

**STRONG COUPLING CONSTANT FROM τ DECAY
WITHIN A DISPERSIVE APPROACH TO
PERTURBATIVE QCD**

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ABSTRACT. We present a new dispersive framework for the extraction of the strong coupling constant α_s from τ -lepton decays. A new feature of our procedure is the use of the quark-hadron duality on the limited region $s_d < s < m_\tau^2$. The duality point s_d and the $\overline{\text{MS}}$ strong coupling constant $\alpha_s(m_\tau^2)$ are self-consistently extracted from the τ data for the non-strange vector spectral function. We use the 2005 ALEPH and 1998 OPAL experimental data on the vector spectral function. We compare the new framework with the contour improved perturbation theory up to order α_s^5 . The new procedure yields systematically lower values for α_s .

რეზიუმე. შემოთავაზებულია ახალი დისპერსიული მიდგომა ძლიერი ურთიერთმოქმედების α_s კონსტანტას რიცხვითი მნიშვნელობის განსაზღვრისათვის τ ლეპტონის დაშლებიდან. კვარკ-ჰადრონული დუალობა გამოყენებულია შეზღუდულ არეში $s_d < s < m_\tau^2$. დუალობის წერტილი s_d და $\overline{\text{MS}}$ სქემის ბმის მუდმივა თვით-შეთანხმებულად განისაზღვრა τ ლეპტონის დაშლების ანალიზით ვექტორულ არეში. გამოყენებულია 2005 ALEPH და 1998 OPAL კოლაბორაციების გაზომვების შედეგები ვექტორული სპექტრალური ფუნქციისათვის “არა-უცნაურ” არეში. ახალი მეთოდი შედარდა “კონტურზე გაუმჯობესებულ შემოთავაზების თეორიას” α_s^5 რიგის ჩათვლით. ახალი პროცედურით მიღებულია შედარებით უფრო დაბალი რიცხვითი მნიშვნელობები ბმის მუდმივასათვის.

1. INTRODUCTION

As is well known, the inclusive hadronic decays of the τ -lepton may be reliably studied within perturbative QCD (see seminal work [1] and the references therein). A general approach to analyzing the perturbative and non-perturbative aspects of the τ -system observables is the renormalization group improved perturbation theory augmented with the Wilson’s Operator Product Expansion (OPE) [2]. The characteristic energy scale is relatively

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small, $m_\tau \approx 1.8 \text{ GeV}$ (m_τ being the mass of the τ -lepton). Hence, the non-perturbative effects of QCD cannot be completely ignored. The original study [1] has shown that they are small and under control within the OPE. In the following years, the inclusive hadronic quantities of the τ system have been intensively exploited to precisely determine the strong coupling constant $\alpha_s(m_\tau^2)$. This became feasible because the observables of the τ system are sensitive to the concrete value of α_s and the accuracy of the experimental data for a variety of the observables has been considerably improved (for the recent review see [3]).

In the past few years, the determination of the strong coupling constant from non-strange hadronic τ -data has received a renewed interest. It was pointed out [4] that there is no good agreement between recent two highest precision low-energy determinations of α_s . These determinations come from the finite energy sum rule (FESR) analysis of hadronic τ decay data [5] and from a lattice perturbation theory analysis of ultraviolet-sensitive lattice observables [6].

$$\alpha_s(M_z^2) = 0.1212 \pm 0.0011 \quad (\tau \text{ decay}), \quad (1)$$

$$\alpha_s(M_z^2) = 0.1170 \pm 0.0012 \quad (\text{lattice}). \quad (2)$$

Moreover, different determinations of α_s from the same τ data [7, 8, 9] are not fully consistent within their mutual errors (see work [10] and the references therein). This discrepancy has stimulated a number of new theoretical investigations on the application of the FESR in τ decays (see works [4, 5, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]). The standard FESR technique based on the truncated OPE series has been reconsidered. The small but still significant non-perturbative effects have been included into analysis [19]. On the one hand, the impact of the higher order terms of the OPE (neglected in the standard analyzes) has been estimated [4, 15, 16, 18]. It was confirmed that their influence on the extracted value of α_s is not small in the separate vector and axial vector channels. To suppress these contributions in the FESR the so-called pinched weights introduced [4, 15, 16, 18]. On the second hand, using the physically motivated model [10, 11, 19] the impact of the non-perturbative corrections coming from the possible duality violations (DVs) [20] has been estimated. In the separate vector and axial vector channels the DVs was found to be appreciable (see recent work [19] and references therein). The pinched weights have also been employed to reduce the effects of DVs [18]. Possible non-perturbative corrections to the FESR (direct instantons, duality violation and tachyonic gluon mass) which cannot be described within the OPE have been estimated in [13].

As is well known, in the time-like region the renormalization group (RG) invariance cannot be used unambiguously. For this reason, two different methods are used to perform the RG resummation within the FESR. These

are fixed order perturbation theory (FOPT) and contour improved perturbation theory (CIPT) [21, 22]. These two approaches lead to differing results. The values of α_s extracted from τ decays employing CIPT have always been higher. A critical comparison of these two approaches may be found in recent works [23, 24]. In [24] FOPT was approved as a better approximation to the true result. In contrast, the authors of [4] and [5] favored CIPT.

