_{t,T}^p(M^Q)|F_t) \leq E(\eta \mathcal{E}_{t,T}^p(M^{\tilde{Q}})|F_t) \leq Ck_2.$$

To prove this part of the theorem, we need to show that any solution Y satisfying the two-sided inequality (2.2.11) belongs to the class $S(X)$. Since any bounded positive process belongs to the class D_p (see Remark 2.3.1), we need to show that the process

$$\left(\mathcal{E}_t\left(-\lambda \cdot M - \frac{1}{(p-1)Y} \cdot \tilde{L}\right),\ t \in [0,T]\right)$$

is a martingale. According to [41, Theorem 2.3], it suffices to prove that the process $-\lambda \cdot M - \frac{1}{(p-1)Y} \cdot \tilde{L}$ belongs to the class BMO. Since the minimal martingale measure satisfies the reverse Hölder condition, [21, Proposition 6] implies that $-\lambda \cdot M \in \mathrm{BMO}$. On the other hand, since $Y \geq k_1$ and $\langle \tilde{L} \rangle \prec \langle L \rangle$, it suffices to show that $L \in \mathrm{BMO}$.

Now let us show that if the random variable η is bounded and if there is an equivalent local martingale measure Q satisfying the reverse Hölder condition, or, if the associated local martingale M^Q belongs to BMO, then the martingale part L of any bounded solution Y of (2.3.5), (2.3.6) belongs to the class BMO.

By the Itô formula,

$$Y_t^2 = Y_0^2 + 2\int_0^t Y_s dY_s + \langle L \rangle_t. \tag{2.3.33}$$

Since $Y_T = \eta$ and $Y_\tau \geq c$, we have from (2.3.33)

$$\langle L \rangle_T - \langle L \rangle_\tau + 2\int_\tau^T Y_s d(B_s + L_s) = \eta^2 - Y_\tau^2 \leq k_2^2. \tag{2.3.34}$$

Since Y satisfies (2.3.5), the process

$$B_t + \frac{p(p-1)}{2}\int_0^t Y_s d\langle M^Q \rangle_s + p\langle M^Q, L \rangle_t$$

is increasing and (2.3.32) implies that

$$\langle L \rangle_T - \langle L \rangle_\tau + 2\int_\tau^T Y_s dL_s - p(p-1)\int_\tau^T Y_s^2 d\langle M^Q \rangle_s - 2p\int_\tau^T Y_s d\langle M^Q, L \rangle_s) \leq k_2^2. \tag{2.3.35}$$

Without loss of generality, we may assume that L is a square integrable martingale; otherwise we can use the localization arguments. Therefore, if we take conditional expectations and take the inequality $Y_t \leq C$ into account, we obtain

$$E(\langle L \rangle_T - \langle L \rangle_\tau | F_\tau) \leq C^2 p(p-1)E(\langle M^Q \rangle_T - \langle M^Q \rangle_\tau | F_\tau) + k_2^2 + 2pCE\left(\int_\tau^T |d\langle M^Q, L \rangle_s||F_\tau\right). \tag{2.3.36}$$

Now using the Kunita–Watanabe inequality

$$E\left(\int_\tau^T |d\langle M^Q, L\rangle_s||F_\tau\right) \le E^{1/2}(\langle M^Q\rangle_T - \langle M^Q\rangle_\tau|F_\tau)E^{1/2}(\langle L\rangle_T - \langle L\rangle_\tau|F_\tau) \tag{2.3.37}$$

and that $M^Q \in \mathrm{BMO}$, we obtain from (2.3.34) that

$$E(\langle L\rangle_T - \langle L\rangle_\tau|F_\tau) \le c_1 + c_2 E^{1/2}(\langle L\rangle_T - \langle L\rangle_\tau|F_\tau) \tag{2.3.38}$$

for some positive constants c_1 and c_2 that do not depend on τ. The last inequality implies that $E(\langle L\rangle_T - \langle L\rangle_\tau|F_\tau)$ is bounded for every stopping time τ by the same constant, and hence $L \in \mathrm{BMO}$. □

Remark 2.3.2. In particular, if $M^Q \in M^2$ and η is square integrable, the same arguments imply that m is a square integrable martingale.

Remark 2.3.3. If Condition (A*) is satisfied, then the p-optimal martingale measure satisfies the reverse Hölder inequality $R_p(P)$, since for any stopping time τ,

$$\begin{aligned} E(\mathcal{E}^p_{\tau,T}(M^{Q^*})|F_\tau) &\le \frac{1}{k_1}E(\eta\mathcal{E}^p_{\tau,T}(M^{Q^*})|F_\tau) \\ &= \frac{1}{k_1}\operatorname*{ess\,inf}_{Q\in\mathcal{M}^e_p} E(\eta\mathcal{E}^p_{\tau,T}(M^Q)|F_\tau) \le \frac{1}{k_1}E(\eta\mathcal{E}^p_{\tau,T}(M^{\tilde{Q}})|F_\tau) \le C\frac{k_2}{k_1}. \end{aligned}$$

Proposition 2.3.3. *Equation* (2.3.5), (2.3.6) *is equivalent to the equation*

$$\frac{\mathcal{E}_T((\bar\psi - p\lambda)\cdot M)}{\mathcal{E}^{p-1}_T(\bar L)} = \bar c\eta\mathcal{E}^p_T(-\lambda\cdot M), \tag{2.3.39}$$

i.e., if Y is a solution of (2.3.5), (2.3.6), *then the triple $(\bar c, \bar\psi, \bar L)$, where*

$$\bar c = \frac{1}{Y_0}, \quad \bar\psi = \frac{\psi}{Y}, \quad \bar L = -\frac{1}{p-1}\int_0^t \frac{1}{Y_s}d\tilde L_s,$$

is a solution of (2.3.37). *Conversely, if $(\bar c, \bar\psi, \bar L)$ solves* (2.3.37), *then Y defined by*

$$Y_t = \frac{1}{\bar c}\mathcal{E}_t((\overline{\psi} - p\lambda)\cdot M)\mathcal{E}^{1-p}_t(\bar L)\mathcal{E}^{-p}_t(-\lambda\cdot M) \tag{2.3.40}$$

satisfies (2.3.5), (2.3.6).

Proof. Let Y be a solution of (2.3.5), (2.3.6) which admits the decomposition (2.2.7), (2.2.8).

It follows from (2.3.23) that

$$Y_t = Y_0 - \frac{p(p-1)}{2}\int_0^t Y_s\lambda'_s d\langle M\rangle_s\lambda_s + p\int_0^t \lambda'_s d\langle M\rangle_s\psi_s + \frac{p}{2(p-1)}\int_0^t \frac{1}{Y_s}d\langle\tilde L\rangle_s + \int_0^t \psi'_s dM_s + \tilde L_t. \tag{2.3.41}$$

We introduce

$$\overline{\psi_t} = \frac{\psi_t}{Y_t}, \quad \overline{L}_t = -\frac{1}{p-1}\int_0^t \frac{1}{Y_s}d\tilde L_s.$$

Then

$$\psi_t = \overline{\psi_t}Y_t, \quad \tilde L_t = -(p-1)\int_0^t Y_s d\overline{L}_s$$

and from (2.3.39), we have

$$dY_t = Y_t\left[-\frac{p(p-1)}{2}\lambda_t' d\langle M\rangle_t\lambda_t + p\lambda_t' d\langle M\rangle_t\overline{\psi_t} + \frac{p(p-1)}{2}d\langle\overline{L}\rangle_t + \overline{\psi_t}dM_t - (p-1)d\overline{L_t}\right], \quad Y_T = \eta. \tag{2.3.42}$$

Solving this linear equation with respect to Y, we obtain

$$Y_t = Y_0\exp\Bigg[-\frac{p(p-1)}{2}\int_0^t \lambda_s' d\langle M\rangle_s\lambda_s + p\int_0^t \lambda_s' d\langle M\rangle_s\overline{\psi}_s + \frac{p(p-1)}{2}\langle\overline{L}\rangle_t$$
$$-\frac{1}{2}\int_0^t \overline{\psi}_s' d\langle M\rangle_s\overline{\psi}_s - \frac{(p-1)^2}{2}\langle\overline{L}\rangle_t + \int_0^t \overline{\psi_s}dM_s - (p-1)\overline{L_t}\Bigg], \tag{2.3.43}$$

which can be expressed by means of Doleans-Dade exponentials

$$Y_t = Y_0\mathcal{E}_t((\bar{\psi} - p\lambda)\cdot M)\mathcal{E}_t^{1-p}(\overline{L})\mathcal{E}_t^{-p}(-\lambda\cdot M). \tag{2.3.44}$$

Now, using the boundary condition $Y_T = \eta$, we see that (2.3.37) is satisfied for $\bar{c} = 1/Y_0$.

Conversely, if a triple $(\overline{c}, \overline{\psi}, \overline{L})$ satisfies (2.3.37), then it is also obvious that Y defined by (2.3.38) is a solution of (2.3.5), (2.3.6). □

Corollary 2.3.1. *The semimartingale Bellman equation* (2.3.5), (2.3.6) *coincides with the equation*

$$V_t(p) = V_0(p) - \int_0^t \left(\frac{p(p-1)}{2}V_s(p)\lambda_s' d\langle M\rangle_s\lambda_s - p\lambda_s' d\langle M\rangle_s\varphi_s\right)$$
$$+\frac{p}{2(p-1)}\int_0^t \frac{1}{V_s(p)}d\langle\tilde{m}\rangle_s + \int_0^t \varphi_s' dM_s + \tilde{m}_t, \quad V_T(p) = \eta, \tag{2.3.45}$$

which is the same as (2.3.39) *written for* $V(p)$ *instead of* Y*. The equation*

$$R_t = R_0 - \int_0^t \frac{1}{2}(\bar{\varphi}_s - p\lambda_s)' d\langle M\rangle_s(\bar{\varphi}_s - p\lambda_s) + \int_0^t \frac{p}{2}\lambda_s' d\langle M\rangle_s\lambda_s$$
$$+\frac{p-1}{2}\langle\bar{m}\rangle_t + \int_0^t p\lambda_s' dM_s + \int_0^t (\bar{\varphi}_s - p\lambda_s)' dM_s - (p-1)\bar{m}_t, \quad R_T = \ln\eta, \tag{2.3.46}$$

with respect to $(R, \overline{\varphi}, \overline{m})$*, which admits a unique solution in the class* $\mathcal{S}_+\times L^2_{\mathrm{loc}}(\langle M\rangle)\times\mathcal{N}(X)$*, is also equivalent to* (2.3.5), (2.3.6).

Corollary 2.3.2. *A martingale measure* Q^* *is p-optimal if and only if*

$$\eta\mathcal{E}_T^{p-1}(M^{Q^*}) = c + \int_0^T h_s' dX_s \tag{2.3.47}$$

for a constant c *and an* X*-integrable predictable process* h *such that*

$$\left(\int_0^t h_s' dX_s,\ t\in[0,T]\right)$$

is a Q*-martingale for every* $Q \in \mathcal{M}_p^e$.

Proof. Let Q^* be a p-optimal martingale measure. According to Theorem 2.3.1, M^{Q^*} admits representation (2.3.7), and hence

$$\mathcal{E}_T^{p-1}(M^{Q^*}) = \mathcal{E}_T^{p-1}(-\lambda \cdot M)\mathcal{E}_T^{p-1}\left(-\frac{1}{p-1}\frac{1}{V(p)} \cdot \tilde{m}\right). \tag{2.3.48}$$

Therefore, using (2.3.38) and the relation $\frac{\mathcal{E}(X)}{\mathcal{E}(Y)} = \mathcal{E}(X - Y - \langle X - Y, Y\rangle)$ valid for continuous semimartingales X and Y, we obtain

$$V_t(p)\mathcal{E}_t^{p-1}(M^{Q^*}) = V_0(p)\frac{\mathcal{E}_t((\overline{\varphi} - p\lambda) \cdot M)}{\mathcal{E}_t(-\lambda \cdot M)} = V_0(p)\mathcal{E}_t((\overline{\varphi} + (1-p)\lambda) \cdot X).$$

Thus, the boundary condition $V_T(p) = \eta$ implies that $\eta\mathcal{E}_T^{p-1}(M^Q)$ is of the form (2.3.45) with

$$h_s = \big(\overline{\varphi}_s + (1-p)\lambda_s\big)\mathcal{E}_s((\overline{\varphi} + (1-p)\lambda) \cdot X), \quad s \in [0,T]. \tag{2.3.49}$$

Moreover, it follows from (2.3.47) that

$$V_0(p) + \int_0^t h'_s dX_s = V_t(p)\mathcal{E}_t^{p-1}(M^{Q^*})$$

and hence $\int_0^t h'_s dX_s$ is a Q^*-martingale by the optimality principle. The latter relation implies

$$\int_0^t h'_s dX_s \geq -V_0(p).$$

Since $\int_0^t h'_s dX_s$ is a Q-local martingale, it is also a supermartingale and

$$E^Q \int_0^t h'_s dX_s \leq 0$$

(for any $Q \in \mathcal{M}_p^e$). On the other hand, since Q^* is optimal, from Proposition 1.7.1 of the Appendix, we have

$$E^Q \int_0^T h'_s dX_s = E^Q\eta\mathcal{E}_T^{p-1}(M^{Q^*}) - V_0(p) = E\eta\mathcal{E}_T^{p-1}(M^{Q^*})\big(\mathcal{E}_T(M^Q) - \mathcal{E}_T(M^{Q^*})\big) \geq 0,$$

which implies that

$$E^Q \int_0^T h'_s dX_s = 0,$$

and hence $\int_0^t h'_s dX_s$ is a martingale for all $Q \in \mathcal{M}_p^e$.

Conversely, if Q^0 is a martingale measure satisfying relation (2.3.45) and the process

$$\left(\int_0^t h'_s dX_s,\ t \in [0,T]\right)$$

is a Q-martingale for every $Q \in \mathcal{M}_p^e$, then

$$E^Q\eta\mathcal{E}_T^{p-1}(M^{Q^0}) = E^{Q^0}\eta\mathcal{E}_T^{p-1}(M^{Q^0})$$

for any Q, which implies that Q^0 is optimal by Proposition 1.7.1 of the Appendix. □

Corollary 2.3.3. *The minimal martingale measure is p-optimal if and only if*

$$\eta \mathcal{E}_T^p(-\lambda \cdot M) = c + \int_0^T g_s' dM_s \tag{2.3.50}$$

for some M-integrable predictable g, and the process

$$\left(\int_0^t g_s' dX_s, \ t \in [0,T] \right)$$

is a P-martingale.

Remark 2.3.4. Obviously, if $\langle \lambda \cdot M \rangle$ is deterministic and $\eta = \text{const}$, then

$$\begin{aligned}\mathcal{E}_T^p(-\lambda \cdot M) &= \mathcal{E}_T(-p\lambda \cdot M) \exp\left\{ \frac{p(p-1)}{2} \langle \lambda \cdot M \rangle_T \right\} \\ &= \exp\left\{ \frac{p(p-1)}{2} \langle \lambda \cdot M \rangle_T \right\} \left(1 - p \int_0^T \mathcal{E}_s(-p\lambda \cdot M) \lambda_s' dM_s \right),\end{aligned}$$

and (2.3.48) is satisfied.

The semimartingale backward equation for the value process $\bar{V}_t(p)$ defined by (2.2.6) can be derived in a similar way. Here, we give only the corresponding theorem and remark some differences.

