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ON REGULARITY OF DYNAMIC VALUE FUNCTION RELATED TO THE UTILITY MAXIMIZATION PROBLEM

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Abstract. We study the regularity properties of both the dynamic value function and the optimal solution to the utility maximization problem for utility functions defined on the whole real line. These properties are needed to show that the value function satisfies the corresponding backward stochastic partial differential equation. In particular, in the case of complete markets we give conditions on the utility function when this equation admits a solution.

რეზიუმე. შესწავლილია სარგებლიანობის მაქსიმიზირების ამოცანასთან დაკავშირებული დინამიური ფასის ფუნქციისა და ოპტიმალური კაპიტალის პროცესის რეგულარობის თვისებები მთელ რიცხვით ღერძზე განსაზღვრული სარგებლიანობის ფუნქციებისთვის. ამ თვისებებზე დაყრდნობით ნაჩვენებია, რომ ფასის ფუნქცია აკმაყოფილებს შესაბამის შექცეულ სტოქასტურ კერძო წარმოებულებიან დიფერენციალურ განტოლებას. კერძოდ, სრული ფინანსური ბაზრის შემთხვევაში მოყვანილია ზემოაღნიშნული განტოლების ამონახსნის არსებობის პირობები.

1. Introduction

We consider a financial market model, where the dynamics of asset prices is described by the continuous semimartingale S defined on the complete probability space $(\Omega, \mathcal{F}, P)$ with continuous filtration $F = (F_t, t \in [0,T])$, where $\mathcal{F} = F_T$ and $T < \infty$. We work in discounted terms, i.e. the bond is assumed to be a constant.

Denote by $\mathcal{M}^e$ a set of probability measures Q equivalent to P on F_T such that S is a local martingale under Q.

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Throughout the paper, we assume that the filtration F is continuous (i.e. all F-local martingales are continuous) and

$$\mathcal{M}^e \neq \emptyset. \tag{1}$$

The continuity of F and the existence of an equivalent martingale measure imply that the structure condition is satisfied, i.e. S admits the decomposition

$$S_t = M_t + \int_0^t \lambda_s \, d\langle M\rangle_s, \quad \int_0^t \lambda_s^2 \, d\langle M\rangle_s < \infty$$

for all t P–a.s., where M is a continuous local martingale and λ is a predictable process.

Let $U = U(x) : R \to R$ be a utility function taking finite values at all points of real line R such that U is continuously differentiable, increasing, strictly concave and satisfies the Inada conditions

$$U'(\infty) = \lim_{x\to\infty} U'(x) = 0, \quad U'(-\infty) = \lim_{x\to-\infty} U'(x) = \infty. \tag{2}$$

We also assume that U satisfies the condition of reasonable asymptotic elasticity (see [6] and [13]), i.e.

$$\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1, \quad \liminf_{x\to-\infty} \frac{xU'(x)}{U(x)} > 1. \tag{3}$$

We consider the utility maximization problem, i.e. the problem of finding a trading strategy $(\pi_t, t \in [0,T])$ such that the expected utility of terminal wealth $X_T^{x,\pi}$ becomes maximal. The wealth process, determined by a self-financing trading strategy π and initial capital x, is defined as a stochastic integral

$$X_t^{x,\pi} = x + \int_0^t \pi_u dS_u, \quad 0 \le t \le T.$$

The value function V associated to the problem is given by

$$V(x) = \sup_{\pi\in\Pi_x} E\left[U\left(x + \int_0^T \pi_u \, dS_u\right)\right], \tag{4}$$

where Π_x is a class of admissible strategies (to be specified later).

For the utility function U, we denote by $\widetilde{U}$ its convex conjugate

$$\widetilde{U}(y) = \sup_x (U(x) - xy), \quad y > 0. \tag{5}$$

The problem dual to (4) is

$$\widetilde{V}(y) = \inf_{Q\in\mathcal{M}^e} E[\widetilde{U}(yZ_T^Q)], \quad y > 0, \tag{6}$$

where $Z_t^Q = dQ_t/dP_t$ is the density process of the measure $Q \in \mathcal{M}^e$ relative to the basic measure P.

Let us introduce the dynamic value function of the problem (4) defined as

$$V(t,x) = \operatorname*{ess\,sup}_{\pi\in\Pi_x} E\Big(U\Big(x + \int_t^T \pi_u dS_u\Big)/F_t\Big), \tag{7}$$

and the value function of the dual problem is

$$\widetilde{V}(t,y) = \operatorname*{ess\,inf}_{Q\in\mathcal{M}^e} E\Big[\widetilde{U}\Big(y\frac{Z_T^Q}{Z_t^Q}\Big)/F_t\Big],$$

Our goal is to study the properties of the dynamic value function $V(t,x)$ and the optimal wealth process $X_t(x)$. It is well known (see e.g., [10]) that for any $x \in R$ the process $(V(t,x), t \in [0,T])$ is a supermartingale admitting an RCLL (right-continuous with left limits) modification.