Note that the non-physical singularities of the perturbative running coupling (the Landau pole problem) which occur at small space-like momenta may, supposedly, deteriorate the extracted values of the parameters [25]. In particular, CIPT suffers from this shortcoming [24]. To cope with this problem dispersive or analytic approaches to perturbative QCD have been developed [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39]. In works [27] and [28], the τ lepton decay rate has been analyzed within a simple and effective dispersive technique, Analytic Perturbation Theory (APT) (for reviews see [29, 38, 39]). However, the minimal analytic QCD model (the same APT) predicts, from the non-strange τ lepton decay data, too large value for the strong coupling constant, $\alpha_s(m_\tau^2) = 0.403 \pm 0.015$ [28]. The advantages and shortcomings of the three approaches to the τ decays (FOPT, CIPT and APT) were thoroughly analyzed in [30]. It should be noted that APT as well as its generalized versions suggested more later [31, 33, 34, 35, 36, 37] proved to be very useful from the phenomenological point of view. A remarkable feature of these modified expansions is the better convergence and improved stability property with respect to change of the renormalization scheme. Nevertheless, one should keep in mind that an analytic approach based only on perturbation theory cannot be defined unambiguously, in fact, there is no a unique recipe for removing the Landau singularities from the running coupling.

In our earlier work [40] we have suggested a dispersive approach to analyze the τ decay data. In contrast to CIPT, the new approach is based on the improved approximations to the Adler function which incorporate correct analyticity and RG invariance properties of the exact function. Moreover, the approximations correctly reproduce the required ultraviolet and infrared properties of the exact Adler function. Another feature of the new framework is the use of the quark-hadron duality in the limited region $s_d < s < m_\tau^2$. The QCD scale parameter $\Lambda_{\overline{\text{MS}}}$ and the duality point s_d may be determined, self-consistently, from the experimental data [40].

In the present article, we investigate the new framework more thoroughly. We revise part of the results of work [40]. We present a more accurate test of the convergence of the numerical results in perturbation theory. The numerical value of the duality point s_d is found to be remarkable stable with respect to higher order QCD corrections. More importantly, we study

the stability of the results with respect to small change of the experimental data.

In Section 2 we critically analyze the FOPT and CIPT approaches to the τ -decay. A dispersive modification of the CIPT suggested in [40] is discussed in more detail. In Section 3 we give corrected numerical values for α_s and s_d extracted from the 2005 ALEPH data. We thoroughly investigate the stability of the results comparing the new and CIPT determinations of α_s order by order in perturbation theory. In addition, we analyze the ALEPH non-strange data employing the renormalization scheme invariant (RSI) framework suggested in [41]. We also analyze the 1998 OPAL [9] vector data within the new dispersive framework. Conclusions are summarized in Section 4.

2. THEORETICAL FRAMEWORK

Let us briefly recall some basic facts about the QCD analysis of the hadronic decays of the τ -lepton through the FESR [42]. The non-strange vector component of the τ -hadronic width is determined as

$$R_{\tau,V} = 6|V_{ud}|^2 S_{EW} \int_0^{m_\tau^2} w_\tau(s) v_1(s) ds, \quad (3)$$

where

$$w_\tau(s) = \frac{1}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + 2\frac{s}{m_\tau^2}\right),$$

V_{ud} is the flavor CKM matrix element, S_{EW} denotes a short-distance electroweak correction¹ and $v_1(s)$ is the vector spectral function defined through the correlation function²

$$v_1(s) = 2\pi \text{Im} \Pi_{\bar{u}d,V}(-s). \quad (4)$$

It is more convenient to define a renormalization scale invariant quantity, the Adler function

$$D(Q^2) = -4\pi^2 Q^2 \frac{d}{dQ^2} \Pi_{\bar{u}d,V}(Q^2), \quad (5)$$

here, we have defined $s = q^2 = -Q^2$. In the exact theory, the correlation function $\Pi_{\bar{u}d,V}(z)$ and the Adler function are analytic functions except the cut running along negative z -axis. This implies the FESR relation

$$R_{\tau,V} = -\frac{3i}{4\pi} |V_{ud}|^2 S_{EW} \oint_{-s_0-i\epsilon}^{-s_0+i\epsilon} \left(1 - \frac{z}{s_0}\right) \left(1 + \frac{z}{s_0}\right)^3 D(z) \frac{dz}{z}, \quad (6)$$

¹In what follows, we neglect the small additive electroweak correction δ'_{EW} .

²We use the normalization of the spectral function with the naive parton prediction $v_{1,\text{naive}} = 1/2$.

the integration contour of the integral in (6) is a circle of radius s_0 ($s_0 = m_\tau^2$). In the case of massless quarks, the Adler function has the perturbation theory expansion [23]

$$D(Q^2) = \sum_{n=0}^{\infty} a_s(\mu^2)^n \sum_{k=1}^{n+1} k c_{nk} L^{k-1}, \quad \text{where } L = \ln \frac{Q^2}{\mu^2}, \quad (7)$$

$a_s(\mu^2) = \alpha_s(\mu^2)/\pi$ and $\alpha_s(\mu^2)$ denotes the strong coupling constant normalized at the scale μ . It follows from the renormalization scale invariance of the Adler function that only the coefficients c_{n1} are independent. All other coefficients are determined in terms of the c_{n1} and β -function coefficients through the RG equation [23, 24]. In practice the series (7) is truncated at some finite order.