Assume that the following conditions are satisfied:

(A′) $\mathcal{M}^e \neq \emptyset$;

(B) all P-local martingales are continuous;

(C′) η is a strictly positive F_T-measurable random variable such that

$$E\eta^{\frac{1}{1-p}} < \infty.$$

Theorem 2.3.2 (Theorem 2.3.1′)**.** *Let* $0 < p < 1$ *and conditions* (A′), (B), *and* (C′) *be satisfied. Then the following assertions hold.*

(a) *The value process V is a solution of the semimartingale backward equation*

$$Y_t = Y_0 - \operatorname*{ess\,sup}_{N \in \mathcal{N}(X)} \left[\frac{1}{2} p(p-1) \int_0^t Y_s d\langle -\lambda \cdot M + N \rangle_s + p \langle \lambda \cdot M + N, L \rangle_t \right] + L_t, \quad t < T, \tag{2.3.51}$$

with the boundary condition

$$Y_T = \eta. \tag{2.3.52}$$

This solution is unique in the class S(X) of semimartingales. Moreover, the martingale measure Q^* is p-optimal if and only if it is given by the density $dQ^* = \mathcal{E}_T(M^{Q^*})dP$, where

$$M_t^{Q^*} = -\int_0^t \lambda_s' dM_s + \frac{1}{1-p} \int_0^t \frac{1}{V_s(p)} d\tilde{m}_s.$$

(b) *If, in addition, the conditions $k_1 \leq \eta \leq k_2$, $\lambda \cdot M \in$ BMO are satisfied and there is a constant c_1 such that*

$$E(\mathcal{E}_{\tau,T}^p(-\lambda \cdot M) | F_\tau) \geq c_1 \tag{2.3.53}$$

for any stopping time τ, then the value process $\bar{V}(p)$ is a unique solution of the semimartingale backward equation (2.3.49), (2.3.50) in the class of semimartingales Y satisfying the two-sided inequality

$$c \leq Y_t \leq C \quad \textit{for all } t \in [0,T] \textit{ a.s.}.$$

for some constants $0 < c < C$.

The proof is essentially similar to the proof of Theorem 2.3.1. In this case, the process $\bar{V}_t(p)\mathcal{E}_t^p(M^Q)$ is a P-supermartingale for all $Q \in \mathcal{M}^e$, and the classes D_p and $S(X)$ are defined similarly. From condition (C$'$) and the Hölder inequality, we have that $\sup_Q E\eta\mathcal{E}_T^p(M^Q) < \infty$, and the existence of an optimal martingale measure Q^* in the class $\mathcal{M}^{abs}$ follows from the same arguments. We only show that conditions (A$'$)–(C$'$) imply that Q^* is equivalent to P. Since Q^* is optimal, for the optimal density Z^{Q^*} and the stopping times τ_n defined by (2.3.14), we have the inequality

$$E\left[\eta\frac{Z_T^p(Q^*)}{Z_{\tau_n}^p(Q^*)}|F_{\tau_n}\right] \geq E\left[\eta\frac{\mathcal{E}_T^p(M^{\tilde{Q}})}{\mathcal{E}_{\tau_n}^p(M^{\tilde{Q}})}/F_{\tau_n}\right] \quad \text{a.s.} \tag{2.3.54}$$

By the Hölder inequality

$$E\left[\eta\frac{Z_T^p(Q^*)}{Z_{\tau_n}^p(Q^*)}|F_{\tau_n}\right] = E\left[\frac{Z_T^p(Q^*)}{Z_{\tau_n}^p(Q^*)}\eta I_{(Z_\tau(Q^*)\neq 0)}/F_{\tau_n}\right] \leq E^{1-p}\left(\eta^{\frac{1}{1-p}}I_{(Z_\tau(Q^*)\neq 0)}/F_{\tau_n}\right). \tag{2.3.55}$$

Condition (C$'$) and the Lévy theorem imply that

$$E^{1-p}\big(\eta^{\frac{1}{1-p}}I_{(Z_\tau(Q^*)=0)}|F_{\tau_n}\big)$$

tends to zero on the set $(Z_\tau(Q^*) = 0)$, hence the left-hand side of (2.3.53) tends to zero on the same set. On the other hand,

$$P\left(\sup_{t\leq T}\mathcal{E}_t^p(M^{\tilde{Q}}) \geq N\right) \leq \frac{1}{N}$$

by the Doob inequality for the supermartingale $\mathcal{E}_t^p(M^{\tilde{Q}})$ and

$$P\left(\inf_{t\leq T} E(\eta\mathcal{E}_T^p(M^{\tilde{Q}})|F_t) > 0\right) = 1,$$

since $\eta\mathcal{E}_T^p(M^{\tilde{Q}}) > 0$. Therefore, the limit of the right-hand side of (2.3.53) is strictly positive, which implies that $P(Z_\tau(Q^*) = 0) = 0$ and Q^* is equivalent to P.

Note that it follows from (2.3.51) that the value process $\bar{V}(p)$ is bounded from below, but this condition (unlike the reverse Hölder condition $R_p(P)$ for $p > 1$) does not imply that $\lambda \cdot M \in \mathrm{BMO}$. Therefore, we assume in part (b) that $\lambda \cdot M \in \mathrm{BMO}$ in order to guarantee

$$E\mathcal{E}_T\left(-\lambda\cdot M - \frac{1}{(p-1)\bar{Y}}\cdot\tilde{L}\right) = 1.$$

2.4. The Itô Process Model

2.4.1. Non-Markovian case. Let X be an Itô process

$$dX_t = \mu(t,\xi)dt + \delta(t,\xi)dw_t, \tag{2.4.1}$$

where ξ_t is the state process satisfying SDE

$$d\xi = b(t,\xi)dt + \sigma(t,\xi)dw_t. \tag{2.4.2}$$

Here, $w = (w_t)$ is an n-dimensional Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$, and $F = (\mathcal{F}_t, t \in [0,T])$ is the P-augmentation of the filtration generated by the Wiener process W. The coefficients μ, δ, b, and σ are nonantisipative functions

$$\mu : [0,T]\times C[0,T];\mathbb{R}^n) \to \mathbb{R}^m, \quad \delta : [0,T]\times C[0,T];\mathbb{R}^n) \to \mathbb{R}^{m\times n},$$
$$b : [0,T]\times C[0,T];\mathbb{R}^n) \to \mathbb{R}^n, \quad \sigma : [0,T]\times C[0,T];\mathbb{R}^n) \to \mathbb{R}^{n\times n}, \quad m < n.$$

We assume that the following conditions hold:

(C1) the coefficients b, σ are bounded continuous and such that Eq. (2.4.2) admits a unique strong solution;

(C2) the coefficients μ and δ are such that the structure condition is satisfied.

Sometimes, we use the following stronger condition:

(C^*2) the matrix $\delta\delta'$ is nonsingular and the function $\theta = \delta'(\delta\delta')^{-1}\mu$ is bounded and continuous.

The process X defines a stock price process by

$$dS_t^j = S_t^j dX_t^j, \quad j = 1, \dots, n. \tag{2.4.3}$$

There are two important particular cases of the model (2.4.1)–(2.4.3).

Example 2.4.1. $b = 0$, $\sigma = 1$, and $\xi_t = w_t$. Then (2.4.1) is of the form

$$dX_t = \mu(t, w)dt + \delta(t, w)dw_t. \tag{2.4.4}$$

Example 2.4.2. If the stock price process is described by SDE

$$\frac{dS_t^i}{S_t^i} = \bar{\mu}_i(t, S)dt + \sum_{j=1}^{n} \bar{\delta}_{ij}(t, S)dw_t, \quad j = i, \dots, m, \quad m = n, \tag{2.4.5}$$

and $S = (\bar{S}, S^{\perp})$, where $\bar{S} \in \mathbb{R}^m$ and $S^{\perp} \in \mathbb{R}^{n-m}$ denote the tradable and nontradable asset price processes, then we can obtain the system (2.4.1)–(2.4.3) from (2.4.5) when $\xi^i = \ln S^i$

$$\mu(t, Y) = \bar{\mu}(t, e^Y), \quad \delta(t, Y) = \bar{\delta}(t, e^Y),$$

$$b_i(t, Y) = \bar{\mu}_i(t, e^Y) + \frac{1}{2}\sum_{j=1}^{n} \bar{\delta}_{ij}^2(t, e^Y), \quad (t, Y) \in [0, T] \times C^n,$$

and $dX_t^j = dS_t^j / S_t^j$, $j = 1, \dots, m$. Here, C^n is the space of $\mathbb{R}^n$-valued continuous functions.

Denote by $L^2[0, T]$ the class of predictable processes ψ such that

$$\int_0^T \|\psi_t\|^2 dt < \infty,$$

a.s., and let $\mathcal{K}_p(\delta)$ be the subset of $L^2[0, T]$ defined by

$$\nu \in \mathcal{K}_p(\delta) \leftrightarrow \nu \in L^2[0, T] : \nu_s \in \ker \delta_s \quad \forall t \in [0, T] \text{ a.s.}$$

and

$$Z_t^{\nu} = \mathcal{E}_t\left(\int (-\theta(s, \xi) + \nu_s)' dw_s\right), \quad t \in [0, T],$$

is a p-integrable P-martingale.

Then the subclass $\mathcal{M}_p^e$ of equivalent martingale measures for (2.4.4) is given by

$$\mathcal{M}_p^e = \{P^{\nu} : dP^{\nu}/dP = Z_T^{\nu}, \ \nu \in \mathcal{K}_p(\delta)\},$$

where $\theta = \delta'(\delta\delta')^{-1}\mu = \delta'\lambda$. Here,

$$M_t = \int_o^t \delta_s dw_s, \quad \langle M \rangle_t = \int_0^t \delta_s \delta_s' ds, \quad \int_0^t \mu_s ds = \int_0^t d\langle M \rangle_s \cdot \lambda_s = \int_0^t \delta_s \delta_s' \lambda_s ds,$$

$$\int_0^t \lambda_s' dM_s = \int_0^t \lambda_s' \delta_s dw_s = \int_0^t \theta_s' dw_s.$$

By the martingale representation theorem, the martingale part of the value process is expressed as a stochastic integral

$$m_t = \int_0^t \zeta_s' dw_s.$$

It is easy to show that in this case,

$$\operatorname*{ess\,inf}_{N\in\mathcal{N}^p(X)}\left[\frac{1}{2}p(p-1)\int_0^t Y_s d\langle -\lambda'\cdot M+N\rangle_s + p\langle \lambda'\cdot M+N, L\rangle_t\right]$$

$$= \int_0^t \operatorname*{ess\,inf}_{\nu\in\mathcal{K}_p(\delta)}\left[\frac{p(p-1)}{2}V_s\big(-\theta_s(\xi)+\nu_s\big)^2 + p(-\theta_s(\xi)+\nu_s)'\zeta_s\right] ds. \quad (2.4.6)$$

Therefore, Eq. (2.3.5)–(2.3.6) takes the form

$$V_t = V_0 - \int_0^t \operatorname*{ess\,inf}_{\nu}\left[\frac{p(p-1)}{2}V_s\big(-\theta_s(\xi)+\nu_s\big)^2 + p\big(-\theta_s(\xi)+\nu_s\big)'\zeta_s\right] ds + \int_0^t \zeta_s' dw_s, \quad (2.4.7)$$

$$V_T = 1, \quad (2.4.8)$$

and according to Theorem 2.3.1, the process V is a unique solution of the BSDE (2.4.7) in the class $S(X)$ of the Itô processes (in the class of bounded strictly positive processes if C^*2 is satisfied).

Remark 2.4.1. Using the properties of exponential martingales, we can rewrite the value process in the form

$$V_t = \operatorname*{ess\,inf}_{\nu} E\left[\mathcal{E}_{tT}^p\left(\int(-\theta_s(\xi)+\nu_s)'dw_s\right)/\mathcal{F}_t\right]$$

$$= \operatorname*{ess\,inf}_{\nu} E^\nu\left[\exp\left(\frac{p(p-1)}{2}\int_t^T\big(|\theta_s(\xi)|^2+|\nu_s|^2\big)ds\right)/\mathcal{F}_t\right],$$

where E^ν is the expectation relative to the measure P^ν given by

$$\mathcal{E}_T\left(p\int(-\theta_s(\xi)+\nu_s)'dw_s\right).$$

By the Girsanov theorem, V_t is the value for the optimization problem

$$E\exp\left[\int_0^T\big(|\theta(s,\xi^\nu)|^2+|\nu_s|^2\big)ds\right] \to \min$$

with the controlled system described by

$$d\xi^\nu = \big[b(t,\xi^\nu) - p\sigma(t,\xi^\nu)(\theta(t,\xi^\nu)-\nu_t)\big]dt + \sigma(t,\xi^\nu)dw_t.$$

Now taking the infimum in expression (2.4.6), we obtain

$$V_t = V_0 + \int_0^t\left(-\frac{p(p-1)}{2}V_s|\theta_s(\xi)|^2 + p\theta_s'(\xi)\zeta_s + \frac{p}{2(p-1)}\frac{1}{V_s}|\Pi_{\ker\delta_s}\zeta_s|^2\right)ds + \int_0^t \zeta_s' dw_s, \quad (2.4.9)$$

$$V_T = 1. \quad (2.4.10)$$

Here and in what follows, Π_H denotes the orthogonal projection on subspace $H\in\mathbb{R}^n$.

If we change the variable $z_t = \zeta_t / V_t$ and solve the resulting linear equation

$$V_t = V_0 + \int\limits_0^t \left(-\frac{p(p-1)}{2} V_s |\theta_s(\xi)|^2 + p\theta_s'(\xi) z_s V_s + \frac{p}{2(p-1)} V_s |\Pi_{\ker \delta_s} \zeta_s|^2 \right) ds + \int\limits_0^t V_s \zeta_s' dw_s \quad (2.4.11)$$

with respect to V, we obtain the following BSDE for $(R = \ln V, z)$:

$$R_t = R_0 + \int\limits_0^t \left(-\frac{p(p-1)}{2} |\theta_s(\xi)|^2 + p\theta_s'(\xi) z_s + \frac{p}{2(p-1)} |\Pi_{\ker \delta_s(\xi)} z_s|^2 - \frac{1}{2}|z_s|^2 \right) ds + \int\limits_0^t z_s' dw_s, \quad V_T = 1, \quad (2.4.12)$$

or, equivalently,

$$R_t = R_0 + \int\limits_0^t \left[-\frac{1}{2} |\Pi_{\ker \delta_t(\xi)} z_s|^2 + \frac{1}{2(p-1)} |\Pi_{Ran\delta_t^*(\xi)} z_s|^2 + p\theta_s^*(\xi) z_s - \frac{p(p-1)}{2} |\theta_s(\xi)|^2 \right] ds + \int\limits_0^t z_s^* dw_s, \quad (2.4.13)$$

where $R_T = 0$. Equation (2.4.13) can be simplified if, instead of (2.4.1), we consider the equation

$$dX_t = \mu(t,\xi)dt + \bar{\delta}(t,\xi) d\bar{w}_t, \quad (2.4.14)$$

where $\bar{\delta}(t,y)$ is an $(m \times m)$-matrix.

Remark 2.4.2. Equation (2.4.1) can be reduced to (2.4.14) by using the Gram decomposition of a matrix.

In this case, we have

$$R_t = R_0 + \int\limits_0^t \left[-\frac{1}{2} |\bar{z}_s|^2 + \frac{p-1}{2} |z_s^\perp|^2 + p\bar{\theta}_s(\xi)' \bar{z}_s - \frac{p(p-1)}{2} |\bar{\theta}_s(\xi)|^2 \right] ds + \int\limits_0^t z_s' dw_s, \quad R_T = 0, \quad (2.4.15)$$

where $\bar{z} = (z_1, \ldots, z_m)$, $z^\perp = -\dfrac{1}{p-1}(z_{m+1}, \ldots, z_n)$.

Introducing the variable $\tilde{z} = \bar{z} - p\bar{\theta}$ we obtain

$$R_t = R_0 + \int\limits_0^t \left[-\frac{1}{2} |\tilde{z}_s|^2 + \frac{p-1}{2} |z_s^\perp|^2 \right] ds + \int\limits_0^t z_s' dw_s, \quad (2.4.16)$$

$$R_T = -\int\limits_0^T p\bar{\theta}_s(\xi)' d\bar{w}_s - \int\limits_0^T \frac{p}{2} |\bar{\theta}_s(\xi)|^2 ds \quad (2.4.17)$$

This can be written as the equation

$$\frac{\mathcal{E}_T(\int \bar{z}_s' d\bar{w}_s)}{\mathcal{E}_T^{p-1}(\int z_s^{\perp\prime} dw_s^\perp)} = c\mathcal{E}_T^p \left(-\int \bar{\theta}_s' d\bar{w}_s \right), \quad (2.4.18)$$

where $c > 0$ is some constant.