Therefore, using the Galchouk–Kunita–Watanabe (GKW) decomposition, the value function is represented as

$$V(t,x) = V(0,x) - A(t,x) + \int_0^t \psi(s,x)\, dM_s + L(t,x),$$

where for any $x \in R$ the process $A(t,x)$ is increasing and $L(t,x)$ is a local martingale, orthogonal to M.

Let us consider the following assumptions:

a) $V(t,x)$ is two-times continuously differentiable at x P–a.s. for any $t \in [0,T]$,

b) for any $x \in R$, the process $V(t,x)$ is a special semimartingale with bounded variation part, absolutely continuous with respect to $\langle M\rangle$, i.e.

$$A(t,x) = \int_0^t a(s,x)\, d\langle M\rangle_s,$$

for some real-valued function $a(s,x)$ which is predictable and $\langle M\rangle$ is integrable for any $x \in R$,

c) for any $x \in R$, the process $V'(t,x)$ is a special semimartingale with the decomposition

$$V'(t,x) = V'(0,x) - \int_0^t a'(s,x)\, d\langle M\rangle_s + \int_0^t \psi'(s,x)\, dM_s + L'(t,x),$$

where V', a', ψ' and L' are partial derivatives at x of V, a, ψ and L, respectively.

We shall say that $(V(t,x), t \in [0,T])$ is a regular family of semimartingales if for V the conditions a), b) and c) are satisfied.

We shall consider also the conditions

d) the conditional optimization problem (7) admits a solution, i.e., for any $t \in [0,T]$ and $x \in R$ there exists a strategy $\pi(t,x)$ such that

$$V(t,x) = E\big(U(x + \int_t^T \pi_u(t,x) dS_u)|F_t\big), \tag{8}$$

e) for each $s \in [t,T]$, the function $(X_s(t,x) = x + \int_t^s \pi_u(t,x)dS_u,\ s \geq t)$ is continuous at (t,x) $P-$ a.s.

It was shown in [8, 9, 10] that if the value function satisfies the conditions a)–e), then it solves the following backward stochastic partial differential equation (BSPDE)

$$V(t,x) = V(0,x) + \frac{1}{2}\int_0^t \frac{(\varphi'(s,x) + \lambda(s)V'(s,x))^2}{V''(s,x)}\, d\langle M\rangle_s + \\ + \int_0^t \varphi(s,x)\, dM_s + L(t,x), \quad V(T,x) = U(x). \tag{9}$$

Our aim is to study the conditions on the basic objects (on the asset price model and on the objective function U) which will guaranty that the value function $V(t,x)$ is a regular family of semimartingales and the conditions d) and e) are also satisfied, in order to show that the solution of equation (9) exists. In Theorem 1 below we provide such type conditions in the case of complete markets.

Condition d) is satisfied if following [13] we assume that

d′) For each $y > 0$, the dual value function $\widetilde{V}$ is finite and the minimizer $Q^*(y) \in \mathcal{M}^e$ (called the minimax martingale measure) exists.

Let $Z_T(y) \equiv yZ_T^{Q^*(y)}$, where $Z_t^{Q^*(y)}$ is a density process of the minimax martingale measure with respect to the measure P, i.e. $Z_t(y)$ is a P-martingale with $Z_0(y) = y = V'(x)$.

Let Π_x be the class of predictable S–integrable processes π such that $U(x + (\pi \cdot S)_T) \in L^1(P)$ and $\pi \cdot S$ is a martingale under the minimax martingale measure $Q^*(y)$, where $y = V^{'}(x)$ and the notation $\pi \cdot S$ stands for the stochastic integral.

It was shown in [13] that under assumptions d′) and an assumption of reasonable asymptotic elasticity (3) there exists the optimal strategy $\pi(x)$ of the problem (4) in the class Π_x. Denote by $X_t(x) = x + \int_0^t \pi_u(x)dS_u$ the optimal solution to (4).

It follows from [13] that Assumption d′) likewise implies the existence of an optimal solution to the conditional optimization problem (7) (where the optimal strategy may depend on t, in general), i.e., for any $t \in [0,T]$ there exists a strategy $\pi(t,x) \in \Pi_x$ such that (8) is satisfied.