The approximations to the Adler function obtained by truncation of the series (7) have correct analytical properties of the exact function. In the case of FOPT, the series (7) is inserted into contour integral (6) and integrated term-by-term. Afterwards, the normalization scale is determined choosing $\mu = m_\tau$ [23]. However, we could start from the original formula (3) with perturbation theory expansion for the spectral function. The expansion for the spectral function is obtained by insertion of the series (7) into inversion formula (15) (see below) and integrating term-by-term. So, we could achieve the same result without using the FESR relation (6). Thus, within FOPT, formulas (3) and (6) are equivalent. However, the approximations to the Adler function employed within FOPT do not describe correctly the asymptotic behavior of the exact function for $Q^2 \rightarrow \infty$. In the standard analysis of the τ data this fact is irrelevant. However, as we shall see latter, this is not the case for the new framework accepted in this paper.

Since the Adler function is the renormalization scale invariant quantity, one may choose $\mu^2 = Q^2$ in series (7). Thus, one obtains the RG improved expansion

$$D(Q^2)|_{\text{RG}} = 1 + d(Q^2)|_{\text{RG}} = 1 + \sum_{n=1}^{\infty} d_n a_s^n(Q^2) \quad (8)$$

where $d_n = c_{n1}$, $a_s(Q^2) = \alpha_s(Q^2)/\pi$, $\alpha_s(Q^2)$ being the running coupling. The first two coefficients in the expansion (8) are the renormalization scheme invariant. The known coefficients in the $\overline{\text{MS}}$ scheme for $n_f = 3$ quark flavors take values $d_1 = 1$, $d_2 \simeq 1.6398$, $d_3 \simeq 6.3710$ and $d_4 \simeq 49.0757$. The last coefficient was calculated recently by the authors of [12] by using powerful computational techniques. The approximations to the Adler function constructed by truncation the series (8) have correct ultraviolet asymptotical behavior ($d(Q^2) \rightarrow 0$ as $Q^2 \rightarrow \infty$), however they violate the cut-plane analyticity of the exact Adler function due to the non-physical Landau singularities of the perturbative running coupling. One may assume, without

loss of generality, that the running coupling has only one Landau singularity located on the positive Q^2 axis [40]. This is true for the asymptotic solutions and for more accurate Lambert-W solutions to the RG equation [43, 44, 45]. In [40], we have derived the violated dispersion relation for the QCD correction to the Adler function:

$$d(Q^2)|_{\text{RG}} = d(Q^2)|_{\text{APT}} + d_L(Q^2), \quad (9)$$

where the function $d(Q^2)|_{\text{APT}}$ satisfies the normal DR

$$d(Q^2)|_{\text{APT}} = \frac{1}{\pi} \int_0^{\infty} \frac{\rho_{\text{eff}}(\sigma)}{\sigma + Q^2} d\sigma, \quad (10)$$

with the effective spectral density determined as

$$\rho_{\text{eff}}(\sigma) = \text{Im}\{d(-\sigma - i\epsilon)|_{\text{RG}}\}. \quad (11)$$

It is to be noted here that the function

$$D(Q^2)|_{\text{APT}} = 1 + d(Q^2)|_{\text{APT}} \quad (12)$$

is the analytic image of the perturbative Adler function determined in the sense of the Analytic Perturbation Theory (APT) approach of Shirkov and Solovtsov [26, 27]. The second term in (9) is the contribution coming from the Landau singularity. It is represented by the contour integral [40]

$$d_L(Q^2) = -\frac{1}{2\pi i} \oint_{C_L^+} \frac{d(\zeta)|_{\text{RGI}}}{\zeta - Q^2} d\zeta, \quad (13)$$

here, the integral is taken round the circle $\{\zeta : \zeta = s_L + s_L \exp(i\phi), -\pi < \phi \leq \pi\}$ in the positive (anti-clockwise) direction, with s_L being the Landau singular point.

In the popular framework, referred to as contour improved perturbation theory (CIPT) [21, 22], the (truncated) expansion (8) is inserted into the FESR integral (6) and then integrated term by term. At this point, one ignores the fact that with the approximation (8) formulas (3) and (6) are not equivalent. Indeed, the FESR relation (6) cannot be derived because of violated analytical properties of the approximation (8). This inadequacy was repeatedly discussed in the literature (see for example [24] and [30]). Nevertheless, CIPT has been very successful from the phenomenological point of view. On the other hand, APT is free from this drawback. However, the analysis of the τ decay data based on APT with massless quarks gave too large value for the strong coupling constant [28]. Furthermore, in the infrared region, the Adler function can not be reproduced correctly within APT, CIPT or FOPT. Thus, in APT the running coupling $\alpha_s(Q^2)$ has a finite limit as $Q^2 \rightarrow 0$ [26]. This leads to the apparent contradiction in the case of the Adler function. In fact, the Adler function vanishes at $Q^2 = 0$,

as is manifested by Chiral Perturbation Theory [47]. In work [31], APT has been modified by considering the quark mass threshold effects for the light quarks. In this way, correct descriptions of the Adler function and τ data were achieved. However, too large values for the effective quark masses ($m_u \sim m_d \sim 330$ MeV) were predicted.