Now we consider cases where this equation can be solved explicitly. Assume that the state process ξ coincides with w and θ does not depend (a) on $w^\perp$ or (b) on $\bar{w}$.

(a) Equation (2.4.18) is solved by

$$z^\perp = 0, \quad \bar{z}_t = \frac{h_t}{\int_0^t h_s' dw_s},$$

where

$$\int_0^t h_s' dw_s = E[c\mathcal{E}_T^p\left(\int -\bar{\theta}_s' d\bar{w}_s\right)|\mathcal{F}_t] - Ec\mathcal{E}_T^p\left(\int -\bar{\theta}_s' d\bar{w}_s\right).$$

(b) Since

$$c\mathcal{E}_T^p\left(\int -\bar{\theta}_s' d\bar{w}_s\right) = c\mathcal{E}_T\left(\int -p\bar{\theta}_s' d\bar{w}_s\right)\exp\left(\frac{p(p-1)}{2}\int_0^t |\bar{\theta}_s|^2 ds\right),$$

we need to take $\bar{z} = -p\bar{\theta}$ and define $z^\perp$ from the equation

$$\mathcal{E}_t\left(\int z_s^{\perp\prime} dw_s^\perp\right) = E\left[c^{-1+p}\exp\left(-\frac{p-1}{2}\left(\int_0^T |\bar{\theta}_s|^2 ds\right)\right)|\mathcal{F}_t^\perp\right]$$

i.e.,

$$z_t^\perp = \frac{f_t}{\bar{c} + \int_0^t f_s' dw_s},$$

where

$$\int_0^t f_s' dw_s^\perp = E\left[c^{-1+p}\exp\left(-\frac{p-1}{2}\left(\int_0^T |\bar{\theta}_s|^2 ds\right)\right)|\mathcal{F}_t^\perp\right] - E\left[c^{-1+p}\exp\left(-\frac{p-1}{2}\left(\int_0^T |\bar{\theta}_s|^2 ds\right)\right)\right].$$

2.4.2. Markovian case. Let us consider the Markovian case, i.e., assume that the coefficients of (2.4.4)–(2.4.5) are of the form $\mu(t,\xi_t)$, $\delta(t,\xi_t)$, $b(t,\xi_t)$, $\sigma(t,\xi_t)$, where μ, δ, b, and σ are functions defined on the set $[0,T]\times\mathbb{R}^n$.

Let us introduce the value function

$$V(t,y) = \inf_{\nu\in\ker\delta} E^{t,y}\exp\left(\frac{p(p-1)}{2}\int_t^T (|\theta_s|^2 + |\nu_s|^2)ds\right).$$

Since the state process ξ is Markovian and the feedback controls (i.e., controls μ_t expressed in the form $\nu(t,\xi_t)$ for some measurable function $\nu(t,x)$) are sufficient, we can represent the value process V_t in the form

$$V_t = V(t,\xi_t) \quad \text{a.s.} \tag{2.4.19}$$

Since the value process V is a solution of Eq. (2.3.45) and the square characteristic of any martingale is absolutely continuous relative to Lebesgue measure, we have that the value V is an Itô process. Moreover, it follows from assumption (C2*) and from the proof of Theorem 2.3.1.b that the martingale part m of the value process belongs to the class BMO. Therefore, from expression (2.3.45) of the value process, we have that the finite variation part of the value process is of integrable variation. Thus, Eq. (2.4.19) implies that $V(t,\xi_t)$ is an Itô process of the form (1.7.15) and according to Proposition 1.7.4, it admits the representation

$$V(t,\xi_t) = V(0,\xi_0) + \int_0^t \mathcal{A}V(s,\xi_s)ds + \int_0^t V_y(t,\xi_t)\sigma'(t,\xi_t)dW_s, \tag{2.4.20}$$

where $(\mathcal{A}V)(t,y) = (\mathcal{L}V)(t,y) + b'(t,y)V_y(t,y)$.

Now, comparing Eq. (2.4.20) with (2.3.5) and using Eq. (2.4.19) and the uniqueness of the canonical decomposition of semimartingales, as a corollary we obtain from Theorem 2.3.1 the following assertion.

Theorem 2.4.1. *Assume that conditions* (C1), (C2*), *and* (A1)–(A3) *are satisfied. Then the value function V admits a generalized L-operator $\mathcal{L}V$, the first-order generalized derivatives V_y, and it is a unique bounded solution of the equation*

$$(\mathcal{L}V)(t,y) + \inf_{\nu\in\ker\delta(t,y)} \left[\frac{p(p-1)}{2}V(t,y)|\nu-\theta(t,y)|^2 + p(\nu-\theta(t,y))'\sigma'(t,y)V_y(t,y)\right] = 0 \quad ds\,dy\text{-a.s.}, \tag{2.4.21}$$

$$V(T,y) = 1. \tag{2.4.22}$$

Using the relation $\inf_{\lambda\in H}(-1/2|\lambda|^2 - b'\lambda) = -|\Pi_H b|^2/2$, from (2.4.20) we obtain the equation

$$\begin{aligned}(\mathcal{L}V)(t,y) &- p\theta'(t,y)\sigma'(t,y)V_y(t,y)\\ &- \frac{p}{2(p-1)}\left|\Pi_{Ran\delta'(t,y)}\sigma'(t,y)R_y(t,y)\right|^2 V(t,y) + \frac{p(p-1)}{2}|\theta(t,y)|^2 V(t,y) = 0.\end{aligned} \tag{2.4.23}$$

Denoting $R(t,y) = \ln V(t,y)$, we have

$$\begin{aligned}(\mathcal{L}R)(t,y) &- p\theta'(t,y)R_y(t,y) + \frac{1}{2}|\sigma'(t,y)R_y(t,y)|^2\\ &- \frac{p}{2(p-1)}|\Pi_{\ker\delta'(t,y)}\sigma'(t,y)R_y(t,y)|^2 + \frac{p(p-1)}{2}|\theta(t,y)|^2 = 0\end{aligned} \tag{2.4.24}$$

or using $|b|^2 = |\Pi_{\ker\delta}b|^2 + |\Pi_{Ran\delta'}b|^2$, we obtain

$$\begin{aligned}(\mathcal{L}R)(t,y) &- p\theta'(t,y)R_y(t,y) + \frac{1}{2}|\Pi_{\ker\delta(t,y)}\sigma'(t,y)R_y(t,y)|^2\\ &- \frac{1}{2(p-1)}|\Pi_{Ran\delta'(t,y)}\sigma'(t,y)R_y(t,y)|^2 + \frac{p(p-1)}{2}|\theta(t,y)|^2 = 0.\end{aligned} \tag{2.4.25}$$

The infimum in (2.4.20) is attained at

$$\nu(t,y) = \frac{1}{1-p}\ker\delta(t,x)\Pi_{\ker\delta(t,x)}\sigma'(t,y)\frac{V_y(t,y)}{V(T,y)} = \frac{1}{1-p}\Pi_{\ker\delta(t,x)}\sigma'(t,y)R_y(t,y),$$

i.e., the p-optimal martingale measure can be given by the density

$$\begin{aligned}&\mathcal{E}_T\left(\int(-\theta(s,\xi_s) + \frac{1}{1-p}\nu'(s,\xi_s))dw_s\right)\\ &\qquad = \mathcal{E}_T\left(-\int\lambda'(s,\xi_s)\delta(s,\xi_s)dw_s + \frac{1}{1-p}\int\Pi_{\ker\delta(t,\xi_s)}R_y'(s,\xi_s)\sigma(s,\xi_s)dw_s\right).\end{aligned}$$

In the case of Eq. (2.4.14), we can take $\bar\theta = \bar\delta^{-1}\mu$, $\ker\delta = \{0\}\times\mathbb{R}^{n-m}$, and (2.4.24) is transformed to the equation

$$\begin{aligned}(\mathcal{L}R)(t,y) &- p\bar\theta'(t,y)\bar R_y(t,y) + \frac{1}{2}|\bar\sigma(t,y)'R_y(t,y)|^2\\ &- \frac{p-1}{2}|\sigma^\perp(t,y)'R_y(t,y)|^2 + \frac{p(p-1)}{2}|\bar\theta(t,y)|^2 = 0.\end{aligned} \tag{2.4.26}$$

If, in addition, ξ is a Wiener process, i.e., $b=0$ and $\sigma = I$, then we can write

$$R_t(t,y) + \frac{1}{2}\Delta R(t,y) - p\overline{\theta}(t,y)'\bar R_y(t,y) + \frac{1}{2}|\bar R_y(t,y)|^2 - \frac{p-1}{2}|R_y^\perp(t,y)|^2 + \frac{p(p-1)}{2}|\overline{\theta}(t,y)|^2 = 0, \tag{2.4.27}$$

$$R(T,y) = 0, \tag{2.4.28}$$

where $\sigma = (\bar\sigma, \frac{1}{1-p}\sigma^\perp)$, $\bar\sigma$ and $\sigma^\perp$ are $(n\times m)$- and $n\times(n-m)$-matrices, respectively. Therefore, $\bar\nu(t,x) = \sigma^\perp(t,x)' R_x(t,x)$ defines variance-optimal martingale measures by the density

$$\mathcal{E}_T\left(-\int \bar\theta(s,\xi_s)' d\bar w_s + \frac{1}{1-p}\int R_x(s,\xi_s)'\sigma^\perp(s,\xi_s) dw_s^\perp\right).$$

Now let us apply this result to the stochastic volatility model considered by Laurent and Pham [49]. Assume that $p=2$. Let an asset price process S be described by the SDE

$$\begin{aligned} dS_t &= \operatorname{diag}(S_t)\big[\bar\mu(t,S_t,Y_t)dt + \bar\delta(t,S_t,Y_t)\big] d\bar w_t,\\ dY_t &= \mu^\perp(t,S_t,Y_t)dt + \delta_1(t,S_t,Y_t)d\bar w_t + \delta_2(t,S_t,Y_t)dw_t^\perp, \end{aligned} \tag{2.4.29}$$

where

$$\operatorname{diag}(S_t) = \begin{pmatrix} S_t^{(1)} & \dots & 0\\ \dots & \dots & \dots\\ 0 & \dots & S_t^{(m)} \end{pmatrix}, \quad S_t = (S_t^{(1)},\dots,S_t^{(m)}) \in \mathbb{R}_+^m, \quad Y_t \in \mathbb{R}^{n-m}.$$

Let us introduce $\xi = (\ln S, Y)$ and rewrite (2.4.29) in the form (2.4.1)–(2.4.3) assuming

$$\begin{aligned} b(t,x) &= \begin{pmatrix} \bar\mu(t,s,y) + \frac12 \tilde{\operatorname{dg}}(\bar\delta(t,s,y)\bar\delta'(t,s,y))\\ \mu^\perp(t,s,y)\end{pmatrix}, \quad \sigma(t,x) = \begin{pmatrix}\bar\delta(t,s,y) & 0\\ \delta_1(t,s,y) & \delta_2(t,s,y)\end{pmatrix},\\ \mu(t,x) &= \bar\mu(t,s,y), \quad \delta(t,x) = \bar\delta(t,s,y), \end{aligned} \tag{2.4.30}$$

where $x = (\ln s, y)$ and by $\tilde{\operatorname{dg}}(\Gamma) = (\gamma_{11},\dots,\gamma_{mm})'$ we denote the vector of diagonal entries of a matrix $\Gamma = (\gamma_{ij})_{i,j\le m}$.

Assume that the following conditions hold:

(D1) the coefficients $\bar\mu$, $\mu^\perp$, δ_1, δ_2, and $\bar\delta$ are bounded continuous functions satisfying the local Lipschitz condition;

(D2) there exists a constant $c>0$ such that

$$\big(\sigma\sigma'(s,x)\lambda,\lambda\big) \ge c|\lambda|^2$$

for all $s\in[0,T]$, $x\in\mathbb{R}^m$, and $\lambda\in\mathbb{R}^m$, where σ is defined by (2.4.30).

It is easy to see that (D1) and (D2) imply that conditions (C1), (C2*), and (A3) are satisfied.

The processes X (with coefficients (2.4.30)) and S from (2.4.1) and (2.4.29), respectively, admit the same martingale measures

$$\mathcal{E}_T\left(-\int \widetilde\theta(s,S_s,Y_s)'d\bar w_s - \int \nu_s' dw_s^\perp\right) = \mathcal{E}_T\left(-\int \bar\theta(s,\xi_s)'d\bar w_s - \int \nu_s' dw_s^\perp\right),$$

where

$$\widetilde\theta(s,S_s,Y_s) = \bar\delta^{-1}(s,S_s,Y_s)\bar\mu(s,S_s,Y_s) = \bar\theta(s,\xi_s).$$

Therefore, the value processes corresponding to models (2.4.30) and (2.4.29) coincide and, by the Markov property, it can be represented as $\widetilde V(t,S_t,Y_t) = V(t,\xi_t)$. Thus, $\widetilde V(t,s,y) = V(t,\ln s,y) = V(t,x)$, and the Bellman equation derived by Laurent and Pham for (2.4.29) (see [49, Eq. (6.14)]) can be obtained from (2.4.9) for coefficients (2.4.30) by changing the variables $(\ln s, y)\to x$. Hence we obtain the existence and uniqueness of a solution of (2.4.26) from [49] in the sense of Theorem 2.4.1.

Equation (2.4.26) for (2.4.30) can be rewritten as

$$\begin{aligned} (\mathcal{L}R)(t,x) - 2\overline{\mu}'(t,e^{\bar x},y)\bar R_x(t,x) - \frac12\left|\delta_2(t,e^{\bar x},y)'R_x^\perp(t,x)\right|^2&\\ + \frac12\left|\bar\delta(t,e^{\bar x},y)'\bar R_x(t,x) - \delta_1(t,e^{\bar x},y)'R_x^\perp(t,x)\right|^2 + |\overline\theta(t,e^{\bar x},y)|^2 &= 0,\\ (t,x) = (t,\bar x,y)\in[0,T]\times\mathbb{R}^m\times\mathbb{R}^{n-m}, & \end{aligned} \tag{2.4.31}$$

since, in this case, $\bar{\sigma}' R_x = \bar{\delta}' \bar{R}_x - \delta_1' R_x^\perp$ and $\sigma^{\perp\prime} R_x = \delta_2' R_x^\perp$, where $R_x = (\bar{R}_x, -R_x^\perp)$, $\bar{R}_x = (R_{x_1}, \dots, R_{x_m})$, and $R_x^\perp = -(R_{x_{m+1}}, \dots, R_{x_n})$.

Let us consider the following two cases:

Case I The coefficients $\mu^\perp$, $\bar{\delta}$, δ_1, δ_2, and $\bar{\theta}$ are independent of the variable y, i.e., we have the functions $\mu^\perp(t,s)$, $\bar{\delta}(t,s)$, $\delta_1(t,s)$, $\delta_2(t,s)$, and $\bar{\theta}(t,s)$. Then the solution of (2.4.31) is independent of y and $\bar{\nu} = 0$.