The following duality relation holds true almost surely (see e.g., [12])

$$U'(X_T(x)) = Z_T(y), \ \text{ or equivalently } \ X_T(x) = -\widetilde{U}'(Z_T(y)), \tag{10}$$

where $X_T(x) = x + \int_0^T \pi_u(x)\, dS_u$ and $y = V'(x)$;

Similarly to [12], one can show that the value process $V(t,x)$ is differentiable at x P–a.s. for any $t \in [0,T]$ (see Proposition A3 from [11]) and it follows from [13] that

$$V'\left(t, x + \int_0^t \pi_u(x)\, dS_u\right) = Z_t(y), \quad t \in [0,T], \tag{11}$$

where $y = V'(x)$. Hereafter we shall use these results without further comments.

The main example, where all conditions a)–e) are satisfied is the case of an exponential utility function

$$U(x) = -e^{-\gamma x}$$

with a risk aversion parameter $\gamma \in (0,\infty)$. In this case $\widetilde{U}(y) = \frac{y}{\gamma}\big(\ln \frac{y}{\gamma} - 1\big)$ and Assumption d$'$) is equivalent to the existence of $Q \in \mathcal{M}^e$ with a finite relative entropy $EZ_T^Q \ln Z_T^Q$ (see e.g. [1]).

In this case, the corresponding value function is of the form $V(t,x) = -e^{-\gamma x} V_t$, and

$$V_t = \operatorname*{essinf}_{\pi \in \Pi_x} E(e^{-\gamma(\int_t^T \pi_u dS_u)} | \mathcal{F}_t) \tag{12}$$

is a special semimartingale. Therefore, it is evident that conditions a)–c) are satisfied and the BSPDE (9) is transformed into a usual backward stochastic differential equation (BSDE). In particular, Theorem 3.1 from [10] implies that $V(t,x) = -e^{-\gamma x} V_t$, where V_t satisfies the BSDE

$$V_t = V_0 + \frac{1}{2} \int_0^t \frac{(\varphi_s + \lambda_s V_s)^2}{V_s} d\langle M \rangle_s + \int_0^t \varphi_s dM_s + L_t, \quad V_T = 1, \tag{13}$$

where L is a local martingale, strongly orthogonal to M, and the optimal wealth process is expressed as

$$X_t(x) = x + \int_0^t \frac{\varphi_u + \lambda_u V_u}{\gamma V_u} dS_u. \tag{14}$$

The problem related with condition a) was studied in [5] for utility functions defined on the positive real line for value functions at time 0 and in [11] for dynamic value function $V(t,x)$ corresponding to utility functions defined on the whole real line.

The problems related with conditions b) and c) we connect with the existence of the inverse flow $X_t^{-1}(x)$ of the optimal wealth. In [11], the conditions are given where for any t the optimal wealth is an increasing function of x P–a.s. and an adapted inverse of $X_t(x)$ exists.

In Proposition 1 we derive a stochastic differential equation for the inverse of the optimal wealth $\psi_t(x) = X_t^{-1}(x)$ and based on this result, we give in Proposition 2 sufficient conditions when b) and c) are fulfilled.

Let

$$R_1(x) = -\frac{U''(x)}{U'(x)}, \quad R_2(x) = -\frac{U'''(x)}{U''(x)}, \quad x \in R. \tag{15}$$

Assuming that the market is complete, we shall use one of the following conditions:

r1) U is three-times differentiable, $R_1(x)$ is bounded away from zero and infinity and $R_2(x)$ is bounded and Lipschitz continuous.

r2) U is four-times differentiable and the density Z_T of the unique martingale measure is bounded.

As a corollary of Propositions 1–3 we get

Theorem 1. *Assume that the market is complete and one of the conditions* r1) *or* r2) *is satisfied. Then conditions* a)–e) *are fulfilled and the value function* $V(t,x)$ *satisfies* $BSPDE$ (9).

In the paper [3] a new approach was developed, where the solution of the problem (4) was reduced to the solvability of a system of Forward–Backward equations which is also a heavy task. Note that they showed that in case of complete markets this system admits a solution under the conditions similar to condition r1).

In the work [4], the wealth inverse process and the duality relations are used to derive some type of $SPDE$ and SDE for the progressive dynamic utility, its derivative and Fenchel conjugate. From their results it follows that there exists a whole class of the dynamic value functions satisfying regularity conditions of the present paper.

2. The Main Results

It has been shown in [11] (Theorem 1.4) that if the filtration F is continuous, condition d$'$) is satisfied and there are the constants $c_2 > c_1 > 0$ such that

$$c_1 \le R_1(x) \le c_2, \tag{16}$$

then for any $t \in [0,T]$ there exists a modification of the optimal wealth process $(X_t(x), x \in R)$ almost all paths of which are strictly increasing and absolutely continuous with respect to dx. Besides,

$$X'_t(x) > 0, \quad \lim_{x\to\infty} X_t(x) = \infty, \quad \lim_{x\to-\infty} X_t(x) = -\infty \tag{17}$$

P–a.s. for any $t \in [0,T]$ and the adapted inverse $X_t^{-1}(x)$ of the optimal wealth process exists.

