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L^2-HEDGING UNDER PARTIAL OBSERVATION

Mania M.,* Tevzadze R.**

* A. Razmadze Mathematical Institute
** Institute of Cybernetics

Abstract. We consider the mean-variance hedging problem under partial information in the case where the flow of observable events does not contain the full information on the underlying asset price process. We introduce a certain type martingale equation and characterize the optimal strategy in terms of the solution of this equation.

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We consider the mean-variance hedging problem under partial information, where the flow of observable events does not necessarily contain the full information on the underlying asset prices.

Assume that the dynamics of the price process of the asset traded on a market is described by a continuous semimartingale $S = (S_t, t \in [0,T])$ defined on a filtered probability space $(\Omega, F, \mathcal{F} = (\mathcal{F}_t, t \in [0,T], P)$, satisfying the usual conditions, where $F = \mathcal{F}_T$ and $T < \infty$ is the fixed time horizon. Suppose that the interest rate is equal to zero and the asset price process admits the decomposition

$$S_t = S_0 + M_t + \int_0^t \lambda_u d\langle M\rangle_u, \quad \langle \lambda \cdot M\rangle_T < \infty \quad a.s., \tag{1}$$

where M is a local martingale and λ is an $\mathcal{F}$-predictable process.

Let us introduce an additional filtration $\mathcal{G}$ such that $\mathcal{G}_t \subseteq \mathcal{F}_t, \quad \text{for every} \quad \mathrm{t} \in [0,\mathrm{T}]$. The filtration $\mathcal{G}$ represents the information that the hedger has at his disposal.

Let H be a square integrable $\mathcal{F}_T$-measurable random variable, representing the payoff of a contingent claim at time T.

We consider the mean-variance hedging problem

$$\text{to minimize} \quad \mathrm{E}[(\mathrm{X}_\mathrm{T}^{\mathrm{x},\pi} - \mathrm{H})^2] \quad \text{over all} \quad \pi \in \Pi(\mathcal{G}), \tag{2}$$

where $\Pi(\mathcal{G})$ is a class of $\mathcal{G}$-predictable S-integrable processes. Here $X_t^{x,\pi} = x + \int_0^t \pi_u dS_u$ is the wealth process starting from initial capital x, determined by the self-financing trading strategy $\pi \in \Pi(\mathcal{G})$.

In the case $\mathcal{G} = \mathcal{F}$ of complete information the mean-variance hedging problem was introduced by Föllmer and Sondermann [1986] in the case when S is a martingale and then developed by several authors for price process admitting a trend. The mean-variance hedging problem under partial information was first studied by Di Masi, Platen and Runggaldier (1995) when the stock price process is a martingale and the prices are observed only at discrete time moments. For a general filtration and when the asset price process is a martingale this problem was solved by Schweizer (1994). Pham (2001) considered the mean-variance hedging problem for a general semimartingale model, assuming that $\mathcal{F}_t^S \subseteq \mathcal{G}_t$, where $\mathcal{F}^S$ is the filtration generated by S. We focus our attention to the case when the filtration $\mathcal{G}$ of observable events does not contain the full information about the asset price process S. In this case S is not a $\mathcal{G}$-semimartingale in general and the problem is more involved.

Let $X^* = X^{0,\pi^*}$ be the wealth process corresponding to the optimal strategy π^*. Let

$$H_t = E(H|\mathcal{F}_t) = EH + \int_0^t h_u dM_u + N_t \tag{3}$$

be the Galtchouk-Kunita-Watanabe (GKW) decomposition of H_t, where N is a martingale orthogonal to M and h is $\mathcal{F}$-predictable M-integrable process.

Denote by $\Pi(\mathcal{G})$ the class of all $\mathcal{G}$-predictable processes π such that $\pi \cdot S$ is in the S^2 space of semimartingales.

Denote by $\hat{Y}$ and ${}^pY$ the $\mathcal{G}$- optional and $\mathcal{G}$ - predictable projections of the process Y.