Case II The coefficients $\mu^\perp$, δ_2, and $\bar{\theta}$ are independent of s and $\delta_1 = 0$. Then the solution of (2.4.31) is independent of s, and we have

$$(\mathcal{L}R)(t,x) - \frac{1}{2}|\delta_2(t,y)R_x^\perp|^2 + |\bar{\theta}(t,y)|^2 = 0, \quad R(T,y) = 0. \tag{2.4.32}$$

For $U(t,y) = e^{-R(t,y)}$, we obtain the linear SDE

$$(\mathcal{L}U)(t,y) - |\bar{\theta}(t,y)|^2 U(t,y) = 0, \quad U(T,y) = 1.$$

Therefore,

$$U(t,y) = E^{t,y} \exp\left(-\int_t^T |\bar{\theta}(s,Y_s)|^2 ds \right), \quad R(t,y) = -\ln U(t,y), \quad \bar{\nu}(t,y) = \delta_2'(t,y) R_y^\perp(t,y).$$

Remark 2.4.3. In Case II, Laurent and Pham [49], under some smoothness conditions on the coefficients (using the results from Krylov [47] and Friedman), showed that Eq. (2.4.32) admits a unique solution of class $C^{1,2}$. In [49], Laurent and Pham also derived a Bellman equation equivalent to (2.4.26) for a more general case (2.4.29). As was mentioned in [49], the solvability of (2.4.26) in the class $C^{1,2}$ is an open question and the value function can be characterized only in terms of viscosity solutions. We solve Eq. (2.4.26) in the class V_μ^L of functions which, in contrast to viscosity solutions, admit all generalized first-order derivatives.

2.5. Minimal Entropy Martingale Measure

The minimal entropy martingale measure minimizes the relative entropy of a martingale measure with respect to the measure P. It is known (see [30, 71]) that for a locally bounded process X, the minimal entropy martingale measure always exists, is unique, and if there is a martingale measure with finite relative entropy, then the minimal entropy martingale measure is equivalent to P.

The aim of this section is to give the construction of the minimal entropy martingale measure when the dynamics of the discounted assets price process is governed by a continuous semimartingale. We obtain a description of the minimal entropy martingale measure in terms of the value function of a suitable problem of an optimal equivalent change of measure and show that this value process uniquely solves the corresponding semimartingale backward stochastic differential equation (BSDE). We show that in two specific extreme cases (already studied in [5, 49, 73] in connection with the variance-optimal martingale measures), this semimartingale BSDE admits an explicit solution, which gives an explicit construction of the minimal entropy martingale measure. In particular, we give a necessary and sufficient condition for the minimal entropy martingale measure to coincide with the minimal martingale measure, as well as with the martingale measure appearing in the second above-mentioned extreme case.

Let

$$\mathcal{M}^e_{\mathcal{E}nt} = \{Q \in \mathcal{M}^e : EZ_T^Q \ln Z_T^Q < \infty\}.$$

We assume that the following conditions hold:

(A) all (F,P)-local martingales are continuous;

(B) there is an equivalent martingale measure Q such that $EZ_T^Q \ln Z_T^Q < \infty$, i.e.,

$$\mathcal{M}^e_{\mathcal{E}nt} \neq \emptyset. \tag{2.5.1}$$

Note that conditions (A) and (B) imply that X is a continuous semimartingale satisfying the structure condition. This means that X admits the decomposition

$$X_t = X_0 + \Lambda_t + M_t, \tag{2.5.2}$$

where M is a continuous local martingale and there exists a predictable $\mathbb{R}^d$-valued process λ such that $d\Lambda = d\langle M\rangle\lambda$ with $K_T = \int_0^T \lambda_s' d\langle M\rangle_s \lambda_s < \infty$, where $'$ denotes the transposition. The process K is called the mean-variance tradeoff process of X (see [85] for the interpretation of the process K).

Since X is continuous, any element Q of $\mathcal{M}^e$ is given by the density Z_t^Q, which is expressed as an exponential martingale of the form

$$\mathcal{E}_t(-\lambda\cdot M + N), \tag{2.5.3}$$

where N is a local martingale strongly orthogonal to M and the notation $\lambda\cdot M$ stands for the stochastic integral.

If the local martingale $\hat{Z} = \mathcal{E}(-\lambda\cdot M)$ is a true martingale, then $d\hat{P}/dP = \hat{Z}_T$ defines an equivalent probability measure called the minimal martingale measure for X.

We denote by $\mathcal{N}_{\mathcal{E}nt}(X)$ the class of local martingales N strongly orthogonal to M such that the process $(\mathcal{E}_t(-\lambda\cdot M+N), t\in[0,T])$ is a strictly positive P-martingale with $E\mathcal{E}_T(-\lambda\cdot M+N)\ln\mathcal{E}_T(-\lambda\cdot M+N)<\infty$. Then

$$\mathcal{M}^e_{\mathcal{E}nt} = \left\{ Q\sim P : \frac{dQ}{dP}\Big|_{F_T} = \mathcal{E}_T(-\lambda\cdot M+N),\ N\in\mathcal{N}_{\mathcal{E}nt}(X) \right\}. \tag{2.5.4}$$

We recall the definition of BMO-martingales and the reverse Hölder $L\ln L$-condition.

The square integrable continuous martingale M belongs to the class BMO iff there is a constant $C>0$ such that

$$E^{1/2}(\langle M\rangle_T - \langle M\rangle_\tau | F_\tau) \le C \tag{2.5.5}$$

for every stopping time τ. The smallest constant with this property is called the BMO norm of M and is denoted by $\|M\|_{\mathrm{BMO}}$.

Let Z be a strictly positive uniformly integrable martingale.

Definition 2.5.1. The process Z satisfies the $R_{\mathcal{E}nt}(P)$ inequality if there is a constant C_1 such that

$$E\left(\frac{Z_T}{Z_\tau}\ln\frac{Z_T}{Z_\tau}|F_\tau\right) \le C_1 \tag{2.5.6}$$

for every stopping time τ.

The proof of the following assertion can be found in [78] (see [21, 41] for the case x^p, $p>1$).

Proposition 2.5.1. *Let $\mathcal{E}(M)$ be an exponential martingale associated with the continuous local martingale M. Then if $\mathcal{E}(M)$ is a uniformly integrable martingale and satisfies the $R_{\mathcal{E}nt}(P)$ inequality, then M belongs to the class BMO.*

Also, let us recall the concept of relative entropy (see [12] about the basic properties of the relative entropy).

The relative entropy, or the Kullback–Leibler distance, $I(Q,R)$ of the probability measure Q with respect to the measure R is defined as

$$I(Q,R) = E^R\frac{dQ}{dR}\ln\frac{dQ}{dR}. \tag{2.5.7}$$

The minimal entropy martingale measure Q^* is a solution of the optimization problem

$$\inf_{Q\in\mathcal{M}^{abs}} I(Q,P) = I(Q^*,P),$$

where $\mathcal{M}^{abs}$ is the set of measures Q absolutely continuous with respect to P such that X is a local martingale under Q.

Proposition 2.5.2. *If X is locally bounded and there exists $Q \in \mathcal{M}^{abs}$ such that $I(Q,P) < \infty$, then the minimal entropy martingale measure exists and is unique. Moreover if $I(Q,P) < \infty$ for some $Q \in \mathcal{M}^e$, then the minimal entropy martingale measure is equivalent to P.*

Remark 2.5.1. This assertion is proved in [30] under the assumption that X is bounded and defines the class $\mathcal{M}^e$ as the set of equivalent measures Q such that X is a martingale (and not a local martingale) under Q. The proof is the same if X is locally bounded and $\mathcal{M}^e$ is defined as in the Introduction.

Since any continuous process is locally bounded, under assumptions (A) and (B), the minimal entropy martingale measure always exists and is equivalent to the basic measure P. Therefore, hereafter, we consider only equivalent martingale measures and focus our attention on the construction and properties of optimal martingale measures.

Thus, we consider the optimization problem

$$\inf_{Q \in \mathcal{M}^e_{\mathcal{E}nt}} E\mathcal{E}_T(M^Q) \ln \mathcal{E}_T(M^Q). \tag{2.5.8}$$

Let us introduce the following notation:

$$\mathcal{E}_{tT}(M^Q) = \frac{\mathcal{E}_T(M^Q)}{\mathcal{E}_t(M^Q)}, \quad \langle M^Q \rangle_{tT} = \langle M^Q \rangle_T - \langle M^Q \rangle_t,$$

and let

$$V_t = \operatorname*{ess\,inf}_{Q \in \mathcal{M}^e_{\mathcal{E}nt}} E(\mathcal{E}_{tT}(M^Q) \ln \mathcal{E}_{tT}(M^Q)|F_t) = \operatorname*{ess\,inf}_{N \in \mathcal{N}_{\mathcal{E}nt}(X)} E^Q\big(\ln \mathcal{E}_{tT}(-\lambda \cdot M + N)|F_t\big) \tag{2.5.9}$$

be the value process corresponding to the problem (2.5.8).

Also, let us introduce the process

$$\bar{V}_t = \frac{1}{2} \operatorname*{ess\,inf}_{Q \in \mathcal{M}^e_{\mathcal{E}nt}} E^Q(\langle M^Q \rangle_{tT}|F_t). \tag{2.5.10}$$

Remark 2.5.2. We see later that $V_t = \bar{V}_t$ if there exists an equivalent martingale measure satisfying the $R_{\mathcal{E}nt}$ inequality.

The optimality principle, which is proved in a standard way (see, e.g., [23, 41, 49]), takes the following form in this case.

Proposition 2.5.3. (a) *There exists an RCLL semimartingale, still denoted by V_t, such that for each $t \in [0,T]$,*

$$V_t = \operatorname*{ess\,inf}_{Q \in \mathcal{M}^e_{\mathcal{E}nt}} E^Q(\ln \mathcal{E}_{tT}(M^Q)|F_t).$$

V_t is the largest RCLL process equal to 0 at time T such that $V_t + \ln \mathcal{E}_t(M^Q)$ is a Q-submartingale for every $Q \in \mathcal{M}^e_{\mathcal{E}nt}$.

(b) *The following properties are equivalent*:

(i) *Q^* is optimal, i.e.,*

$$V_0 = \inf_{Q \in \mathcal{M}^e_{\mathcal{E}nt}} E^Q \ln \mathcal{E}_T(M^Q) = E^{Q^*} \ln \mathcal{E}_T(M^{Q^*});$$

(ii) *Q^* is optimal for all conditional criteria, i.e., for each $t \in [0,T]$,*

$$V_t = E^{Q^*}(\ln \mathcal{E}_{tT}(M^{Q^*})|F_t) \quad a.s.;$$

– *(iii) $V_t + \ln \mathcal{E}_t(M^{Q^*})$ is a Q^*-martingale.*

The following statement proved in [15] is a consequence of Proposition 2.5.3b.

Corollary 2.5.1. *If there exists an equivalent martingale measure $\tilde{Q}$ whose density satisfies the $R_{\mathcal{E}nt}(P)$ inequality, then the density of the minimal entropy martingale measure also satisfies the $R_{\mathcal{E}nt}(P)$ inequality.*

Proof. It follows immediately, since for any stopping time τ

$$E(\mathcal{E}_{\tau T}(M^{Q^*})\ln \mathcal{E}_{\tau T}(M^{Q^*})|F_\tau) = \operatorname*{ess\,inf}_{Q\in\mathcal{M}^e_{\mathcal{E}nt}} E(\mathcal{E}_{\tau T}(M^Q)\ln \mathcal{E}_{\tau T}(M^Q)|F_\tau)$$
$$\leq E(\mathcal{E}_{\tau T}(M^{\tilde{Q}})\ln \mathcal{E}_{\tau T}(M^{\tilde{Q}})/F_\tau) \leq C.$$

The corollary is proved. □

2.6. Backward Semimartingale Equation for the Value Process Related to the Minimal Entropy Martingale Measure

We say that a process B strongly dominates a process A and we write $A \prec B$ if the difference $B - A \in \mathcal{A}^+_{\text{loc}}$, i.e., if it is a locally integrable increasing process. Let $(A^Q, Q \in \mathcal{Q})$ be the family of processes of bounded variations, zero at time zero. Denote by $\operatorname{ess\,inf}_{Q\in\mathcal{Q}}(A^Q)$ the largest process of finite variation, zero at time zero, which is strongly dominated by the process A^Q for every $Q \in \mathcal{Q}$, i.e., this is "ess inf" of the family $(A^Q, Q \in \mathcal{Q})$ relative to the partial order $\prec$.

Let us consider the following semimartingale backward equation:

$$Y_t = Y_0 - \operatorname*{ess\,inf}_{Q\in\mathcal{M}^e_{\mathcal{E}nt}} \left[\frac{1}{2}\langle M^Q\rangle_t + \langle M^Q, L\rangle_t\right] + L_t, \quad t < T, \tag{2.6.1}$$

with the boundary condition

$$Y_T = 0. \tag{2.6.2}$$

We say that the process Y is a solution of (2.6.1), (2.6.2) if Y is a special semimartingale with respect to the measure P with the canonical decomposition

$$Y_t = Y_0 + B_t + L_t, \quad B \in \mathcal{A}_{\text{loc}}, \quad L \in \mathcal{M}^2_{\text{loc}}, \tag{2.6.3}$$

such that $Y_T = 0$ and

$$B_t = -\operatorname*{ess\,inf}_{Q\in\mathcal{M}^e_{\mathcal{E}nt}} \left[\frac{1}{2}\langle M^Q\rangle_t + \langle M^Q, L\rangle_t\right]. \tag{2.6.4}$$

Let

$$L_t = \int_0^t \psi'_s dM_s + \tilde{L}_t, \quad \langle \tilde{L}, M\rangle = 0, \tag{2.6.5}$$

be the Galtchouk–Kunita–Watanabe decomposition (G-K-W) of L with respect to the martingale M.

Lemma 2.6.1. *If there exists $Q \in \mathcal{M}^e_{\mathcal{E}nt}$ such that $M^Q \in$ BMO, then the martingale part L of any bounded solution Y of Eq.* (2.6.1), (2.6.2) *belongs to the class BMO and*

$$\|L\|_{\text{BMO}} \leq (2C+1)^2 \|M^Q\|_{\text{BMO}}, \tag{2.6.6}$$

where C is an upper bound of the process Y.

Proof. Using the Itô formula for $Y^2_T - Y^2_\tau$ and the boundary condition $Y_T = 0$, we have

$$\langle L\rangle_T - \langle L\rangle_\tau + 2\int_\tau^T Y_s d(B_s + L_s) \leq 0 \tag{2.6.7}$$

for any stopping time τ. Since Y satisfies (2.6.1)

$$B_t + \frac{1}{2}\langle M^Q\rangle_t + \langle M^Q, L\rangle_t \in \mathcal{A}^+_{\text{loc}} \tag{2.6.8}$$

and, therefore, (2.6.7) implies that

$$\langle L\rangle_T - \langle L\rangle_\tau + 2\int_\tau^T Y_s dL_s - \int_\tau^T Y_s d\langle M^Q\rangle_s - 2\int_\tau^T Y_s d\langle M^Q, L\rangle_s \le 0. \tag{2.6.9}$$

Without loss of generality, we may assume that L is a square integrable martingale; otherwise, one can use localization arguments. Therefore, if we take the conditional expectations in (2.6.9) having inequality $|Y_t| \le C$ in mind, we obtain

$$E\left(\langle L\rangle_T - \langle L\rangle_\tau | F_\tau\right) - CE\left(\langle M^Q\rangle_T - \langle M^Q\rangle_\tau | F_\tau\right) - 2CE\left(\int_\tau^T |d\langle M^Q, L\rangle_s||F_\tau\right) \le 0. \tag{2.6.10}$$

Now using the conditional Kunita–Watanabe inequality from (2.6.10), we have

$$E(\langle L\rangle_T - \langle L\rangle_\tau|F_\tau) - 2C\|M^Q\|_{\mathrm{BMO}}^{1/2} E^{1/2}(\langle L\rangle_T - \langle L\rangle_\tau|F_\tau) - C\|M^Q\|_{\mathrm{BMO}} \le 0. \tag{2.6.11}$$

Solving this quadratic inequality with respect to $x = E^{1/2}(\langle L\rangle_T - \langle L\rangle_\tau|F_\tau)$, we obtain the estimate

$$E(\langle L\rangle_T - \langle L\rangle_\tau|F_\tau) \le (2C+1)^2\|M^Q\|_{\mathrm{BMO}}.$$

Since the right-hand side is independent of τ, estimate (2.6.6) also holds and L belongs to the space BMO. □

The value process of problem (2.5.8) defined by (2.5.9) is a special semimartingale with respect to the measure P with the canonical decomposition

$$V_t = V_0 + m_t + A_t, \quad m \in M^2_{\mathrm{loc}}, \quad A \in \mathcal{A}_{\mathrm{loc}}. \tag{2.6.12}$$

Let

$$m_t = \int_0^t \varphi'_s dM_s + \tilde{m}_t, \quad \langle \tilde{m}, M\rangle = 0, \tag{2.6.13}$$

be the GKW decomposition of m with respect to M.