Under the stronger conditions we shall derive a Stochastic Differential Equation (SDE) for the inverse process $X_t^{-1}(x)$.

For stochastic process $\xi_t(x)$, by $\xi'_t(x)$ (or $\partial\xi_t(x)$) we denote the derivative with respect to x, $\mu^{\langle S\rangle}$ denotes Dolean's measure for $\langle S\rangle$, i.e., the measure $d\langle S\rangle dP$ on $[0,T]\times\Omega$. If $F(t,x)$ is a family of semimartingales, then $\int_0^T F(ds,\xi_s)$ denotes a generalized stochastic integral (see [7]), or stochastic line integral by terminology from [2]. If $F(t,x)=xG_t$, where G_t is a semimartingale, then the generalized stochastic integral coincides with the usual one denoted by $\int_0^T \xi_s dG_s$, or $(\xi\cdot G)_T$.

Now we shall derive an SDE for the inverse of the optimal wealth $\psi_t(x) = X_t^{-1}(x)$ of the form

$$d\psi_t = \sigma_t(\psi_t)dS_t + \mu_t(\psi_t)d\langle S\rangle_t,\ \psi_0 = x, \tag{18}$$

where $\sigma_t(z) = -\frac{\pi_t(z)}{X'_t(z)}$, $\mu_t(z) = \frac{1}{2X'_t(z)}\left(\frac{\pi_t^2(z)}{X'_t(z)}\right)'$.

Proposition 1. *Let $X''_t(x)$, $\pi'_t(x)$ exist $\mu^{\langle S\rangle}-$a.e. and are locally Lipschitz functions with respect to x $\mu^{\langle S\rangle}-$a.e. Then SDE (18), or equivalently*

$$d\psi_t = -\frac{\pi_t(\psi_t)}{X'_t(\psi_t)}dS_t + \frac{\pi'_t(\psi_t)\pi_t(\psi_t)}{X'_t(\psi_t)^2}d\langle S\rangle_t - \frac{1}{2}\frac{X''_t(\psi_t)\pi_t^2(\psi_t)}{X'_t(\psi_t)^3}d\langle S\rangle_t, \tag{19}$$

$$\psi_0 = x \tag{20}$$

admits a unique maximal solution coinciding with $X_t^{-1}(x)$.

Proof. The SDE (18) admits a unique maximal solution up to time $\tau(x)\le T$, where $|\psi_{\tau(x)-}| = \infty$, if $\tau(x) < T$ (see [7]). Applying the Ito-Ventzel formula for $X_t(\psi_t) \equiv X(t,\psi_t)$ (see [7] or [14]) and bearing in mind that ψ_t satisfies (19), we get

$$\begin{aligned} dX(t,\psi_t) &= X(dt,\psi_t) + X'(t,\psi_t)d\psi_t + \frac{1}{2}X''(t,\psi_t)d\langle\psi\rangle_t + \\ &+ d\left\langle \int_0^{\cdot} X'(dr,\psi_r(x)),\psi(x)\right\rangle_t = \pi_t(\psi_t)dS_t + X'_t(\psi_t)\Big[-\frac{\pi_t(\psi_t)}{X'_t(\psi_t)}dS_t + \\ &+ \frac{\pi'_t(\psi_t)\pi_t(\psi_t)}{X'_t(\psi_t)^2}d\langle S\rangle_t - \frac{1}{2}\frac{X''_t(x)\pi_t^2(\psi_t)}{X'_t(\psi_t)^3}d\langle S\rangle_t\Big] + \\ &+ \frac{1}{2}\frac{X''_t(x)\pi_t^2(\psi_t)}{X'_t(\psi_t)^2}d\langle S\rangle_t - \frac{\pi'_t(\psi_t)\pi_t(\psi_t)}{X'_t(\psi_t)}d\langle S\rangle_t = 0, \\ &\psi_0(x) = x. \end{aligned}$$

Hence $X(t,\psi_t(x)) = x$ on $[0,\tau(x))$. Since $|X^{-1}_{\tau(x)}(x)| < \infty$, we have $\tau(x) = T$ P–a.s. and $\psi_t(x) = X_t^{-1}(x)$. □

Remark 1. Let $\pi_t(x) = H_t(X_t(x))$. Then

$$d\psi_t = -\frac{H_t(X_t(\psi_t))}{X_t'(\psi_t)}dS_t + \frac{H_t'(X_t(\psi_t))H_t(X_t(\psi_t))}{X_t'(\psi_t)^2}d\langle S\rangle_t - \\ -\frac{1}{2}\frac{X_t''(\psi_t)H_t^2(X_t(\psi_t))}{X_t'(\psi_t)^3}d\langle S\rangle_t.$$