As is well known, in the exact theory the Adler function satisfies the dispersion relation (DR)

$$D(Q^2) = Q^2 \int_0^{\infty} \frac{2v_1(s)ds}{(s+Q^2)^2}. \quad (14)$$

The hadronic vector spectral function may be calculated in terms of the Adler function via the contour integral

$$v_1(s) = \frac{1}{4\pi i} \oint_{-s-i\epsilon}^{-s+i\epsilon} \frac{D(z)}{z} dz, \quad (15)$$

where the path of integration, connecting the points $-s \mp i\epsilon$ on the complex z -plane, avoids the cut running along the real negative z -axis. The integral being traversed in a positive (anticlockwise) sense. From the violated DR (9), we may also derive the integral representation

$$D(Q^2)|_{\text{RG}} = Q^2 \int_{-s_L}^{\infty} \frac{2v_1^{\text{RG}}(s)ds}{(s+Q^2)^2}, \quad (16)$$

where the singular integral at the lower bound should be treated in the sense of distribution theory³. It is to be noted that the spectral function $v_1^{\text{RG}}(s)$ may be again calculated via the inversion formula (15), but now the integration contour should also avoid the non-physical cut running along the positive interval $0 < z < s_L$ (see Fig. 1).

The dispersion relation (14) may be used to construct the approximations to the Adler function with correct analyticity properties. To approximate the hadronic spectral function, one may use the global duality *ansatz* employed previously in works [46, 47]

$$v_1(s) = \theta(s_d - s)v_1^{\text{np}}(s) + \theta(s - s_d)v_1^{\text{pQCD}}(s), \quad (17)$$

where $v_1^{\text{pQCD}}(s)$ is the perturbation theory approximation to the spectral function, $v_1^{\text{np}}(s)$ denotes the non-perturbative component of the spectral function confined, presumably, in the low energy region, and s_d is the onset of perturbative continuum, an infrared boundary in Minkowski region above

³We have confirmed formula (16) in the case of the one-loop order β -function.

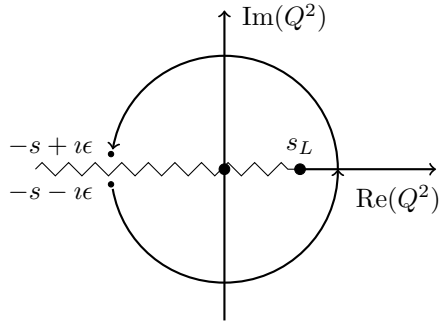


FIGURE 1. The integration contour in the formula (15) in the case of the approximation $D(Q^2)|_{\text{RG}}$ violating the DR. Branch points on the real axis are represented by the blobs and branch cuts by the zigzagging lines. s_L denotes the Landau singularity.

which we trust pQCD ⁴. One may also construct a “semi-experimental” spectral function

$$v_1^{\text{“s.exp”}}(s) = \theta(s_d - s)v_1^{\text{exp}}(s) + \theta(s - s_d)v_1^{\text{pQCD}}(s), \quad (18)$$

where $v_1^{\text{exp}}(s)$ denotes the genuine experimental part of the total “semi-experimental” spectral function. It was measured with high precision by ALEPH [7, 8] and OPAL [9] collaborations in the range $0 < \sqrt{s} < m_\tau = 1.777 \text{ GeV}$. Formula (18) extends the spectral function beyond the range accessible in the experiment. Formulas (17) and (18) provide practical realizations of the concept of the quark-hadron duality (see the original work [46]). In [47], this *ansatz* was used to determine the duality point s_d for a given value of $\Lambda_{\overline{\text{MS}}}$, the QCD scale parameter in the $\overline{\text{MS}}$ scheme. The perturbative component $v_1^{\text{pQCD}}(s)$ was constructed from the FOPT series (7) by choosing the normalization scale $\mu^2 = s_d$. Such a framework may be considered as a modification of FOPT. In [40], we have used the same *ansatz* for the spectral function. However, our strategy was somewhat different. Starting from the *ansatz* (18), we have determined the parameters $\Lambda_{\overline{\text{MS}}}$ and s_d self-consistently from the τ data. In contrast to [47], we have used the RG improved approximation $v_1^{\text{RG}}(s)$ to the spectral function. The function $v_1^{\text{RG}}(s)$ is calculated by insertion of the (truncated) RG improved series (8) into inversion formula (15). For $s > 0$, one finds [40]

$$v_1^{\text{pQCD}}(s) = v_1^{\text{RG}}(s) = v_1^{\text{APT}}(s), \quad (19)$$

⁴It is assumed that $0 < s_d < m_\tau^2$.

where $v_1^{\text{APT}}(s)$ is the spectral function determined in the sense of the Shirkov-Solovtsov APT

$$v_1^{\text{APT}}(s) = \frac{1}{2}(1 + r(s)), \quad \text{where} \quad r(s) = \frac{1}{\pi} \int_s^\infty \frac{\rho_{\text{eff}}(\sigma)}{\sigma} d\sigma. \quad (20)$$