Proposition 1. *Assume that $\langle M\rangle$ is G-adapted. If $\pi^* \in \Pi(\mathcal{G})$ is the optimal strategy of the problem* (2) *then $d\langle M\rangle_t dP$-a.e.*

$$\pi_t^* =^p (h_t + \psi_t + \lambda_t H_t + \lambda_t Y_t - \lambda_t X_t^*) \tag{4}$$

where the triple (Y, ψ, L), $\langle L, M\rangle = 0$ is a solution of BSDE

$$dY_t = \pi_t^* \lambda_t d\langle M\rangle_t + \psi_t dM_t + dL_t, \;\; Y_T = 0. \tag{5}$$

Proof. The variational principle gives that $E(H - X_T(\pi^*))X_T(\pi) = 0, \;\; \forall \pi \in \Pi(\mathcal{G})$.

Since $\pi^* \in \Pi(\mathcal{G})$ we have that $E\big(\int_0^T \pi_u^* \lambda_u d\langle M\rangle_u\big)^2 < \infty$ and by the GKW decomposition

$$-\int_0^T \pi_u^* \lambda_u d\langle M\rangle_u = c + \int_0^T \psi_u dM_u + L_u, \quad \langle M, N\rangle = 0, \tag{6}$$

where $\psi \cdot M$ and L are square integrable martingales. Using the martingale property, it follows from (6) that the triple (Y, ψ, L), where $Y_t = E\big(\int_t^T \pi_u^* \lambda_u d\langle M\rangle_u|\mathcal{F}_t)$ and ψ, L are defined by (6), satisfies the BSDE

$$Y_t = Y_0 + \int_0^t \pi_u^* \lambda_u d\langle M\rangle_u + \int_0^t \psi_u dM_u + L_t, \;\; Y_T = 0. \tag{7}$$

Therefore (taking in mind decompositions (3), (6)) we have

$$\begin{aligned} E(H \;\; - \;\; X_T(\pi^*))X_T(\pi) &= E\left(-\int_0^T \pi_t^* \lambda_t d\langle M\rangle_t - \int_0^T \pi_t^* dM_t + H\right)\left(\int_0^T \pi_t dS_t\right) \\ &= E\left(Y_0 + \int_0^T \psi_t dM_t + L_T - \int_0^T \pi_t^* dM_t + H\right)\left(\int_0^T \pi_t dS_t\right) \end{aligned}$$

$$= \quad E\left(Y_0 + L_T + \int_0^T (\psi_t - \pi_t^*)\, dM_t + H\right)\left(\int_0^T \pi_t\lambda_t d\langle M\rangle_t\right) \tag{8}$$

$$+ \; E\left(Y_0 + L_T + \int_0^T (\psi_t - \pi_t^*)\, dM_t + c^H + \int_0^T h_t dM_t + N_T\right)\left(\int_0^T \pi_t dM_t\right) = 0. \tag{9}$$

Using the formula of integration by parts in (8) and properties of mutual characteristics of martingales in (9) we obtain the equality

$$E\int_0^T\left(Y_0 + L_t + \int_0^t (\psi_u - \pi_u^*)\, dM_t + H_t\right)\pi_t\lambda_t d\langle M\rangle_t + E\int_0^T (\psi_t + h_t - \pi_t^*)\,\pi_t d\langle M\rangle_t = 0.$$

Inserting the solution Y of BSDE (7) in the latter equality gives

$$E\int_0^T\left(Y_0 + H_t + Y_t - \int_0^t \lambda_u\pi_u^* d\langle M\rangle_u - \int_0^t \pi_u^* dM_t\right)\pi_t\lambda_t d\langle M\rangle_t + E\int_0^T (\psi_t + h_t - \pi_t^*)\,\pi_t d\langle M\rangle_t$$

$$= E\int_0^T \big((H_t + Y_t - X_t^*)\lambda_t + \psi_t + h_t - \pi_t^*\big)\pi_t d\langle M\rangle_t = 0.$$