Using equalities $X_t(\psi_t(x)) = x$, $\frac{1}{X_t'(\psi_t(x))} = \psi_t'(x)$, and $-\frac{X_t''(\psi_t(x))}{X_t'(\psi_t(x))} = \frac{\psi_t''(x)}{\psi_t'(x)^2}$ we obtain the linear Partial SDE (linear PSDE)

$$d\psi_t(x) = -H_t(x)\psi_t'(x)dS_t + H_t'(x)H_t(x)\psi_t'(x)d\langle S\rangle_t + \frac{1}{2}H_t^2(x)\psi_t''(x)d\langle S\rangle_t$$

or a PSDE in the divergence form

$$d\psi_t(x) = -H_t(x)\psi_t'(x)dS_t + \frac{1}{2}(H_t^2(x)\psi_t'(x))'d\langle S\rangle_t.$$

Let us define martingale random fields

$$\mathcal{M}(t,x) = E[U(X_T(x)|F_t],$$
$$\overline{\mathcal{M}}(t,x) = E[U'(X_T(x)|F_t].$$

Proposition 2. *Let the conditions of Proposition* 1 *be satisfied.*

a) *If $\mathcal{M}(t,x)$ is two times continuously differentiable with respect to x, then the finite variation part of $V(t,x) = \mathcal{M}(t,\psi_t(x))$ is absolutely continuous with respect to $\langle S\rangle$.*

b) *If $\overline{\mathcal{M}}(t,x)$ is two times continuously differentiable with respect to x, then $V^{'}(t,x)$ is a special semimartingale, and the finite variation part of $V'(t,x) = \overline{\mathcal{M}}(t,\psi_t(x))$ is absolutely continuous with respect to $\langle S\rangle$.*

Proof. a) By the optimality principle, $V(t, X_t(x))$ is a martingale, and since $V(T,x) = U(x)$, we find that $V(t,X_t(x)) = E[U(X_T(x))|F_t] = \mathcal{M}(t,x)$. Therefore, by the duality relation (10),

$$\mathcal{M}'(t,x) = V'(t,X_t(x))X_t'(x) = Z_t(y)X_t'(x) \tag{21}$$

is a martingale and let

$$\mathcal{M}'(t,x) = V'(x) + \int_0^t h_r(x)dM_r + L_t(x), \quad L(x)\perp M$$

be the GKW decomposition of $\mathcal{M}'(t,x)$. From (19), we have

$$\left\langle \int_0^\cdot \mathcal{M}'(dr,\psi_r(x)),\psi(x)\right\rangle_t = -\int_0^t h_r(\psi_r(x))\frac{\pi_r(\psi_r(x))}{X_r'(\psi_r(x))}d\langle S\rangle_r. \tag{22}$$

Since $V(t,x)=\mathcal{M}(t,X_t^{-1}(x))$, by the Ito-Ventzel formula we get

$$V(t,x)=V(0,x)+\int_0^t\mathcal{M}(ds,\psi_s)+\int_0^t\mathcal{M}'(s,\psi_s)d\psi_s+$$

$$+\frac{1}{2}\int_0^t\mathcal{M}''(s,\psi_s)d\langle\psi\rangle_s+\left\langle\int_0^\cdot\mathcal{M}'(dr,\psi_r(x)),\psi(x)\right\rangle_t. \tag{23}$$

In view of (19) and (22), one can verify that all finite variation members of (23) are integrals with respect to $\langle S\rangle$.

b) By the duality relation (11), we have

$$\overline{\mathcal{M}}(t,x)=E[U'(X_T(x))|F_t]=E[Z_T(y)|F_t]=Z_t(y)=V'(t,X_t(x)), \tag{24}$$

which (together with (21)) implies that $\mathcal{M}$ and $\overline{\mathcal{M}}$ are related as

$$\mathcal{M}'(t,x)=\overline{\mathcal{M}}(t,x)X_t'(x) \tag{25}$$

and $V'(t,x)=\overline{\mathcal{M}}(t,X_t^{-1}(x))$. It follows from (24) that $\overline{\mathcal{M}}'(t,x)=Z_t'(y)V''(x)$ is a martingale and

$$\left\langle\int_0^\cdot\overline{\mathcal{M}}'(dr,\psi_r(x)),\psi(x)\right\rangle_t=-\int_0^t\bar{h}_r(\psi_r(x))\frac{\pi_r(\psi_r(x))}{X_r'(\psi_r(x))}d\langle S\rangle_r, \tag{26}$$

where $\overline{\mathcal{M}}'(t,x)=\bar{V}''(x)+\int_0^t\bar{h}_r(x)dM_r+\bar{L}_t(x)$, $\bar{L}(x)\perp M$ is the GKW decomposition of $\overline{\mathcal{M}}'(t,x)$. Therefore the Ito–Ventzel formula implies that $V^{'}(t,x)=\overline{M}(t,X_t^{-1}(x))$ is a special semimartingale, and similarly to a), one can show that the finite variation part of $V^{'}(t,x)$ is absolutely continuous with respect to $\langle S\rangle$. □

We provide the sufficient conditions which guarantee the existence of all needed derivatives.