Now we formulate the main statement of the paper.

Theorem 2.6.1. . *Let conditions* (A) *and* (B) *be satisfied. Then the following assertions hold.*

(a) *The value process V is a solution of the semimartingale backward Eq.* (2.6.1)–(2.6.2)*. Moreover, a martingale measure Q^* is the minimal entropy martingale measure if and only if it is given by the density $dQ^* = \mathcal{E}_T(M^{Q^*})dP$, where*

$$M_t^{Q^*} = -\int_0^t \lambda'_s dM_s - \tilde{m}_t. \tag{2.6.14}$$

(b) *If, in addition, the minimal martingale measure exists and satisfies the reverse Hölder $R_{\mathcal{E}nt}$-inequality, then the value process V is a unique bounded solution of* (2.6.1)–(2.6.2)*.*

Proof. (a) By Condition (B), there exists $\tilde{Q} \in \mathcal{M}^e_{\mathcal{E}nt}$, and according to Proposition 2.5.3, the process $Z_t = V_t + \ln\mathcal{E}_t(M^{\tilde{Q}})$ is a $\tilde{Q}$-submartingale; hence it is a P-semimartingale by the Girsanov theorem. Since $\mathcal{E}_T(M^{\tilde{Q}})$ is strictly positive and continuous, the process $\ln\mathcal{E}_t(M^{\tilde{Q}})$ is a semimartingale; consequently, the value process V is also a semimartingale under P. Condition (A) implies that any adapted RCLL process is predictable (see [76]), and hence any semimartingale is special. Therefore, V is a P-special semimartingale admitting decomposition (2.6.12).

The processes $m - \langle m, M^Q\rangle$ and $M^Q - \langle M^Q\rangle$ are Q-local martingales by the Girsanov theorem. Therefore, since

$$\begin{aligned} Z_t = V_t + \ln \mathcal{E}_t(M^Q) &= V_0 + m_t + A_t + M_t^Q - \frac{1}{2}\langle M^Q\rangle_t \\ &= V_0 + (m_t - \langle m, M^Q\rangle_t) + (M_t^Q - \langle M^Q\rangle_t) + A_t + \frac{1}{2}\langle M^Q\rangle_t + \langle m, M^Q\rangle_t \end{aligned} \tag{2.6.15}$$

and since $V_t + \ln \mathcal{E}_t(M^Q)$ is a Q-submartingale for every $Q \in \mathcal{M}^e_{\mathcal{E}nt}$, we have

$$A_t + \frac{1}{2}\langle M^Q\rangle_t + \langle m, M^Q\rangle_t \in \mathcal{A}^+_{\text{loc}} \tag{2.6.16}$$

for every $Q \in \mathcal{M}^e_{\mathcal{E}nt}$.

On the other hand, according to Proposition 2.5.2, the optimal martingale measure Q^* exists and is equivalent to P. Therefore, by the optimality principle, the process $V_t + \ln \mathcal{E}_t(M^{Q^*})$ is a Q^*-martingale, and using the Girsanov theorem once again, we obtain

$$A_t + \frac{1}{2}\langle M^{Q*}\rangle_t + \langle m, M^{Q*}\rangle_t = 0. \tag{2.6.17}$$

Relations (2.6.16) and (2.6.17) imply

$$A_t = -\operatorname*{ess\,inf}_{Q \in \mathcal{M}^e_{\mathcal{E}nt}} \left[\frac{1}{2}\langle M^Q\rangle_t + \langle M^Q, m\rangle_t\right], \tag{2.6.18}$$

and hence the value process V satisfies Eq. (2.6.1) and, obviously, $V_T = 0$. Relation (2.6.17) implies that the processes A_t and hence V_t are continuous.

Now let us show that the optimal martingale measure Q^* is given by (2.6.14).

From (2.6.18), we have

$$\begin{aligned} A_t &= -\frac{1}{2}\langle \lambda \cdot M\rangle_t + \langle \lambda \cdot M, m\rangle_t - \operatorname*{ess\,inf}_{N \in N_{\mathcal{E}nt}(X)} \left(\frac{1}{2}\langle N\rangle_t + \langle N, m\rangle_t\right) \\ &= -\frac{1}{2}\langle \lambda \cdot M\rangle_t + \langle \lambda \cdot M, m\rangle_t + \frac{1}{2}\langle \tilde{m}\rangle_t - \frac{1}{2}\operatorname*{ess\,inf}_{N \in N_{\mathcal{E}nt}(X)} (\langle N + \tilde{m}\rangle_t) \\ &= -\frac{1}{2}\langle \lambda \cdot M\rangle_t + \langle \lambda \cdot M, m\rangle_t + \frac{1}{2}\langle \tilde{m}\rangle_t, \end{aligned} \tag{2.6.19}$$

since

$$\operatorname*{ess\,inf}_{N \in N_{\mathcal{E}nt}(X)} (\langle N + \tilde{m}\rangle_t) = 0. \tag{2.6.20}$$

To prove relation (2.6.20), let us define the sequence of stopping times

$$\tau_n = \inf\left\{t : \mathcal{E}_t(\tilde{N}) \geq \frac{1}{n} \quad \text{or} \quad \mathcal{E}_t(-\lambda \cdot M - \tilde{m}) \geq n\right\} \wedge T,$$

where $\tilde{N}$ is a local martingale from the class $\mathcal{N}_{\mathcal{E}nt}(X)$, which exists by condition (B). It is not difficult to see that the local martingale $N^n = -\tilde{m}^{\tau_n} + \tilde{N} - \tilde{N}^{\tau_n}$ belongs to the class $\mathcal{N}_{\mathcal{E}nt}(X)$ and $\tau_n \uparrow T$. Therefore,

$$\operatorname*{ess\,inf}_{N \in N_{\mathcal{E}nt}(X)} (\langle N + \tilde{m}\rangle_t) \leq \langle N^n + \tilde{m}\rangle_t = \langle \tilde{m} - \tilde{m}^{\tau_n} + \tilde{N} - \tilde{N}^{\tau_n}\rangle \leq 2(\langle \tilde{m}\rangle_t - \langle \tilde{m}\rangle_{t \wedge \tau_n} + \langle \tilde{N}\rangle_t - \langle \tilde{N}\rangle_{t \wedge \tau_n})$$

for each $n \geq 1$ and (2.6.20) holds, since the right-hand side of the latter inequality tends to zero as $n \to \infty$. Here, as above, $\tilde{m}$ is the orthogonal martingale part of m in the GKW decomposition (2.6.13) and $\tilde{m}^{\tau_n} = (\tilde{m}_{\tau_n \wedge t}, t \in [0, T])$ is a stopped martingale.

By the optimality principle, $V_t + \ln \mathcal{E}_t(M^{Q^*})$ is a Q^*-martingale. Since V solves Eq. (2.6.1), this implies

$$\operatorname*{ess\,inf}_{Q\in\mathcal{M}^e_{\mathcal{E}nt}} \left[\frac{1}{2}\langle M^Q\rangle_t + \langle M^Q, m\rangle_t\right] = \frac{1}{2}\langle M^{Q^*}\rangle_t + \langle M^{Q^*}, m\rangle_t. \tag{2.6.21}$$

Since M^{Q^*} is represented in the form $-\lambda\cdot M + N^*$ for some $N^* \in \mathcal{N}_{\mathcal{E}nt}(X)$, it follows from (2.6.19) and (2.6.21) that the processes N^* and $\tilde{m}$ and hence the processes M^{Q^*} and $-\lambda\cdot M - \tilde{m}$ are indistinguishable. Therefore, the minimal entropy martingale measure is unique and admits representation (2.6.14).

(b) It is easy to see that the value process satisfies the two-sided inequality for all $t \in [0, T]$:

$$0 \le V_t \le C \quad \text{a.s.} \tag{2.6.22}$$

The positivity of V follows from the Jensen inequality. On the other hand, if there exists a martingale measure $\tilde{Q}$ satisfying the reverse Hölder $R_{\mathcal{E}nt}$ inequality, we have that V is bounded above, since

$$V_t = \operatorname*{ess\,inf}_{Q\in\mathcal{M}^e_{\mathcal{E}nt}} E(\mathcal{E}_{tT}(M^Q)\ln \mathcal{E}_{tT}(M^Q)|F_t) \le E(\mathcal{E}_{tT}(M^{\tilde{Q}})\ln \mathcal{E}_{tT}(M^{\tilde{Q}})|F_t) \le C.$$

Thus, V is a bounded solution of (2.6.1), (2.6.2).

Uniqueness. Let Y be a bounded solution of (2.6.1), (2.6.2). Let us show that the processes Y and V are indistinguishable. Since Y solves (2.6.1) we have

$$Y_t + \ln \mathcal{E}_t(M^Q) = Y_0 + L_t + B_t + M^Q_t - \frac{1}{2}\langle M^Q\rangle_t$$

$$= Y_0 + (L_t - \langle L, M^Q\rangle_t) + (M^Q_t - \langle M^Q\rangle_t) + \frac{1}{2}\langle M^Q\rangle_t + \langle L, M^Q\rangle_t - \operatorname*{ess\,inf}_{Q\in\mathcal{M}^e_{\mathcal{E}nt}} \left[\frac{1}{2}\langle M^Q\rangle_t + \langle L, M^Q\rangle_t\right]. \tag{2.6.23}$$

Therefore, the Girsanov theorem implies that $Y_t + \ln \mathcal{E}_t(M^Q)$ is a Q-local submartingale for every $Q \in \mathcal{M}^e_{\mathcal{E}nt}$.

Thus, the process

$$Y_t\mathcal{E}_t(M^Q) + \mathcal{E}_t(M^Q)\ln \mathcal{E}_t(M^Q)$$

is a local P-submartingale.

Since $(\mathcal{E}_t(M^Q), t \in [0, T])$ is a martingale satisfying the condition $E\mathcal{E}_T(M^Q)\ln \mathcal{E}_T(M^Q) < \infty$, the process $\mathcal{E}_t(M^Q)\ln \mathcal{E}_t(M^Q)$ is from the class D, since a submartingale is bounded from below (by the constant $-1/e$). On the other hand, the process $Y_t\mathcal{E}_t(M^Q)$ is also from the class D, since Y is bounded and $\mathcal{E}_t(M^Q)$ is a martingale (see, e.g., [19]). Thus, $Y_t\mathcal{E}_t(M^Q) + \mathcal{E}_t(M^Q)\ln \mathcal{E}_t(M^Q)$ is a submartingale from the class D, and hence from the boundary condition, we have

$$Y_t\mathcal{E}_t(M^Q) + \mathcal{E}_t(M^Q)\ln \mathcal{E}_t(M^Q) \le E(\mathcal{E}_T(M^Q)\ln \mathcal{E}_T(M^Q)|F_t)$$

for all $Q \in \mathcal{M}^e_{\mathcal{E}nt}$ and

$$Y_t \le \operatorname*{ess\,inf}_{Q\in\mathcal{M}^e_{\mathcal{E}nt}} E[\mathcal{E}_{tT}(M^Q)\ln \mathcal{E}_{tT}(M^Q)|F_t] = V_t. \tag{2.6.24}$$

Let us show the converse inequality.

Similarly to (2.6.19) we have

$$B_t = -\frac{1}{2}\langle \lambda\cdot M\rangle_t + \langle \lambda\cdot M, L\rangle_t + \frac{1}{2}\langle \tilde{L}\rangle_t, \tag{2.6.25}$$

and the infimum is attained for the martingale

$$N_t = -\tilde{L}_t, \tag{2.6.26}$$

where $\tilde{L}$ is the orthogonal martingale part of L in the GKW decomposition (2.6.5).

Let $M^{Q^0} = -\lambda\cdot M - \tilde{L}$. Since the minimal martingale measure satisfies the $R_{\mathcal{E}nt}(P)$ condition, Proposition 2.5.1 implies $-\lambda\cdot M \in \text{BMO}$. On the other hand, for any $s \le t$,

$$\langle \tilde{L}\rangle_t - \langle \tilde{L}\rangle_s \le \langle L\rangle_t - \langle L\rangle_s,$$

and hence Lemma 2.6.1 implies that $M^{Q^0} \in \mathrm{BMO}$. Therefore, from [41], it follows that the process $(\mathcal{E}_t(M^{Q^0}), t \in [0,T])$ is a martingale, and hence $dQ^0 = \mathcal{E}_T(M^{Q^0})dP$ defines an absolutely continuous martingale measure.

It is easy to see that $Y_t + \ln \mathcal{E}_t(M^{Q^0})$ is a local martingale under Q^0. Indeed, (2.6.25) and (2.6.5) imply

$$Y_t + \ln \mathcal{E}_t(M^{Q^0}) = Y_0 + L_t - \frac{1}{2}\langle \lambda \cdot M\rangle_t + \langle \lambda \cdot M, L\rangle_t + \frac{1}{2}\langle \tilde{L}\rangle_t$$
$$- (\lambda \cdot M)_t - \tilde{L}_t - \frac{1}{2}\langle \lambda \cdot M\rangle_t - \frac{1}{2}\langle \tilde{L}\rangle_t = Y_0 + ((\psi - \lambda)\cdot X)_t \quad (2.6.27)$$

which is a Q^0-local martingale, by the Girsanov theorem. Therefore,

$$Z_t = Y_t \mathcal{E}_t(M^{Q^0}) + \mathcal{E}_t(M^{Q^0}) \ln \mathcal{E}_t(M^{Q^0})$$

is a P-local martingale.

Let us show that $Q^0 \in \mathcal{M}^e_{\mathcal{E}nt}$ and that the process Z is a martingale. It is easy to see that

$$Z_t \geq -C\mathcal{E}_t(M^{Q^0}) - \frac{1}{e}.$$

Thus, Z is a local martingale majoring a uniformly integrable martingale, hence it is a supermartingale, and we have

$$Y_t \mathcal{E}_t(M^{Q^0}) + \mathcal{E}_t(M^{Q^0}) \ln \mathcal{E}_t(M^{Q^0}) \geq E(Y_T \mathcal{E}_T(M^{Q^0}) + \mathcal{E}_T(M^{Q^0}) \ln \mathcal{E}_T(M^{Q^0}) | F_t).$$

Therefore, from (2.6.2) and (2.6.24) we obtain

$$E(\mathcal{E}_{tT}(M^{Q^0}) \ln \mathcal{E}_{tT}(M^{Q^0}) | F_t) \leq Y_t \leq V_t \leq C. \quad (2.6.28)$$

The latter inequality implies that $E\mathcal{E}_T(M^{Q^0}) \ln \mathcal{E}_T(M^{Q^0}) < \infty$ and that Q^0 is optimal, and hence by Proposition 2.5.2, Q^0 is equivalent to P and $Q^0 \in \mathcal{M}^e$. Using the same arguments as before, we have that Z is a local martingale of class D and, therefore, it is a martingale (see, e.g., [19]). Now, the martingale property and the boundary condition imply that

$$Y_t = E(\mathcal{E}_{tT}(M^{Q^0}) \ln \mathcal{E}_{tT}(M^{Q^0}) | F_t). \quad (2.6.29)$$

Since $Q^0 \in \mathcal{M}^e_{\mathcal{E}nt}$, the relation $Y_t = V_t$ a.s. for all $t \in [0,T]$ results from (2.6.24) and (2.6.29), hence V is the unique bounded solution of Eq. (2.6.1), (2.6.2). □

Now we formulate Theorem 2.6.1(b) in the following equivalent martingale form.