It follows from the duality relation (18) that one may calculate in QCD perturbation theory the decay rate of the τ lepton into hadrons of invariant mass larger than $\sqrt{s_d}$

$$R_{\tau,V}^{\text{QCD}}|_{s>s_d} = 6|V_{ud}|^2 S_{\text{EW}} \int_{s_d}^{m_\tau^2} w_\tau(s) v_1^{\text{APT}}(s) ds, = R_{\tau,V}^{\text{exp}}|_{s>s_d} \quad (21)$$

so that

$$\Phi_\tau(s_d, \Lambda^2) = \int_{s_d}^{m_\tau^2} w_\tau(s) v_1^{\text{APT}}(s) ds = \int_{s_d}^{m_\tau^2} w_\tau(s) v_1^{\text{exp}}(s) ds. \quad (22)$$

Using relation (20), one may express the left-hand side of (22) in terms of the effective spectral density [40]

$$\begin{aligned} \Phi_\tau(s_d, \Lambda^2) &= (1 - \hat{s}_d)^3 (1 + \hat{s}_d) \frac{(1 + r(s_d))}{4} - \\ &\quad - \frac{1}{4\pi} \int_{\hat{s}_d}^1 y^{-1} (1 - y)^3 (1 + y) \rho_{\text{eff}}(m_\tau^2 y) dy, \end{aligned} \quad (23)$$

where $\hat{s}_d = s_d/m_\tau^2$.

Inserting the duality *ansatz* (18) into DR (14) one constructs the “semi-experimental” Adler function

$$D(Q^2)|_{\text{“s.exp”}} = D(Q^2, s_d)|_{\text{exp}} + D(Q^2, s_d)|_{\text{pQCD}}, \quad (24)$$

where the experimental and QCD components of the Adler function are determined by

$$\begin{aligned} D(Q^2, s_d)|_{\text{exp}} &= Q^2 \int_0^{s_d} \frac{2v_1^{\text{exp}}(s) ds}{(s + Q^2)^2}, \\ D(Q^2, s_d)|_{\text{pQCD}} &= Q^2 \int_{s_d}^\infty \frac{2v_1^{\text{pQCD}}(s) ds}{(s + Q^2)^2}. \end{aligned} \quad (25)$$

In general, the QCD component $v_1^{\text{pQCD}}(s)$ may contain the non-perturbative corrections coming from the OPE as well as the duality violating terms [19] not included into the OPE. Intuitively, it seems to us that the non-perturbative corrections are more essential in the region $0 < s < s_d$. In

what follows, we will ignore these non-perturbative corrections into QCD component of the spectral function and employ the perturbative approximation (19). The power suppressed part of the “semi-experimental” Adler function is defined as

$$D(Q^2)|_{\text{pw.s}} = D(Q^2)|_{\text{“s.exp”}} - D(Q^2)|_{\text{RG}}, \quad (26)$$

it can be represented in the form [40]

$$D(Q^2)|_{\text{pw.s}} = 2 \int_0^{s_p} K(Q^2, s) (v_1^{\text{exp}}(s) - v_1^{\text{APT}}(s)) ds - d_L(Q^2), \quad (27)$$

where $K(Q^2, s) = Q^2/(Q^2 + s)^2$. Formula (27) enables us to derive the asymptotic expansion at large Q^2

$$D(Q^2)|_{\text{pw.s}} \sim \sum_{n=1}^{\infty} \eta_n \left(\frac{\Lambda^2}{Q^2} \right)^n, \quad (28)$$

where $\Lambda \equiv \Lambda_{\overline{\text{MS}}}$ is the QCD scale parameter in the $\overline{\text{MS}}$ scheme and the coefficients η_n depend on the dimensionless ratios Λ^2/m_τ^2 and s_d/m_τ^2 . In the case of massless quarks, the gauge invariant operator of dimension two cannot be constructed. Thus $\eta_1 = 0$. This condition from the OPE leads to the equation relating the parameters s_d and Λ with the experimental spectral function [40]

$$\Phi_{\text{as}}(s_d, \Lambda^2) = \frac{1}{m_\tau^2} \int_0^{s_d} v_1^{\text{exp}}(s) ds, \quad (29)$$

where

$$\Phi_{\text{as}}(s_d, \Lambda^2) = \frac{\hat{s}_d}{2} (1 + r(s_d)) + \frac{1}{2\pi m_\tau^2} \int_0^{s_d} \rho_{\text{eff}}(\sigma) d\sigma + \frac{c_L}{2} \frac{\Lambda^2}{m_\tau^2}, \quad (30)$$

with $\hat{s}_d = s_d/m_\tau^2$ and the coefficient c_L is a positive number independent of Λ

$$c_L = \Lambda^{-2} \frac{1}{2\pi i} \oint_{C_L^+} d(\zeta)|_{\text{RG}} d\zeta = \frac{1}{2\pi} \frac{s_L}{\Lambda^2} \int_{-\pi}^{\pi} d(s_L + s_L e^{i\phi})|_{\text{RG}} d\phi, \quad (31)$$

here s_L being the Landau singularity of the running coupling. It is proportional to Λ^2 ⁵. Numerical values of the coefficient c_L calculated in the $\overline{\text{MS}}$

⁵Analytic expressions for s_L in the $\overline{\text{MS}}$ scheme up to fourth order in perturbation theory may be found in [45].