Hereafter we shall assume that the market is complete, i.e.,

$$dQ=Z_TdP,\quad\text{where}\quad Z_T=\mathcal{E}_T(-\lambda\cdot M)$$

is the unique martingale measure.

Lemma 1. *Let the market be complete and condition* r1) *be satisfied. Then the optimal wealth $X_T(x)$ is two-times differentiable and the derivatives $X_T^{'}(x)$, $X_T^{''}(x)$ are bounded and Lipschitz continuous.*

Proof. Since $\widetilde{U}(y)$ and $U(x)$ are conjugate, $\widetilde{U}(y)$ is also three-times differentiable and

$$\widetilde{U}^{''}(y)=-\frac{1}{U''(x)},\quad\widetilde{U}^{'''}(y)=-\frac{U^{'''}(x)}{(U''(x))^3},\quad y=U^{'}(x). \tag{27}$$

Therefore the functions $B_1(y)$ and $B_2(y)$, where

$$B_1(y) = y\widetilde{U}^{''}(y) = 1/R_1(x), \quad B_2(y) = y^2\widetilde{U}^{'''}(y) = R_2(x)/R_1^2(x), \tag{28}$$

are likewise bounded. This implies that the second and the third order derivatives of $\widetilde{U}(yZ_T)$ are bounded, hence the function $\widetilde{V}(y) = E\widetilde{U}(yZ_T)$ is three-times differentiable and

$$\widetilde{V}^{'''}(y) = E^Q\widetilde{U}^{'''}(yZ_T)Z_T^2.$$

Since $\widetilde{V}(y)$ and $V(x)$ are conjugate, $V(x)$ is also three-times differentiable.

The duality relation (10) takes in this case the following form:

$$U^{'}(X_T(x)) = yZ_T, \quad X_T(x) = -\widetilde{U}^{'}(yZ_T), \quad y = V^{'}(x). \tag{29}$$

This relation implies that the function $X_T(x)$ is two-times differentiable for all $\omega \in \Omega^{'} = (Z_T > 0)$ with $P(\Omega^{'}) = 1$, and differentiating the first equality in (29), we have

$$U^{''}(X_T(x))X_T^{'}(x) = V^{''}(x)Z_T, \tag{30}$$

$$U^{'''}(X_T(x))(X_T^{'}(x))^2 + U^{''}(X_T(x))X_T^{''}(x) = V^{'''}(x)Z_T. \tag{31}$$

From (29) and (30), we obtain

$$X_T^{'}(x) = \frac{V^{''}(x)}{V'(x)} \; \frac{U^{'}(X_T(x))}{U''(X_T(x))}.$$

By condition r1) and Proposition 1.2 from [11], $c_1 \le -\frac{V^{''}(x)}{V'(x)} \le c_2$. Therefore this implies that $X_T^{'}(x)$ is bounded, in particular,

$$\frac{c_1}{c_2} \le X_T^{'}(x) \le \frac{c_2}{c_1}, \tag{32}$$

where c_1 and c_2 are constants from (16).

Comparing equations (30) and (31), we have

$$X_T^{''}(x) + \frac{U^{'''}(X_T(x))}{U''(X_T(x))}(X_T^{'}(x))^2 = \frac{V^{'''}(x)}{V''(x)}X_T^{'}(x). \tag{33}$$

Since $E^QX_T^{'}(x) = 1$ and $E^QX_T^{''}(x) = 0$, taking expectations with respect to the measure Q in equation (33), we get

$$\frac{V^{'''}(x)}{V''(x)} = E^Q\frac{U^{'''}(X_T(x))}{U''(X_T(x))}(X_T^{'}(x))^2, \tag{34}$$

which together with (32) and condition $r1$) implies that $\frac{V^{'''}(x)}{V''(x)}$ is bounded.

Therefore, it follows from (33) that $X_T^{''}(x)$ is likewise bounded, hence $X_T'(x)$ is Lipschitz continuous.

Since the product of bounded Lipschitz continuous functions are Lipschitz continuous, it follows from (34) that $\frac{V^{'''}(x)}{V''(x)}$ is Lipschitz continuous and (33) implies that $X''_T(x)$ is likewise Lipschitz continuous, since all terms in (33) are bounded and Lipschitz continuous. □

Lemma 2. *Let the market be complete and condition* r2) *be satisfied. Then the optimal wealth $X_T(x)$ is three-times differentiable, $X'_T(x)$ is strictly positive and the derivatives $X'_T(x), X''_T(x)$ and $X'''_T(x)$ are uniformly bounded on every compact $[a,b] \in R$.*

Proof. Since $U(x)$ and $\widetilde{U}(y)$ are conjugate, condition r2) implies that $\widetilde{U}(y)$ is also four times differentiable and the derivatives of $\widetilde{U}(yZ_T)$ are bounded for any $y \in R$, hence the function $\widetilde{V}(y) = E\widetilde{U}(yZ_T)$ is four-times differentiable.