Proposition 2.6.1. *Let the conditions of Theorem* 2.6.1(b) *be satisfied, and let*

$$V_t = V_0 + A_t + \int_0^t \varphi'_s dM_s + \tilde{m}_t, \quad \langle M, \tilde{m}\rangle = 0, \quad (2.6.30)$$

be the decomposition of the value process. Then the triple $(V_0, \varphi, \tilde{m})$ *is a solution of the martingale equation*

$$c + \int_0^T \psi'_s dM_s + \tilde{L}_T = \frac{1}{2}\langle \lambda \cdot M\rangle_T - \langle \lambda \cdot M, \psi \cdot M\rangle_T - \frac{1}{2}\langle \tilde{L}\rangle_T \quad (2.6.31)$$

and $c \in \mathbb{R}_+$, $\varphi \cdot M$, *and* $\tilde{m} \in \mathrm{BMO}$.

Conversely, if a triple $(c, \psi, \tilde{L})$ *such that* $c \in \mathbb{R}_+$ *and* $\psi \cdot M, \tilde{L} \in \mathrm{BMO}$ *solves* (2.6.31), *then the process* Y *defined by*

$$Y_t = E\left(\frac{1}{2}\langle \lambda \cdot M\rangle_{tT} - \langle \lambda \cdot M, \psi \cdot M\rangle_{tT} - \frac{1}{2}\langle \tilde{L}\rangle_{tT} | F_t\right) \quad (2.6.32)$$

is a bounded solution of (2.6.1)–(2.6.2) *and coincides with the value process.*

Proof. Relation (2.6.19) implies that Eq. (2.6.1), (2.6.2) is equivalent to the backward semimartingale equation

$$Y_t = Y_0 - \frac{1}{2}\langle \lambda \cdot M\rangle_t + \langle \lambda \cdot M, \psi \cdot M\rangle_t + \frac{1}{2}\langle \tilde{L}\rangle_t + \int_0^t \psi_s' dM_s + \tilde{L}_t, \tag{2.6.33}$$

$$Y_T = 0. \tag{2.6.34}$$

Since V solves Eq. (2.6.33), using the boundary condition (2.6.2), we obtain from (2.6.33) that the triple $(V_0, \varphi, \tilde{m})$ satisfies (2.6.31). Moreover, it follows from Lemma 2.6.1 that $\varphi \cdot M$, $\tilde{m} \in \mathrm{BMO}$.

Conversely, let the triple $(c, \psi, \tilde{L})$ solve (2.6.31) and Y be the process defined by (2.6.32). Using the martingale properties of the BMO-martingales $\psi \cdot M$ and $\tilde{L}$, we see that the martingale part of Y coincides with $V_0 + \int_0^t \psi_s' dM_s + \tilde{L}_t$, hence Y satisfies (2.6.1), (2.6.2). Since $\psi \cdot M, \tilde{L} \in \mathrm{BMO}$, the conditional Kunita–Watanabe inequality and (2.6.32) imply that Y is bounded and, therefore, Y coincides with the value process by Theorem 2.6.1(b). □

It is well known (see [30, 78]) that Q^* is the minimal entropy martingale measure if and only if

(i) $\mathcal{E}_T(M^{Q^*}) = e^{c+\int_0^T h_s' dX_s}$ for some constant c and an X-integrable h;
(ii) $E^{Q^*} \int_0^T h_s' dX_s = 0$ and $E^Q \int_0^T h_s' dX_s \geq 0$ for any $Q \in \mathcal{M}^e_{\mathcal{E}nt}$.

The sufficiency part of this assertion is difficult to verify, since condition (ii) involves the optimal martingale measure. The following consequence of Theorem 2.6.1 shows that the integrand h of the minimal entropy martingale measure can be expressed in terms of the value process V, and since V solves Eq. (2.6.1), (2.6.2), condition (ii) is automatically satisfied.

Corollary 2.6.1. *A martingale measure Q^* is the minimal entropy martingale measure if and only if the corresponding density admits representation*

$$\mathcal{E}_T(M^{Q^*}) = \exp\left(V_0 + \int_0^T (\varphi_s - \lambda_s)' dX_s\right), \tag{2.6.35}$$

where φ is the integrand in the GKW decomposition of the martingale part m of the value process.

Proof. It follows from Theorem 2.6.1 and relation (2.6.19) that V satisfies the equation

$$V_t = V_0 - \frac{1}{2}\langle \lambda \cdot M\rangle_t + \langle \lambda \cdot M, \varphi \cdot M\rangle_t + \frac{1}{2}\langle \tilde{m}\rangle_t + (\varphi \cdot M)_t + \tilde{m}_t.$$

Taking the exponentials of both sides of the latter equation and using the definitions of the process X and the Doleans-Dade exponential, we obtain

$$e^{V_t} = \mathcal{E}_t^{-1}(-\lambda \cdot M - \tilde{m}) \exp\left(V_0 + \int_0^t (\varphi_s - \lambda_s)' dX_s\right), \tag{2.6.36}$$

and from the boundary condition (2.6.2), we have

$$\mathcal{E}_T(-\lambda \cdot M - \tilde{m}) = \exp\left(V_0 + \int_0^T (\varphi_s - \lambda_s)' dX_s\right). \tag{2.6.37}$$

Now, since by Theorem 2.6.1, Q^* is the minimal entropy martingale measure if and only if it satisfies (2.6.14), the representation (2.6.35) follows from (2.6.14) and (2.6.37).

Note that for the process $\varphi - \lambda$, condition (ii) is satisfied. Indeed, (2.6.36) implies

$$V_t + \ln \mathcal{E}_t(-\lambda \cdot M - \tilde{m}) = V_0 + \int_0^t (\varphi_s - \lambda_s)' dX_s, \tag{2.6.38}$$

and by the optimality principle,

$$\left(\int_0^t (\varphi_s - \lambda_s)' dX_s,\ t \in [0, T] \right)$$

is a Q^*-martingale, and hence

$$E^{Q^*} \int_0^T (\varphi_s - \lambda_s)' dX_s = 0.$$

For any $Q \in \mathcal{M}^e_{\mathcal{E}nt}$ and $x \in [0, 1]$, we set $Q_x = xQ + (1 - x)Q^*$. Then

$$Z^x_T = x\mathcal{E}_T(M^Q) + (1 - x)\mathcal{E}_T(M^{Q^*})$$

is the corresponding density and according to [30, Lemma 2.1], the function $f(x) = EZ^x_T \ln Z^x_T$ is differentiable in x and

$$\frac{d}{dx} EZ^x_T \ln Z^x_T|_{x=0} = E \ln \mathcal{E}_T(M^{Q^*}) \big(\mathcal{E}_T(M^Q) - \mathcal{E}_T(M^{Q^*})\big).$$

Moreover, Q^* is optimal if and only if $\frac{d}{dx} f|_{x=0} \geq 0$. Therefore, from (2.6.38) and the latter inequality, we obtain

$$E^Q \int_0^T (\varphi_s - \lambda_s)' dX_s = E^Q \ln \mathcal{E}_T(-\lambda \cdot M - \tilde{m}) - V_0$$

$$\geq E \ln \mathcal{E}_T(M^{Q^*}) \big(\mathcal{E}_T(M^Q) - \mathcal{E}_T(M^{Q^*})\big) \geq 0 \tag{2.6.39}$$

for any $Q \in \mathcal{M}^e_{\mathcal{E}nt}$, and hence (ii) is satisfied. □

Corollary 2.6.2. *If there exists a martingale measure $\tilde{Q}$ whose density satisfies the reverse Hölder inequality $R_{\mathcal{E}nt}(P)$, then*

$$V_t = \bar{V}_t \tag{2.6.40}$$

Proof. Denote by $\mathcal{R}_{\mathcal{E}nt}(X)$ the set of martingale measures Q whose densities Z^Q satisfy the $R_{\mathcal{E}nt}(P)$ inequality. By Corollary 2.6.1, the minimal entropy martingale measure Q^* is in $\mathcal{R}_{\mathcal{E}nt}(X)$. Therefore,

$$\begin{aligned} V_t &= \operatorname*{ess\,inf}_{Q \in \mathcal{M}^e_{\mathcal{E}nt}} E^Q(\ln \mathcal{E}_{tT}(M^Q)|F_t) = \operatorname*{ess\,inf}_{Q \in \mathcal{R}_{\mathcal{E}nt}(X)} E^Q(\ln \mathcal{E}_{tT}(M^Q)|F_t) \\ &= \operatorname*{ess\,inf}_{Q \in \mathcal{R}_{\mathcal{E}nt}(X)} E^Q\left(M^Q_{tT} - \langle M^Q\rangle_{tT} + \frac{1}{2}\langle M^Q\rangle_{tT}|F_t\right) = \frac{1}{2} \operatorname*{ess\,inf}_{Q \in \mathcal{R}_{\mathcal{E}nt}(X)} E^Q(\langle M^Q\rangle_{tT}|F_t) \end{aligned}$$

since $Q \in \mathcal{R}_{\mathcal{E}nt}(X)$ implies $M^Q \in \text{BMO}$ (Proposition 2.5.1), and according to [21, Proposition 7], from $M^Q \in \text{BMO}(P)$ we have that the process $M^Q - \langle M^Q\rangle$ is a BMO-martingale with respect to the measure Q, and hence $E^Q(M^Q_{tT} - \langle M^Q\rangle_{tT}|F_t) = 0$.

We recall that $M_{tT} = M_T - M_t$ and $\langle M\rangle_{tT} = \langle M\rangle_T - \langle M\rangle_t$. □

This expression of the value process allows us to determine easily the minimal entropy martingale measure in some particular cases.

Proposition 2.6.2. *Assume that the minimal martingale measure Q^{min} belongs to the class $\mathcal{M}^e_{\mathcal{E}nt}$ and $\lambda \cdot X$ is a martingale with respect to any $Q \in \mathcal{M}^e_{\mathcal{E}nt}$. Then the following assertions are equivalent:*

(1) *the minimal entropy martingale measure* Q^* *coincides with the minimal martingale measure* $Q^{\min}$;
(2) *the mean variance tradeoff admits the representation*

$$\langle\lambda \cdot M\rangle_T = c + \int_0^T \psi'_s dX_s \tag{2.6.41}$$

for some constant c *and* X*-integrable process* ψ *such that*

$$E^{\min} \int_0^T \psi'_s dX_s = 0, \quad E^Q \int_0^T \psi'_s dX_s \geq 0$$

for any $Q \in \mathcal{M}^e_{\mathcal{E}nt}$.

Proof. (1)⇒(2). Let $Q^* = Q^{\min}$. Then by Corollary 2.6.2,

$$\mathcal{E}_T(-\lambda \cdot M) = \exp\left(V_0 + \int_0^T (\varphi_s - \lambda_s)' dX_s\right), \tag{2.6.42}$$

where φ is defined by (2.6.30). It follows from (2.6.42) that

$$\exp\left\{-\int_0^T \lambda'_s dM_s - \frac{1}{2}\langle\lambda \cdot M\rangle_T\right\} = \exp\left\{V_0 + \int_0^T \varphi'_s dX_s - \int_0^T \lambda'_s dM_s - \langle\lambda \cdot M\rangle_T\right\},$$

which implies

$$\frac{1}{2}\langle\lambda \cdot M\rangle_T = V_0 + \int_0^T \varphi'_s dX_s, \tag{2.6.43}$$

and hence (2.6.41) is satisfied with $\psi = 2\varphi$ and $c = 2V_0$.

Since

$$E^Q \int_0^T \lambda'_s dX_s = 0$$

for any $Q \in \mathcal{M}^e_{\mathcal{E}nt}$, it follows from (2.6.39) that

$$E^Q \int_0^T \varphi'_s dX_s \geq E^Q \int_0^T \lambda'_s dX_s = 0$$

for any $Q \in \mathcal{M}^e_{\mathcal{E}nt}$. Moreover, (2.6.36) implies

$$V_t + \ln \mathcal{E}_t(-\lambda \cdot M) = V_0 + \int_0^t (\varphi_s - \lambda_s)' dX_s \tag{2.6.44}$$

and by the optimality principle,

$$\left(\int_0^t (\varphi_s - \lambda_s)' dX_s,\ t \in [0, T]\right)$$

is a $Q^{\min}$-martingale, hence

$$E^{\min}\int_0^T \varphi_s' dX_s = E^Q \int_0^T \lambda_s' dX_s = 0.$$

(2)⇒(1). If (2.6.41) is satisfied, then

$$\mathcal{E}_T(-\lambda\cdot M) = \exp\left\{-\int_0^T \lambda_s' dX_s + \frac{1}{2}\langle\lambda\cdot M\rangle_T\right\} = \exp\left\{\frac{c}{2} + \int_0^T (\tfrac{1}{2}\psi_s - \lambda_s)' dX_s\right\},$$

and it is obvious that

$$E^Q \int_0^T (\tfrac{1}{2}\psi_s - \lambda_s)' dX_s \geq 0$$

for any $Q \in \mathcal{M}^e_{\mathcal{E}nt}$ and

$$E^{\min}\int_0^T (\tfrac{1}{2}\psi_s - \lambda_s)' dX_s = 0,$$

hence $Q^* = Q^{\min}$ by Theorem 2.3 of Frittelli (given above). □

Corollary 2.6.3. *Assume that the mean variance tradeoff $\langle\lambda\cdot M\rangle_T$ is bounded. Then $Q^* = Q^{\min}$ if and only if* (2.6.41) *is satisfied for some constant c and X-integrable process ψ such that*

$$\left(\int_0^t \psi_s' dX_s,\ t\in[0,T]\right)$$

is $Q^{\min}$-martingale.

The proof follows from Proposition 2.3.2 since the boundedness of $\langle\lambda\cdot M\rangle_T$ implies that $\lambda\cdot X$ is a martingale with respect to any $Q \in \mathcal{M}^e_{\mathcal{E}nt}$. Moreover, if equality (2.6.41) is satisfied and if

$$\left(\int_0^t \psi_s' dX_s,\ t\in[0,T]\right)$$

is a martingale with respect to some $Q \in \mathcal{M}^e_{\mathcal{E}nt}$, then this process is bounded and is a martingale with respect to any $Q \in \mathcal{M}^e_{\mathcal{E}nt}$.

Remark 2.6.1. Condition (2.6.41) is satisfied in the case of "almost complete" diffusion models (see, e.g., [73]) where the market price of risk is measurable with respect to the filtration generated by the asset price process.

Corollary 2.6.4. *The mean variance tradeoff $\langle\lambda\cdot M\rangle_T$ is deterministic if and only if the minimal entropy martingale measure coincides with the minimal martingale measure and $\varphi = 0$ $\mu^{\langle M\rangle}$-a.e., where φ is defined by* (2.6.13) *and $\mu^{\langle M\rangle}$ is the Dolean measure of $\langle M\rangle$.*

The proof immediately follows from Proposition 2.6.1.

Proposition 2.6.3. *Assume that the minimal martingale measure exists and satisfies the reverse Hölder $R_{\mathcal{E}nt}$-inequality. Then the density of the minimal entropy martingale measure is of the form*

$$Z_T^{Q^*} = \frac{\exp\{-\int_0^T \lambda_s' dX_s\}}{E\exp\{-\int_0^T \lambda_s' dX_s\}} \tag{2.6.45}$$

if and only if

$$\exp\left\{-\frac{1}{2}\langle\lambda\cdot M\rangle_T\right\} = c + \hat{m}_T \tag{2.6.46}$$

for some constant c and a martingale $\hat{m}$ strongly orthogonal to M.