TABLE 1. Numerical values of the coefficient c_L in the $\overline{\text{MS}}$ scheme calculated by means of the four-loop order exact numeric running coupling. For the coupling, the RG equation is solved numerically.

	Approximations to the Adler function				
	LO	NLO	N ² LO	N ³ LO	N ⁴ LO
c_L	0.301262	0.453421	0.555401	0.651373	0.721687

scheme are listed in Table 1. In the calculations we have used the approximations to the Adler function of increasing order ⁶. All approximations have been constructed with the four-loop order exact (numeric) running coupling. For the unknown $\mathcal{O}(\alpha_s^5)$ correction to the Adler function, we use the geometric estimate $d_5 = d_4(d_4/d_3) = 378$ [5].

An important remark is in order here. The advantage of the approximation $v_1^{\text{APT}}(s)$ is that it correctly describes asymptotic behavior of the exact function as $s \rightarrow \infty$; in this limit $v_1^{\text{APT}}(s) \rightarrow 1/2$. In contrast, in the same limit, the FOPT approximation $v_1^{\text{FOPT}}(s)$ increases as a polynomial of logarithm. This shortcoming of FOPT is irrelevant as far as the duality relation (22) is concerned. However, Eq. (29) depends on the ultraviolet properties of the Adler function. This discussion suggests that a more consistent framework should be constructed in the contour improved scheme. In this work, we will refer to the new framework as the dispersive contour improved perturbation theory (DCIPT) ⁷. Although technically the DCIPT resembles APT, there are significant differences between the two frameworks. Thus, in the DCIPT we do not mention modifications of the QCD β -function and running coupling.

3. NUMERICAL RESULTS

The parameters s_d and Λ may be extracted from the data by solving the system of equations

$$\Phi_\tau(s_d, \Lambda^2) = \int_{s_d}^{m_\tau^2} w_\tau(s) v_1^{\text{exp}}(s) ds, \quad (32)$$

$$\Phi_{\text{as}}(s_d, \Lambda^2) = \frac{1}{m_\tau^2} \int_0^{s_d} v_1^{\text{exp}}(s) ds, \quad (33)$$

⁶We use the abbreviation N^kLO to denote the order $\mathcal{O}(\alpha_s^{k+1})$ approximation to the Adler function.

⁷In [40], we used the abbreviation APT⁺.

TABLE 2. Central values for the parameters in the $\overline{\text{MS}}$ scheme extracted from the 2005 ALEPH vector τ data order-by-order within the DCIPT. These results correspond to the four-loop order running coupling.

Observable	Approximation to the Adler function				
	LO	NLO	N ² LO	N ³ LO	N ⁴ LO
$s_d \text{ GeV}^2$	1.707	1.710	1.709	1.707	1.705
$\Lambda \text{ GeV}$	0.486	0.378	0.348	0.332	0.323
$\alpha_s(m_\tau^2)$	0.401	0.337	0.321	0.313	0.308

where the functions Φ_τ and Φ_{as} are defined in formulas (23) and (30). The right-hand sides of Eqs. (32)-(33) are determined in terms of the empirical function $v_1^{\text{exp}}(s)$. We have reconstructed the experimental vector spectral function from the 2005 ALEPH spectral data for the vector invariant mass squared distribution [8]. This was done, with the values $|V_{ud}| = 0.9746 \pm 0.0006$ and $S_{\text{EW}} = 1.0198$ quoted in [7]. To interpolate the spectral function between the fixed experimental values of the energy squared, we use cubic splines. Evidently, the mean values of the parameters should be determined from the mean value of $v_1^{\text{exp}}(s)$. The error analysis is based on the system of equations (32)-(33) [40]. To determine the experimental uncertainties on the extracted values of the parameters, we use covariance matrices provided by ALEPH. Unfortunately, in the earlier work [40], we used (inconsistently) the N²LO value $c_L = 0.555401$ (see Table 1) in all other orders. In this work, we present corrected results.

In general, the system (32)-(33) has more than one solution. For phenomenological reasons, we look for a solution in the limited region $280 \text{ MeV} < \Lambda < 420 \text{ MeV}$. In this region, the system has only one solution. In Table 2, we give the central values for the parameters extracted from the 2005 ALEPH data within the new (DCIPT) framework. Formally, we can write a series for the numerical value of the coupling constant as follows:

$$\alpha_s(m_\tau^2)|_{\text{N}^4\text{LO}} = \alpha_s(m_\tau^2)|_{\text{LO}} + \sum_{k=1}^4 \Delta_k,$$

where $\Delta_k = \alpha_s(m_\tau^2)|_{\text{N}^k\text{LO}} - \alpha_s(m_\tau^2)|_{\text{N}^{k-1}\text{LO}}$. Using the numbers listed in Table 2, we obtain the series

$$\alpha_s(m_\tau^2)|_{\text{N}^4\text{LO}}^{\text{DCIPT}} = 0.401 - 0.064 - 0.016 - 0.009 - 0.005. \quad (34)$$

In [40], from the same data, we have obtained the CIPT series

$$\alpha_s(m_\tau^2)|_{\text{N}^4\text{LO}}^{\text{CIPT}} = 0.485 - 0.095 - 0.023 - 0.013 - 0.007. \quad (35)$$

TABLE 3. The changes of the leading term induced by the consecutive corrections in the series (34) and (35).