Then $V(x)$ is likewise four-times differentiable, since $V'(x)$ is the inverse of $-\widetilde{V}'(y)$. Therefore, the duality relation

$$X_T(x) = -\tilde{U}'(V'(x)Z_T)$$

implies that the optimal wealth $X_T(x)$ is three-times differentiable and the derivatives $X'_T(x), X''_T(x)$ and $X'''_T(x)$ are bounded on every compact $[a,b] \in R$. Therefore the derivatives $X'_T(x)$ and $X''_T(x)$ satisfy the local Lipschitz condition.

Besides,

$$X'_T(x) = -V''(x)Z_T\widetilde{U}''(V'(x)Z_T)) > 0,$$

since $V''(x) < 0$ and $\widetilde{U}''(y) > 0$. □

Corollary 1. *The process $(X''_t(x),\ (t,x) \in [0,T] \times R)$ admits a continuous modification.*

Proof. Since $X''_t(x)$ is a $Q-$martingale, by the Doob inequality and the mean value theorem we get

$$E^Q \sup_{t \le T} |X''_t(x_1) - X''_t(x_2)|^2 \le c_1 E^Q |X''_T(x_1) - X''_T(x_2)|^2 \le$$

$$\le c_1|x_1 - x_2| E^Q \sup_{\alpha \in [0,1]} |X'''_T(\alpha x_1 + (1-\alpha)x_2)|^2 \le c_2|x_1 - x_2|^2$$

for some constants c_1, c_2. By the Kolmogorov theorem, the map

$$R \ni x \to X''_\cdot(x) \in C[0,T]$$

admits a continuous modification which implies the continuity of $X''_t(x)$ with respect to the variables (t,x), P–a.s. □

Proposition 3. *Assume that the market is complete and one of the conditions* r1)*, or* r2) *is satisfied.*

Then the optimal wealth $X_t(x)$, the optimal strategy $\pi_t(x)$ ($\mu^{\langle S\rangle}$–a.e.), martingale flows $\mathcal{M}(t,x)$ and $\overline{\mathcal{M}}(t,x)$ are two-times continuously differentiable at x for all t, $P-$a.s. and the coefficients of equation (19) *satisfy the local Lipschitz condition.*

Proof. Let us first assume that condition r1) is satisfied. According to Lemma 1, the optimal wealth $X_T(x)$ is two-times differentiable and the derivatives $X_T^{'}(x)$ and $X_T^{''}(x)$ are bounded and Lipschitz continuous.

To show the existence of $\pi'(x)$, we use the decomposition $X_T'(x) = 1 + \int_0^T \pi_r^{(x)} dS_r$ with some predictable S–integrable integrand $\pi^{(1)}(x)$ and inequalities

$$E^Q \int_0^T \left(\pi_t^{(1)}(x+\varepsilon) - \pi_t^{(1)}(x)\right)^2 d\langle S\rangle_t = E^Q \left\langle X'(x+\varepsilon) - X'(x)\right\rangle_T =$$
$$= E^Q \left(X_T'(x+\varepsilon) - X_T'(x)\right)^2 \le \varepsilon^2 E^Q \max_{0\le s\le 1} |X_T''(x+s\varepsilon)|^2 \le$$
$$\le \varepsilon^2 \text{Const}.$$

By the Kolmogorov theorem, $\pi^{(1)}(x)$ is continuous with respect to x $\mu^{\langle S\rangle}$–a.e.

Note that if instead of r1) the condition r2) is satisfied, then we shall have that there exists a $\mu^{\langle S\rangle}$–a.e. continuous modification of $\pi^{(1)}(x)$ on each compact of R which will imply the existence of a continuous modification on the whole real line.

Thus, by the stochastic Fubini's Theorem, (see [14])

$$x_2 - x_1 + \int_0^T (\pi_r(x_2) - \pi_r(x_1)) dS_r = X_T(x_2) - X_T(x_1) =$$
$$= \int_{x_1}^{x_2} X_T'(x)dx = x_2 - x_1 + \int_0^T \int_{x_1}^{x_2} \pi_r^{(1)}(x) dx dS_r$$

and consequently, $\pi_r(x_2) - \pi_r(x_1) = \int_{x_1}^{x_2} \pi_r^{(1)}(x)dx$ $\mu^{\langle S\rangle}$–a.e. Hence $\pi^{(1)}(x) = \pi'(x)$ $\mu^{\langle S\rangle}$–a.e. and

$$X_T'(x) = 1 + \int_0^T \pi_r'(x) dS_r, \tag{35}$$

for all x P–a.s.