Proof. Let $Z_T(Q^*)$ be of the form (2.6.45). By (2.6.14), we have

$$\mathcal{E}_T(-\lambda\cdot M - \tilde{m}) = \frac{\exp\{-\int_0^T \lambda'_s dX_s\}}{E\exp\{-\int_0^T \lambda'_s dX_s\}},$$

which implies that

$$\exp\left\{-\frac{1}{2}\langle\lambda\cdot M\rangle_T\right\} = c\exp\left\{-\tilde{m}_T - \frac{1}{2}\langle\tilde{m}\rangle_T\right\} = c\mathcal{E}_T(\tilde{m}) = c + c\int_0^T \mathcal{E}_s(\tilde{m})d\tilde{m}_s,$$

where the martingale $\tilde{m}$ orthogonal to M is defined by (2.6.13) and belongs to the class BMO according to Lemma 2.6.1. Therefore, (2.6.46) is satisfied with

$$\hat{m}_t = c\int_0^t \mathcal{E}_s(\tilde{m})d\tilde{m}_s,$$

which is a martingale according to [41].

Conversely, let (2.6.46) be satisfied. Then using the Itô formula for $\ln(c+\hat{m}_t)$, from (2.6.46), we have

$$\ln c + \int_0^T \frac{1}{c+\hat{m}_s}d\hat{m}_s = -\frac{1}{2}\langle\lambda\cdot M\rangle_T + \frac{1}{2}\left\langle\int_0^{\cdot} \frac{1}{c+\hat{m}_s}d\hat{m}_s\right\rangle_T,$$

which implies that the triple

$$-\ln c, \quad \psi = 0, \quad \tilde{L} = -\int_0^t \frac{1}{c+\hat{m}_s}d\hat{m}_s$$

is a solution of the martingale equation (2.6.31). The martingale $\frac{1}{c+\hat{m}}\cdot\hat{m}$ belongs to the class BMO, since by (2.6.46), $c+\hat{m}_t \le 1$ and Proposition 2.5.1 with the Jensen inequality imply

$$c + \hat{m}_t = E(e^{-\frac{1}{2}\langle\lambda\cdot M\rangle_T}/\mathcal{F}_t) \ge E(e^{-\frac{1}{2}\langle\lambda\cdot M\rangle_{tT}}/\mathcal{F}_t) \ge e^{-\frac{1}{2}E(\langle\lambda\cdot M\rangle_{tT}/\mathcal{F}_t)} \ge e^{\frac{1}{2}C}.$$

Since the solution of (2.6.31) is unique in the class $\mathbb{R}_+\times$BMO $\times$ BMO, we obtain that $\varphi = 0$. Therefore, it follows from Corollary 2.6.2 that $Z_T(Q^*)$ is of the form (2.6.45). $\square$

2.7. The Itô Process Model

We consider the diffusion model for the financial market as in Karatzas et al. [40] and Laurent and Pham [49]. Let $W = (W^1,\dots,W^n)$ be an n-dimensional standard Brownian motion defined on a complete probability space (Ω, F, P) equipped with the P-augmentated filtration generated by W, $F = (F_t, t\in[0,T])$. Denote by $W^l = (W^1,\dots,W^d)$ and $W^\perp = (W^{d+1},\dots,W^n)$ the d- and $(n-d)$-dimensional Brownian motions, respectively.

Assume that there are d risky assets (stocks) and a bond traded in the market. For simplicity, the bond price is assumed to be 1 at all times and the stock price dynamics is given by

$$dX_t = \operatorname{diag}(X_t)(\mu_t dt + \sigma_t dW_t^l), \quad t\in[0,T], \tag{2.7.1}$$

where $\operatorname{diag}(X)$ denotes the diagonal $(d\times d)$-matrix with diagonal entries $(X^1,\dots,X^d)$.

The market coefficients: the d-dimensional vector process μ of stock appreciation rates and the volatility $(d \times d)$-matrix σ are progressively measurable with respect to F. We also require that for any $t \in [0,T]$, the volatility matrix is nonsingular almost surely. We take $n > d$, so that there are more sources of uncertainty than stocks available for trading and the market is incomplete in the Harrison and Pliska sense (see [35]).

Straightforward calculations yield that in this case, $\lambda = \mathrm{diag}(X^{-1})(\sigma\sigma')^{-1}\mu$, where σ' denotes the transposition of σ,

$$\int_0^t \lambda_s' dM_s = \int_0^t \theta_s' dW_s^l, \quad \langle \lambda \cdot M \rangle_t = \int_0^t \|\theta_s\|^2 ds$$

is the mean variance tradeoff, and $\theta = \sigma^{-1}\mu$ is the market price of risk. As before, we denote by $\mathcal{M}^e$ the set of equivalent martingale measures of X. Let $\mathcal{K}(\sigma)$ be the class of F-predictable $\mathbb{R}^{n-d}$-valued processes ν such that $\int_0^T \|\nu_t\|^2 dt < \infty$, a.s. Since σ is nonsingular, by the Itô representation theorem, any local martingale N strongly orthogonal to $M = \mathrm{diag}(X)\sigma \cdot W^l$ admits the integral representation

$$N_t = \int_0^t \nu_s' dW_s^{\perp}$$

for some $\nu \in \mathcal{K}(\sigma)$, and from (2.5.3) the density of any martingale measures is expressed as

$$Z_t^{\nu} = \mathcal{E}_t\left(-\int_0^t \theta_s' dW_s^l + \int_0^t \nu_s' dW_s^{\perp}\right), \quad t \in [0,T], \tag{2.7.2}$$

for some $\nu \in \mathcal{K}(\sigma)$. Let

$$\mathcal{K}_{\mathcal{E}nt}(\sigma) = \{\nu \in \mathcal{K}(\sigma) : EZ_T^{\nu} = 1,\ EZ_T^{\nu} \ln Z_T^{\nu} < \infty\}.$$

Then the subclass $\mathcal{M}^e_{\mathcal{E}nt}$ of equivalent martingale measures is given by

$$\mathcal{M}^e_{\mathcal{E}nt} = \{P^{\nu} : dP^{\nu}/dP = Z_T^{\nu},\ \nu \in \mathcal{K}_{\mathcal{E}nt}(\sigma)\}, \tag{2.7.3}$$

and condition (B) is equivalent to $\mathcal{K}_{\mathcal{E}nt}(\sigma) \neq \emptyset$.

Assume that the following condition holds.

(C) the mean variance tradeoff is bounded, i.e.,

$$\int_0^T \|\theta_s\|^2 ds \leq C \quad \text{a.s. for some } C > 0.$$

Remark 2.7.1. This condition is satisfied if the market price of risk θ is bounded. Note that condition (C) implies that the minimal martingale measure exists, i.e., $E\mathcal{E}_T(-\int_0^\cdot \theta_s' dW_s^l) = 1$, and satisfies the reverse Hölder $R_{\mathcal{E}nt}(P)$ inequality, since for any stopping time τ,

$$E(\mathcal{E}_{\tau T}(-\lambda \cdot M) \ln \mathcal{E}_{\tau T}(-\lambda \cdot M)|F_\tau) = E(\mathcal{E}_{\tau T}(-\lambda \cdot M)\langle \lambda \cdot M \rangle_{\tau T}|F_\tau) \leq C. \tag{2.7.4}$$

According to Corollary 2.6.3 and (2.7.2), problem (2.5.10) is equivalent to

$$\frac{1}{2} \inf_{\nu \in \mathcal{K}_{\mathcal{E}nt}(\sigma)} E^{\nu} \int_0^T (\|\theta_s\|^2 + \|\nu_s\|^2) ds, \tag{2.7.5}$$

and the corresponding value process takes the form

$$V_t = \frac{1}{2} \operatorname*{ess\,inf}_{\nu \in \mathcal{K}_{\mathcal{E}nt}(\sigma)} E^{\nu}\left(\int_t^T (\|\theta_s\|^2 + \|\nu_s\|^2) ds | F_t\right). \tag{2.7.6}$$

By the martingale representation theorem, the martingale part of the value process is expressed as a stochastic integral

$$m_t = \int_0^t \varphi_s' dW_s^l + \int_0^t \varphi_s^{\perp\prime} dW_s^\perp \tag{2.7.7}$$

and it is easy to show (e.g., since the essential infimum in both expressions is attained) that in this case,

$$\operatorname*{ess\,inf}_{Q\in\mathcal{M}^e_{\mathcal{E}nt}} \left[\frac{1}{2}\langle M^Q\rangle_t + \langle M^Q, m\rangle_t\right] = \int_0^t \inf_{\nu\in\mathbb{R}^{n-d}} \left[\frac{1}{2}\|\theta_s\|^2 + \frac{1}{2}\|\nu\|^2 - \theta_s'\varphi_s + \nu'\varphi_s^\perp\right] ds. \tag{2.7.8}$$

Since condition (C) implies that the minimal martingale measure satisfies the $R_{\mathcal{E}nt}(P)$ inequality and the filtration F is continuous, the following statement follows from Theorem 2.6.1(b) and Eq. (2.7.8) as a corollary.

Theorem 2.7.1. *Let condition* (C) *be satisfied. Then the value process V is a unique bounded positive solution of the BSDE*

$$Y_t = Y_0 - \int_0^t \inf_{\nu\in\mathbb{R}^{n-d}} \left[\frac{1}{2}\|\theta_s\|^2 + \frac{1}{2}\|\nu\|^2 - \theta_s'\psi_s + \nu'\psi_s^\perp\right] ds + \int_0^t \psi_s' dW_s^l + \int_0^t \psi_s^{\perp\prime} dW_s^\perp, \quad Y_T = 0. \tag{2.7.9}$$

Moreover, ν^* is optimal if and only if

$$\nu_t^* = -\varphi_t^\perp \quad dt \times dP\text{-a.e.}, \tag{2.7.10}$$

i.e., the density of the minimal entropy martingale measure has the form

$$Z_T^{\nu^*} = \mathcal{E}_T\left(-\int_0^\cdot \theta_s' dW_s^l - \int_0^\cdot \varphi_s^{\perp\prime} dW_s^\perp\right). \tag{2.7.11}$$

Remark 2.7.2. Since the essential infimum in (2.7.8) is attained for $\nu_t^* = -\varphi_t^\perp$, Eq. (2.7.9) is equivalent to

$$Y_t = Y_0 - \int_0^t \left[\frac{1}{2}\|\theta_s\|^2 - \theta_s'\psi_s - \frac{1}{2}\|\psi_s^\perp\|^2\right] ds + \int_0^t \psi_s' dW_s^l + \int_0^t \psi_s^{\perp\prime} dW_s^\perp, \quad Y_T = 0. \tag{2.7.12}$$

Note that the martingale equation (2.6.31) equivalent to (2.7.9) takes the form

$$Y_0 + \int_0^T \psi_s' dW_s^l + \int_0^T \psi_s^{\perp\prime} dW_s^\perp = \int_0^T \left[\frac{1}{2}\|\theta_s\|^2 - \theta_s'\psi_s - \frac{1}{2}\|\psi_s^\perp\|^2\right] ds. \tag{2.7.13}$$

Now we consider two extreme cases in which Eq. (2.7.9) can be solved explicitly. These specific examples were already studied by Pham et al. [73] and Laurent and Pham [49] in connection with the variance optimal martingale measure using different methods.

Case 1. An "almost complete" diffusion model. Assume that the market price of risk is adapted to the filtration F^l generated by the Brownian motion W^l, i.e., $\theta = \theta(t, W^l), t \in [0,T]$). Denote by $Q^{\min}$ the minimal martingale measure. Let $E^{\min}$ be the expectation with respect to this measure.

By the Girsanov theorem, the process $\tilde{W}^l$ defined by

$$\tilde{W}_t^l = \int_0^t \theta(s, W^l) ds + W_t^l \tag{2.7.14}$$

is the Brownian motion with respect to the measure $Q^{\min}$, and by the integral representation theorem (see, e.g., [51, Theorem 7.12]), any $Q^{\min}$-local martingale adapted to F^l is represented as a stochastic integral; hence

$$\int_0^T \|\theta_s\|^2 ds = E^{\min} \int_0^T \|\theta_s\|^2 ds + \int_0^T \tilde{\psi}'_s d\tilde{W}^l_s. \tag{2.7.15}$$

Obviously, from condition (C) we have

$$\int_0^t \tilde{\psi}'_s d\tilde{W}^l_s \in \mathrm{BMO}\,.$$

Corollary 2.7.1. *The triple* $(c, \varphi, \varphi^\perp)$, *where*

$$c = \frac{1}{2} E^{\min}\left(\int_0^T \|\theta_s\|^2 ds\right), \quad \varphi = \frac{1}{2}\tilde{\psi}, \quad \varphi^\perp = 0,$$

is a unique solution of the martingale equation (2.7.13) *in the class* $\mathbb{R}_+ \times \mathrm{BMO} \times \mathrm{BMO}$. *The process*

$$\frac{1}{2} E^{\min}\left(\int_t^T \|\theta_s\|^2 ds\right) |F^l_t)$$

coincides with the value process V, *and the minimal entropy martingale measure coincides with the minimal martingale measure, i.e.,* $\nu^* = 0$ *and*

$$Z^{Q^*}_T = \mathcal{E}_T(-\theta \cdot W^l).$$

Proof. Let consider the process

$$Y_t = \frac{1}{2} E^{\min}\left(\int_t^T \|\theta_s\|^2 ds | F^l_t\right). \tag{2.7.16}$$

Obviously, Y is bounded (by condition (C)) and positive. Since θ is F^l adapted, we have

$$Y_t = \frac{1}{2} E^{\min}\left(\int_0^T \|\theta_s\|^2 ds | F^l_t\right) - \frac{1}{2}\int_0^t \|\theta_s\|^2 ds. \tag{2.7.17}$$

Therefore, it follows from (2.7.14), (2.7.15), and (2.7.17) that

$$Y_t = Y_0 - \int_0^t \left(\frac{1}{2}\|\theta_s\|^2 - \frac{1}{2}\theta'_s \tilde{\psi}_s\right) ds + \frac{1}{2}\int_0^t \tilde{\psi}'_s dW^l_s, \tag{2.7.18}$$

which means that Y is a bounded positive solution of (2.7.9) and $(c, \frac{1}{2}\tilde{\psi}, 0)$ is the unique solution of (2.7.13) in the class $\mathbb{R}_+ \times \mathrm{BMO} \times \mathrm{BMO}$ (see Proposition 2.6.1). Therefore, by Theorem 2.6.1(b), Y coincides with the value process

$$V_t = \frac{1}{2} E^{\min}\left(\int_t^T \|\theta_s\|^2 ds | F_t\right)$$

and hence the minimal martingale measure is optimal. $\square$

Remark 2.7.3. Since the market price of risk is adapted to the filtration F^l, the process $\tilde{\psi}$ in (2.7.15) is F^l-predictable. According to Corollary 2.6.4, the necessary and sufficient condition for $Q^* = Q^{\min}$ is that the mean variance tradeoff admits representation (2.7.15) for some F-predictable $\tilde{\psi}$ such that the process

$$\int_0^t \tilde{\psi}_s' d\tilde{W}_s^l$$

is a martingale with respect to $Q^{\min}$.

Case 2. Assume that the market price of risk is adapted to the filtration $F^\perp$ generated by the Brownian motion $W^\perp$, i.e.,

$$\theta = \theta(t, W^\perp),\ t \in [0,T]).$$

Since θ is $F^\perp$ adapted, by the integral representation theorem, there exists an $F^\perp$ adapted process g such that

$$\exp\left\{-\frac{1}{2}\int_0^T \|\theta_s\|^2 ds\right\} = E\exp\left\{-\frac{1}{2}\int_0^T \|\theta_s\|^2 ds\right\} + \int_0^T g_s' dW_s^\perp. \tag{2.7.19}$$

Corollary 2.7.2. *The triple* $(\ln(1/c), \psi, \psi^\perp)$, *where*

$$c = E\exp\left\{-\frac{1}{2}\int_0^T \|\theta_s\|^2 ds\right\}, \quad \varphi = 0, \quad \psi_t^\perp = -\frac{g_t}{E\exp\{-\frac{1}{2}\int_0^T \|\theta_s\|^2 ds\} + \int_0^t g_s' dW_s^\perp} \tag{2.7.20}$$

is a unique solution of (2.7.13) *in the class* $\mathbb{R}_+ \times \mathrm{BMO} \times \mathrm{BMO}$.