It follows from the latter equality that

$$E\int_0^T \pi_t\pi_t^* d\langle M\rangle_t = E\int_0^T \pi_t[h_t + \psi_t + \lambda_t H_t + \lambda_t Y_t - \lambda_t X_t^*]\langle M\rangle_t$$

and by arbitrariness of $\pi \in \Pi(\mathcal{G})$ we obtain (4). □

The forward-backward equation (4)-(5),which gives a necessary condition of optimality, is hard to solve and to give the solution of the problem in a more constructive form we require the following additional assumptions:

A) $\langle M\rangle$ and λ are $\mathcal{G}$-predictable,

B) any $\mathcal{G}$- martingale is an $\mathcal{F}$-local martingale,

D) there exists a martingale measure for S that satisfies the Reverse Hölder condition.

E) $\rho_t^2 \equiv \dfrac{d\langle \widehat{M}\rangle_t}{d\langle M\rangle_t} < 1$ for all $t \in [0, T]$.

By condition A) $m_t = \int_0^t \psi_s dM_s + L_t$ is $\mathcal{G}$-martingale. If we use the GKW decomposition of m with respect to $\widehat{M}$

$$m_t = \int_0^t \widetilde{\psi}_u d\widehat{M}_u + \widetilde{L}_t, \quad \langle \widehat{M}, \widetilde{L}\rangle = 0,$$

then $\widehat{\psi}_t = \rho_t^2\widetilde{\psi}_t$. Besides, conditions A), B) imply that $\widehat{X}_t(\pi^*) = \int_0^t \pi_s^*\lambda_s d\langle M\rangle_s + \int_0^t \pi_s^* d\widehat{M}_s$. Therefore, from (4) and (5) we obtain the Forward-Backward equation for the filtered processes

$$d\widehat{X}_t^* = \left({}^p h_t + \rho_t^2\widetilde{\psi}_t + \lambda_t(\widehat{H}_t + \widehat{Y}_t - \widehat{X}_t^*)\right) d\widehat{S}_t, \quad \widehat{X}_0^* = x \tag{10}$$

$$d\widehat{Y}_t = \lambda_t\left({}^p h_t + \rho_t^2\widetilde{\psi}_t + \lambda_t(\widehat{H}_t + \widehat{Y}_t - \widehat{X}_t^*)\right) d\langle M\rangle_t + \widetilde{\psi}_t d\widehat{M}_t + d\widetilde{L}_t, \;\; \widehat{Y}_T = 0. \tag{11}$$

Let us introduce the operator $(AY)_t = E(\int_0^T \frac{1}{1-\rho_u^2}[Y_u\lambda_u + \rho_u^2\widetilde{\psi}_u](\lambda_u d\langle M\rangle_u + d\widehat{M}_u)|\mathcal{G}_t)$ defined for any $\mathcal{M}^2(\mathcal{G}, P)$. We shall use the following notations;

$$\widetilde{h}_t =^p h_t - \frac{d\langle\widehat{M},\widehat{H}\rangle_t}{d\langle M\rangle_t}, \quad \widetilde{H} = \widehat{H}_T - \int_0^T \frac{\widetilde{h}_t}{1-\rho_u^2} d\widehat{S}_t$$

Let us consider equation

$$\widetilde{Y}_T = \widetilde{H} - \int_0^T \frac{1}{1-\rho_t^2}\left[\lambda_t\widetilde{Y}_t + \rho_t^2\widetilde{\psi}_t\right]\left(\lambda_t d\langle M\rangle_t + d\widehat{M}_t\right). \tag{12}$$

Theorem 1. *Let $E\tilde{H}^2 < \infty$. Then equation (12) admits a unique solution $\widetilde{Y} \in \mathcal{M}^2(\mathcal{G}, P)$ satisfying $E|\widetilde{Y}_T|^2 \le E|\widetilde{H}|^2$. If conditions A), B), D) and E) are satisfied, then the strategy π^* is optimal if and only if it admits the representation*

$$\pi_t^* = \frac{1}{1-\rho_t^2}\left(\widetilde{h}_t + \lambda_t\widetilde{Y}_t + \rho_t^2\widetilde{\psi}_t\right). \tag{13}$$