It follows from (35) and Fubini's theorem that

$$X_t(x_2) - X_t(x_1) = x_2 - x_1 + \int_0^t (\pi_r(x_2) - \pi_r(x_1)) dS_r =$$
$$= x_2 - x_1 + \int_0^t \int_{x_1}^{x_2} \pi_r'(x) dx dS_r = \int_{x_1}^{x_2} X_t'(x)dx$$

for any $x_2 \geq x_1$ $P-$a.s., and Lemma A3 from [11] implies that for each fixed t there exists a modification of $(X_t(x),\, x \in R)$ which is absolutely continuous with respect to the Lebesgue measure dx. Since $(X'_t(x), t \in [0,T])$ is a Q-martingale

$$|X'_t(x_2) - X'_t(x_1)| \leq E^Q(|X'_T(x_2) - X'_T(x_1)|/F_t) \leq C|x_2 - x_1| \qquad (36)$$

for any $x_2 \geq x_1$ $P-$a.s., and Lemma 1 and Corollary 1 imply that there exists $\Omega' \subset \Omega,\ P(\Omega') = 1$, such that at each $\omega \in \Omega'$ the inequality (36) is fulfilled for all (t,x).

Since $EX''_T(x) = 0$ and the market is complete, we have $X''_T(x) = \int_0^T \pi_r^{(2)}(x)dS_r$ for some predictable S-integrable integrand $\pi^{(2)}$. Similarly, one can show that $\pi^{(2)}(x)$ is continuous at x $\mu^{<S>}$–a.e., $\pi^{(2)}(x) = \pi''(x)$ $\mu^{\langle S\rangle}$–a.e. and, hence $X''_t(x)$ admits the representation

$$X''_t(x) = \int_0^t \pi''_r(x)dS_r.$$

Similarly, we can show that one can choose a modification of $X_t(x)$ which is two-times differentiable and such that $X''(x)$ is Lipschitz continuous.

In the case where instead of r1) the condition r2) is fulfilled, $X''(x)$ will satisfy the local Lipschitz condition. Thus, in both cases (i.e., if condition r1) or r2) is satisfied), the coefficients of equation (19) will be locally Lipschitz continuous. □

Since the market is complete, $\overline{\mathcal{M}}(t,x) = V'(x)Z_t$ and it is evident that $\overline{\mathcal{M}}(t,x)$ is two-times continuously differentiable. Besides, equality (25) implies that $\mathcal{M}(t,x)$ is also two-times continuously differentiable at x.

Proof of Theorem 1. It is evident that the boundedness of $B_1(y)$ and $B_2(y)$ (defined by (28)) implies that the dual value function $\widetilde{V}(t,y) = E(\widetilde{U}(y\frac{Z_T}{Z_t})/F_t)$ is two-times continuously differentiable. Since

$$V''(t,x) = -\frac{1}{\widetilde{V}''(t,y)}, \quad y = V'(x),$$

the value function $V(t,x)$ is also two-times continuously differentiable, hence condition a) is fulfilled.

It follows from Proposition 3 that under the presence assumptions all conditions of Propositions 1 and 2 are satisfied, therefore these propositions imply that $V(t,x)$ satisfies conditions b) and c), hence $V(t,x)$ is a regular family of semimartingales.

Let us show that the condition e) is also satisfied. By the optimality principle (see [10]), for any $t \in [0,T]$, the process $(V(s, X_s(t,x)), s \geq t)$ is a martingale, where $X_s(t,x) = x + \int_t^s \pi_u(t,x)dS_u$ is the solution of the conditional optimization problem (9). This implies that P-a.s.

$$V(t,x)=E(V(s,X_s(t,x))/F_t). \tag{37}$$

On the other hand, using again the optimality principle, we have

$$V(t,X_t(x))=E(V(s,X_s(x))/F_t),$$

and substituting in this equality the inverse of the optimal capital $X_t(x)$, we get

$$V(t,x)=E(V(s,X_s(X_t^{-1}(x))/F_t). \tag{38}$$

Since for any t the function $(V(t,x),x\in R)$ is strictly convex, comparing (37) and (38) we obtain that P-a.s $X_s(t,x)=X_s(X_t^{-1}(x))$. By the continuity at (t,x) of $X_t^{-1}(x)$ as a solution of SDE (19), we obtain that condition e) is satisfied.

Thus, all the conditions of Theorem 3.1 from [10] are satisfied which implies that $V(t,x)$ is a solution of the $BSPDE$ (9). □

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