Perturbative orders	NLO	N ² LO	N ³ LO	N ⁴ LO
DCIPT	15.9%	4.0%	2.2%	1.2%
CIPT	19.6%	4.7%	2.7%	1.4%

In Table 3, we give the changes (in percents) of the leading term induced by the consecutive corrections in the DCIPT and CIPT series. One sees that the DCIPT series (34) converges more rapidly.

In this paper, we will estimate only the so-called indicative theoretical errors. These are defined as a half of the last retained term in the series [41]. As is pointed out in [41], this definition of the error is heuristic and indicative. From the DCIPT series (34), we obtain the estimates

$$\begin{aligned}
\alpha_s(m_\tau^2)|_{\text{NLO}} &= 0.337 \pm 0.016_{\text{exp}} \pm 0.032_{\text{th}} \\
\alpha_s(m_\tau^2)|_{\text{N}^2\text{LO}} &= 0.321 \pm 0.016_{\text{exp}} \pm 0.008_{\text{th}} \\
\alpha_s(m_\tau^2)|_{\text{N}^3\text{LO}} &= 0.313 \pm 0.014_{\text{exp}} \pm 0.004_{\text{th}} \\
\alpha_s(m_\tau^2)|_{\text{N}^4\text{LO}} &= 0.308 \pm 0.014_{\text{exp}} \pm 0.002_{\text{th}} \pm (0.0045_{d_5}), \quad (36)
\end{aligned}$$

here we have also included the experimental errors ⁸. Analogically, in the case of CIPT we have found (see appendix to [40])

$$\begin{aligned}
\alpha_s(m_\tau^2)|_{\text{NLO}} &= 0.390 \pm 0.011_{\text{exp}} \pm 0.048_{\text{th}} \\
\alpha_s(m_\tau^2)|_{\text{N}^2\text{LO}} &= 0.367 \pm 0.009_{\text{exp}} \pm 0.012_{\text{th}} \\
\alpha_s(m_\tau^2)|_{\text{N}^3\text{LO}} &= 0.354 \pm 0.008_{\text{exp}} \pm 0.007_{\text{th}} \\
\alpha_s(m_\tau^2)|_{\text{N}^4\text{LO}} &= 0.347 \pm 0.008_{\text{exp}} \pm 0.003_{\text{th}} \pm (0.0065_{d_5}). \quad (37)
\end{aligned}$$

The N⁴LO estimates in (36) and (37) correspond to the central value $d_5 = 378$. The additional theoretical error in the coupling constant induced from the uncertainty in the fifth order unknown coefficient ($d_5 = 378 \pm 378$) takes the values 0.0045 ($\approx 1.5\%$) and 0.0065 ($\approx 1.9\%$) in the new and standard extraction procedures respectively. Comparing formulas (36) and (37), one sees that within the DCIPT the indicative theoretical errors take smaller values. In contrast to this, the experimental errors on the values of α_s increase by the factor of 1.75 within the new procedure. It is remarkable that the more reliable estimate of the theoretical error presented in [5] within CIPT (at N⁴LO) is close to our estimate of the error presented in formula (37).

⁸For the error analysis we use formulas derived in [40].

TABLE 4. Comparison of the DCIPT and CIPT τ decay determinations of the strong coupling constant at the scale $M_Z = 91.187$ GeV. Two errors are given, the experimental (first number) and the error from the evolution procedure (second number).

Approximation	$\alpha_s(M_z^2) _{\text{DCIPT}}$	$\alpha_s(M_z^2) _{\text{CIPT}}$
N ² LO	$0.1187 \pm 0.0019 \pm 0.0005$	$0.1238 \pm 0.0009 \pm 0.0005$
N ³ LO	$0.1176 \pm 0.0018 \pm 0.0005$	$0.1224 \pm 0.0009 \pm 0.0005$
N ⁴ LO	$0.1170 \pm 0.0018 \pm 0.0005$	$0.1217 \pm 0.0009 \pm 0.0005$

Similarly, determining the theoretical and experimental errors on the parameter s_d , we find stable results

$$\begin{aligned}
s_d|_{\text{NLO}} &= 1.710 \pm 0.054_{\text{exp}} \pm 0.002_{\text{th}} \text{ GeV}^2 \\
s_d|_{\text{N}^2\text{LO}} &= 1.709 \pm 0.054_{\text{exp}} \pm 0.001_{\text{th}} \text{ GeV}^2 \\
s_d|_{\text{N}^3\text{LO}} &= 1.707 \pm 0.054_{\text{exp}} \pm 0.001_{\text{th}} \text{ GeV}^2 \\
s_d|_{\text{N}^4\text{LO}} &= 1.705 \pm 0.054_{\text{exp}} \pm 0.001_{\text{th}} \text{ GeV}^2.
\end{aligned} \tag{38}$$

It is seen from (38), that the estimate for the duality point s_d decreases very slowly as the order of the perturbation theory increases. Practically, it is constant, $s_d \approx 1.71 \pm 0.05 \text{ GeV}^2$.