Proof. To prove the first part of the theorem, we need only to show that A is a non-negative operator. Indeed, for $Y_t = c + \int_0^t \varphi_s d\widehat{M}_s + L_t$, $\langle\widehat{M}, L\rangle = 0$ we have

$$(Y, AY) = E\bigg(Y_T\int_0^T \frac{1}{1-\rho_t^2}Y_t\lambda_t^2 d\langle M\rangle_t + Y_T\int_0^T\frac{1}{1-\rho_t^2}Y_t\lambda_t d\widehat{M}_t$$
$$+Y_T\int_0^T\frac{\rho_t^2}{1-\rho_t^2}\varphi_t\lambda_t d\langle M\rangle_t + Y_T\int_0^T\frac{\rho_t^2}{1-\rho_t^2}\varphi_t d\widehat{M}_t\bigg)$$

Since $\langle Y,\widehat{M}\rangle_t = \int_0^t \varphi_u\rho_u^2 d\langle M\rangle_u$ and $EY_T\int_0^T g_u d\langle M\rangle_u = E\int_0^T Y_u g_u d\langle M\rangle_u$ for any $\mathcal{G}$-predictable process g, we obtain that

$$(Y, AY) = E\bigg(\int_0^T\frac{1}{1-\rho_t^2}Y_t^2\lambda_t^2 d\langle M\rangle_t + \int_0^T\frac{1}{1-\rho_t^2}Y_t\lambda_t\varphi_t d\langle\widehat{M}\rangle_t + \int_0^T\frac{\rho_t^2}{1-\rho_t^2}Y_t\varphi_t\lambda_t d\langle M\rangle_t$$
$$+\int_0^T\frac{\rho_t^2}{1-\rho_t^2}\varphi_t^2 d\langle\widehat{M}\rangle_t\bigg) = E\bigg(\int_0^T\frac{1}{1-\rho_t^2}Y_t^2\lambda_t^2 d\langle M\rangle_t + \int_0^T\frac{\rho_t^2}{1-\rho_t^2}Y_t\lambda_t\varphi_t d\langle M\rangle_t$$
$$+\int_0^T\frac{\rho_t^2}{1-\rho_t^2}Y_t\varphi_t\lambda_t d\langle M\rangle_t + \int_0^T\frac{\rho_t^4}{1-\rho_t^2}\varphi_t^2 d\langle M\rangle_t\bigg) = E\int_0^T\frac{1}{1-\rho_t^2}\left(Y_t\lambda_t + \rho_t^2\varphi_t\right)^2 d\langle M\rangle_t \ge 0.$$

Thus $Y + AY$ is a strictly positive operator, $(Id + A)^{-1}$ is bounded with the norm less than one and $Y = (Id + A)^{-1}\widetilde{H}$ is a unique solution of (12).

Here we shall only show that if π^* is optimal, then it is of the form (13). Introducing notations

$$\widetilde{Y}_t = \widehat{Y}_t + \widehat{H}_t - \widehat{X}_t(\pi^*), \quad \widetilde{m}_t = m_t + \widehat{H}_t - \int_0^t \pi_s^* d\widehat{M}_s$$

from (10)-(11) we have

$$\pi_t^* = \widetilde{h}_t + \rho_t^2\widetilde{\psi}_t + \pi_t^*\rho_t^2 + \lambda_t\widetilde{Y}_t, \quad d\widetilde{Y}_t = d\widetilde{m}_t,$$

which gives (since $\rho_t^2 < 1$ for all t) (13) and

$$\widetilde{Y}_T = \widehat{H}_T - \widehat{X}_T(\pi^*).$$

Integrating (13) with respect to $\widehat{S}$ and inserting obtained equality into (14) we receive that the pair $(\widetilde{Y}, \widetilde{\psi})$ satisfies equation (12), hence π^* is of the form (13).

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