The process

$$\frac{1}{2}E\left(\int_t^T (\|\theta_s\|^2 - \|\psi_s^\perp\|^2) ds | F_t^\perp\right)$$

coincides with the value process V, *and the density of the minimal entropy martingale measure is of the form*

$$Z_T^{Q^*} = \frac{\exp\{-\int_0^T \lambda_s' dX_s\}}{E\exp\{-\int_0^T \lambda_s' dX_s\}}. \tag{2.7.21}$$

Proof. By the Itô formula, we have

$$\ln\left(c + \int_0^T g_s' dW_s^\perp\right) = \ln c - \int_0^T \psi_s^{\perp\prime} dW_s^\perp - \frac{1}{2}\int_0^T \|\psi_s^\perp\|^2 ds \tag{2.7.22}$$

and from (2.7.19) we obtain

$$\ln(1/c) + \int_0^T \psi_s^{\perp\prime} dW_s^\perp = \frac{1}{2}\int_0^T (\|\theta_s\|^2 - \|\psi_s^\perp\|^2) ds, \tag{2.7.23}$$

which coincides with Eq. (2.7.13) for $\psi = 0$.

This means that the triple $(\ln(1/c), \psi, \psi^\perp)$ defined by (2.7.20) satisfies (2.7.13). Moreover, condition (C) implies $\psi^\perp W^\perp \in \mathrm{BMO}$, and since there is a unique solution of (2.7.13) in the class $\mathbb{R}_+ \times \mathrm{BMO} \times \mathrm{BMO}$, we obtain $\varphi = \psi = 0$ and $\varphi^\perp = \psi^\perp$, where φ and $\varphi^\perp$ are defined by (2.7.7). Therefore, the process

$$Y_t = \frac{1}{2}E\left(\int_t^T (\|\theta_s\|^2 - \|\psi_s^\perp\|^2) ds | F_t\right)$$

solves Eq. (2.7.9) and by Theorem 2.6.1(b), it coincides with the value process V, since $\int_0^t \psi_s^{\perp\prime} dW_s^{\perp} \in$ BMO and condition (C) imply that the process Y is bounded. Since $\varphi = 0$, Corollary 2.6.2 implies that the density of the minimal entropy martingale measure admits representation (2.7.21). $\square$

2.8. Diffusion Model

We assume that the dynamics of the assets price process is determined by the following system of stochastic differential equations:

$$dX_t = \mathrm{diag}(X_t)(\mu(t, X_t, Y_t)dt + \sigma^l(t, X_t, Y_t)dW_t^l), \tag{2.8.1}$$

$$dY_t = b(t, X_t, Y_t)dt + \delta(t, X_t, Y_t)dW_t^l + \sigma^{\perp}(t, X_t, Y_t)dW_t^{\perp}. \tag{2.8.2}$$

Assume that the following conditions hold:

(D1) the coefficients μ, b, δ, σ^l, and $\sigma^{\perp}$ are measurable and bounded;

(D2) the $(n \times n)$-matrix function $\sigma\sigma'$ is uniformly elliptic, i.e., there is a constant $c > 0$ such that

$$(\sigma(s, x, y)\lambda, \sigma(s, x, y)\lambda) \geq c|\lambda|^2$$

for all $s \in [0, T]$, $x \in \mathbb{R}_+^d$, $y \in \mathbb{R}^{n-d}$, and $\lambda \in \mathbb{R}^n$, where

$$\sigma(t, x, y) = \begin{pmatrix} \sigma^l(t, x, y) & 0 \\ \delta(t, x, y) & \sigma^{\perp}(t, x, y), \end{pmatrix}.$$

In addition, we assume that

(D3) system (2.8.1), (2.8.2) admits a unique strong solution.

Let us introduce the value function

$$V(t, x, y) = \frac{1}{2} \inf_{\nu \in \mathcal{K}_{\mathcal{E}nt}^M(\sigma)} E^{\nu} \left(\int_t^T (\|\theta(s, X_s, Y_s)\|^2 + \|\nu(s, X_s, Y_s)\|^2) ds / X_t = x, Y_t = y \right),$$

where $\theta = \sigma^{l^{-1}} \mu$ and $\mathcal{K}_{\mathcal{E}nt}^M(\sigma)$ is the class of feedback controls from $\mathcal{K}_{\mathcal{E}nt}(\sigma)$, i.e., controls $\nu \in \mathcal{K}_{\mathcal{E}nt}(\sigma)$ expressed in the form $\nu(t, X_t, Y_t)$ for some measurable function $\nu(t, x, y)$, $t \in [0, T]$, $x \in \mathbb{R}_+^d$, $y \in \mathbb{R}^{n-d}$.

Theorem 2.8.1. *Let conditions* (D1)–(D3) *be satisfied. Then the value function $V(t, x, y)$ admits all first order generalized derivatives V_x and V_y and the generalized L-operator LV (in the sense of Definition* 1.7.1 *of the Appendix) is a unique bounded positive solution of the equation*

$$LV(t, x, y) - \theta'(t, x, y)\delta'(t, x, y)V_y(t, x, y) + V_y'(t, x, y)b(t, x, y) + \frac{1}{2}\|\theta(t, x, y)\|^2$$
$$+ \inf_{\nu \in \mathbb{R}^{n-d}} \left[\frac{1}{2}\|\nu\|^2 + \nu' \sigma^{\perp\prime}(t, x, y)V_y(t, x, y) \right] = 0 \quad dt\, dx\, dy\text{-a.s.} \tag{2.8.3}$$

with the boundary condition

$$V(T, x, y) = 0. \tag{2.8.4}$$

Moreover, $\nu^* = -\sigma^{\perp} V_y$, and the density of the minimal entropy martingale measure is of the form

$$Z_T^* = \mathcal{E}_T \left(-\int_0^{\cdot} \theta(s, X_s, Y_s) dW_s^l - \int_0^{\cdot} (\sigma^{\perp} V_y)(s, X_s, Y_s) dW_s^{\perp} \right).$$

Proof. Existence. Since (X, Y) is a Markov process, the feedback controls are sufficient and the value process is expressed by

$$V_t = V(t, X_t, Y_t) \quad \text{a.s.} \tag{2.8.5}$$

(one can show this fact, e.g., similarly to [8]).

Moreover, the value process satisfies Eq. (2.7.9); therefore, it is an Itô process. It follows from assumptions (D1) and (D2) that the value process is bounded and Theorem 2.6.1(b) implies that its martingale part is in BMO. Hence the finite variation part of V_t is of integrable variation. Thus, from (2.8.5), we have that $V(t, X_t, Y_t)$ is an Itô process of the form (1.7.15) (see the Appendix). Therefore, Proposition 1.7.1 of the Appendix implies that the function $V(t, x, y)$ admits a generalized L-operator, all first-order generalized derivatives, and can be represented as

$$\begin{aligned}V(t, X_t, Y_t) = V_0 + \int_0^t \big(V'_x(s, X_s, Y_s)\operatorname{diag}(X_s)\sigma^l(s, X_s, Y_s) + V'_y(s, X_s, Y_s)\delta(s, X_s, Y_s)\big)dW^l_s \\ + \int_0^t V'_y(s, X_s, Y_s)\sigma^{\perp}(s, X_s, Y_s)dW^{\perp}_s + \int_0^t LV(s, X_s, Y_s)ds \\ + \int_0^t (V'_x(s, X_s, Y_s)\operatorname{diag}(X_s)\mu(s, X_s, Y_s) + V'_y(s, X_s, Y_s)b(s, X_s, Y_s))ds, \quad (2.8.6)\end{aligned}$$

where LV is the generalized L-operator defined in the Appendix (Definition 1.7.1).

On the other hand, the value process is a solution of (2.7.9) and by the uniqueness of the canonical decomposition of semimartingales, comparing the martingale parts of (2.8.6) and (2.7.9), we have that $dt \times dP$-a.e.

$$\varphi_t = \sigma^{l'}(t, X_t, Y_t)\operatorname{diag}(X_t)V_x(t, X_t, Y_t) + \delta'(t, X_t, Y_t)V_y(t, X_t, Y_t), \quad (2.8.7)$$

$$\varphi^{\perp}_t = \sigma^{\perp'}(t, X_t, Y_t)V_y(t, X_t, Y_t). \quad (2.8.8)$$

Then, equating the processes of bounded variation of the same equations and taking into account (2.8.7) and (2.8.8), we obtain

$$\begin{aligned}\int_0^t \Big[V'_y(s, X_s, Y_s)b(s, X_s, Y_s) + \frac{1}{2}\|\theta(s, X_s, Y_s)\|^2 - \theta'(s, X_s, Y_s)\delta'(s, X_s, Y_s)V_y(s, X_s, Y_s) \\ + LV(s, X_s, Y_s) + \inf_{\nu\in\mathbb{R}^{n-d}}\Big(\frac{1}{2}\|\nu\|^2 + \nu'\sigma^{\perp'}(s, X_s, Y_s)V_y(s, X_s, Y_s)\Big)\Big]ds = 0, \quad (2.8.9)\end{aligned}$$

which gives that $V(t, x, y)$ solves the Bellman equation (2.8.3).

Uniqueness. Let $\tilde{V}(t, x, y)$ be a bounded positive solution of (2.8.3), (2.8.4) from the class V^L. Then using the generalized Itô formula (Proposition 1.7.1 of the Appendix) and Eq. (2.8.3), we obtain that $\tilde{V}(t, X_t, Y_t)$ is a solution of (2.7.9), and hence $\tilde{V}(t, X_t, Y_t)$ coincides with the value process V by Theorem 2.7.1. Therefore, $\tilde{V}(t, X_t, Y_t) = V(t, X_t, Y_t)$ a.s. and $\tilde{V} = V$, $dtdxdy$-a.e.

It is obvious that Theorem 2.7.1 and Eq. (2.8.8) imply that $\nu^* = -\sigma^{\perp'}V_y$. □

Analogously to Remark 2.7.2, since the infimum is attained for $\nu^* = -\sigma^{\perp'}V_y$, we can rewrite (2.8.3) as

$$\begin{aligned}b'(t, x, y)V_y(t, x, y) + LV(t, x, y) + \frac{1}{2}\|\theta(t, x, y)\|^2 - \theta'(t, x, y)\delta'(t, x, y)V_y(t, x, y) \\ - \frac{1}{2}\|\sigma^{\perp'}(t, x, y)V_y(t, x, y)\|^2 = 0 \quad dt\,dx\,dy\text{-a.s.} \quad (2.8.10)\end{aligned}$$

Now we consider the two particular cases of the previous section.

Case 1. *"Almost complete" diffusion model.* Assume that the price process X is described by the equation

$$dX_t = \operatorname{diag}(X_t)(\mu(t, X_t)dt + \sigma^l(t, X_t)dW^l_t), \quad t \in [0, T], \quad (2.8.11)$$

where σ^l satisfies the uniform ellipticity condition, and μ and σ^l are bounded measurable and such that Eq. (2.8.11) admits a unique strong solution. Then

$$F_t^l = F_t^X \tag{2.8.12}$$

and the market price of risk $\theta(t, X_t)$ is F_t^l-measurable. As is seen in Corollary 2.7.1,

$$V_t = \frac{1}{2} E^{\min} \left(\int_t^T \|\theta(s, X_s)\|^2 ds | F_t^l \right),$$

and (2.8.12) and the Markov property of X imply that $V_t = V(t, X_t)$ a.s., where

$$V(t, x) = \frac{1}{2} E^{\min} \left(\int_t^T \|\theta(s, X_s)\|^2 ds / X_t = x \right).$$

Since the conditions of Theorem 2.8.1 are satisfied, $V(t, x)$ is a unique bounded solution of the equation

$$LV(t, x) + \frac{1}{2}\|\theta(t, x)\|^2 = 0, \quad V(T, x) = 0 \tag{2.8.13}$$

in the class V^L.

Under suitable regularity conditions on μ and σ^l, the value function $V(t, x)$ is a unique bounded solution of (2.8.13) from the class $C^{1,2}$ and

$$LV = \frac{\partial V}{\partial t} + \frac{1}{2} \operatorname{tr}(\operatorname{diag}(x)\sigma^l \sigma^{l'} \operatorname{diag}(x) V_{xx}).$$

Case 2. Let us consider the stochastic volatility model

$$dX_t = \operatorname{diag}(X_t)(\mu(t, Y_t)dt + \sigma^l(t, Y_t)dW_t^l), \quad dY_t = b(t, Y_t)dt + \sigma^{\perp}(t, Y_t)dW_t^{\perp}, \tag{2.8.14}$$

where Eq. (2.8.14) admits a unique strong solution. We assume that the coefficients of (2.8.14) satisfy (D1) and (D2). We have that $F^{\perp} = F^Y$ and the market price of risk $\theta(t, Y_t)$ is $F_t^{\perp}$ adapted. According to Corollary 2.7.2, the value function is independent of x, i.e., $V(t, x, y) = V(t, y)$. Therefore, by Theorem 2.8.1, $V(t, y)$ is the unique bounded solution in the class V^L of the equation

$$LV(t, y) + \frac{1}{2}\|\theta(t, y)\|^2 + V_y'(t, y)b(t, y) - \frac{1}{2}\|\sigma^{\perp\prime}(t, y)V_y(t, y)\|^2 = 0 \quad dt\, dy\text{-a.s.}, \tag{2.8.15}$$

$$V(T, y) = 0. \tag{2.8.16}$$

For $U(t, y) = e^{-V(t,y)}$, Eq. (2.8.15) can be reduced to the linear SDE

$$LU(t, y) + b'(t, y)U_y(t, y) + \frac{1}{2}\|\theta(t, y)\|^2 U(t, y) = 0.$$

Under additional smoothness conditions on the coefficients μ, σ^l, b, and $\sigma^{\perp}$ this equation with the boundary condition $U(T, y) = 1$ has a unique solution in the class $C^{1,2}$ with LV the usual L-operator. By the Feynmann–Kac formula, the value admits the representation

$$V(t, y) = -\ln E \left[\exp \left\{ \frac{1}{2} \int_t^T |\theta(s, Y_s)|^2 ds| \right\} |Y_t = y \right].$$

2.9. Appendix

The proof of the following assertion for the case $p = 2$ can be found in [85]. For all cases we give the proof for $p > 1$.

Proposition 2.9.1. *$\tilde{Z}_T \in \mathcal{M}_p^{abs}$ is p-optimal if and only if*

$$E\eta(Z_T - \tilde{Z}_T)\tilde{Z}_T^{p-1} \geq 0 \tag{2.9.1}$$

for all $Z \in \mathcal{M}_p^{abs}$.

Proof. Let (2.9.1) be satisfied. We consider the function

$$f(x) = E\eta(xZ_T + (1-x)\tilde{Z}_T)^p.$$

Obviously, f is convex and continuously differentiable since the derivative $pE\eta(Z_T - \tilde{Z}_T)(\bar{x}Z_T + (1-\bar{x})\tilde{Z}_T)^{p-1}$ of the function $\eta(xZ_T + (1-x)\tilde{Z}_T)^p$ is majorized by the integrable random variable $2^{p-1}\eta(Z_T + \tilde{Z}_T)(Z_T^{p-1} + \tilde{Z}_T^{p-1})$. According to (2.9.1), $f'(0) \geq 0$. It follows from the convexity of f that $f(\varepsilon) - f(0) \leq f(x) - f(x-\varepsilon)$ for all ε and x such that $0 < \varepsilon < x \leq 1$, which implies that $f'(x) \geq f'(0) \geq 0$. Hence f is a nondecreasing function and $E\eta Z_T^p = f(1) \geq f(0) = E\eta\tilde{Z}_T^p$. Thus, $\tilde{Z}_T$ is p-optimal. Conversely, if $\tilde{Z}$ is p-optimal, then, obviously, $f'(0) \geq 0$ for any $Z \in \mathcal{M}^{abs}$ which gives (2.9.1). $\square$

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