Usually, it is convenient to perform evolution of the α_s results to the reference scale $M_z = 91.187$ GeV. This is done by using the RG equation and appropriate matching conditions at the heavy quark (charm and bottom) thresholds (see [48] and references therein). The three-loop level matching conditions in the $\overline{\text{MS}}$ scheme were derived in [49]. In this paper, we follow the work [50]. We perform the matching at the matching scale $m_{\text{th}} = 2\mu_{\text{h}}$ where μ_{h} is a scale invariant $\overline{\text{MS}}$ mass of the heavy quark $\mu_{\text{h}} = \overline{m}_{\text{h}}(\mu_{\text{h}})$. We assume for the scale invariant $\overline{\text{MS}}$ masses the estimates of the Particle Data Group $\mu_c = 1.29^{+0.05}_{-0.11}$ GeV and $\mu_b = 4.19^{+0.18}_{-0.06}$ GeV [51]. In the evolution procedure, we use the exact numeric four-loop running coupling. In Table 4, we compare the estimates for $\alpha_s(M_z^2)$ obtained from the two (DCIPT and CIPT) τ -decay determinations of the coupling constant.

As is stated above, we have used the $\overline{\text{MS}}$ scheme four-loop running coupling uniformly in all calculations, whereas the order of approximation to the Adler function has been varied consecutively. To perform a more accurate test, let us now employ the same orders to approximate the β and Adler functions. The coefficient c_L is accordingly recalculated. In Table 5, we present the results of the improved test. Comparing the numbers in Tables 2 and 5, we see that the extracted values for the parameters, beyond LO, are very close (the N²LO and N³LO results practically coincide).

TABLE 5. Testing the stability of the results with regard to higher order perturbation theory corrections. Here, the approximations to the β and Adler functions are chosen consistently, at the same orders.

Perturbative order	c_L	$s_d \text{ GeV}^2$	$\alpha_s(m_\tau^2) _{\text{DCIPT}}$
LO	0.444444	1.721	0.394
NLO	0.336798	1.713	0.335
N ² LO	0.527261	1.709	0.321
N ³ LO	0.651373	1.707	0.313

Let us now employ the renormalization scheme invariant extraction method (RSI) of [41] to extract the numerical values of the coupling constant from the 2005 ALEPH $V + A$ spectral data. We shall also include into consideration the recently calculated $\mathcal{O}(\alpha_s^4)$ term in the series expansion of the Adler function. The advantage of this technique is that one starts from the physical quantity, the effective charge defined by

$$a_\tau = \frac{\alpha_\tau}{\pi} = \delta_{\text{th}}^{(0)}, \quad (39)$$

where $\delta^{(0)}$ is the perturbative correction to the τ -decay rate. The running coupling a_τ defines the internal scheme for the physical quantity. The numerical value for the QCD scale in the internal scheme, Λ_τ , is extracted by using the equation $a_\tau(m_\tau^2) = \delta_{\text{exp}}^{(0)}$. The $\overline{\text{MS}}$ scheme scale parameter is determined according to the relation $\Lambda_{\overline{\text{MS}}} = \Lambda_\tau \exp\{-5.20232/(2\beta_0)\}$, where $\beta_0 = 9/2$. Formulas for calculation of the coefficients of the function β_τ (the β -function in the internal scheme) may be found in works [3, 41]. For the experimental value of the perturbative part of the τ decay rate in the non-strange channel, we use the updated value

$$\delta_{\text{exp}}^{(0)}|_{V+A} = 0.2042 \pm 0.0050_{\text{exp}},$$

evaluated recently in [24]. For consistency, we use the same orders to approximate the β and Adler functions in the $\overline{\text{MS}}$ -scheme. In Table (6), we compare the RSI and DCIPT determinations of the coupling constant order-by-order in perturbation theory. The relevant channels which have been used to extract the coupling are indicated by subscripts. It is seen from the Table, that beyond NLO the two determinations of the coupling constant are in good agreement.

As is known, mathematically, the extraction of QCD parameters from experimental data via the sum rules constitutes the so-called ill-posed inverse problem (analytical continuation of an approximately known function) [17]. Small changes in the input data may lead to large changes in the output. In this regard, it is desirable to check the new framework. To do such a test,

TABLE 6. Comparison of the RSI and DCIPT determinations of the $\overline{\text{MS}}$ coupling constant from the τ -decay data. Experimental errors are given only.

Perturbative order	$\alpha_s(m_\tau^2) _{V+A}^{\text{RSI}}$	$\alpha_s(m_\tau^2) _V^{\text{DCIPT}}$
NLO	0.278 ± 0.003	0.335 ± 0.016
N ² LO	0.319 ± 0.004	0.321 ± 0.016
N ³ LO	0.312 ± 0.004	0.313 ± 0.014

one may extract the values of the parameters using the data from different τ -decay experiments. As a different experimental data, let us employ the 1998 OPAL experimental data on the non-strange isovector vector spectral function which is publicly available⁹. The data are arranged in 100 bins with bin size 0.032 GeV^2 , starting from $s = 0.016 \text{ GeV}^2$. Note that, the OPAL data correspond to the branching fractions available in 1998, as well as the then-current values of V_{ud} and the electronic branching fraction B_e . These parameters have been updated since then. The 2005 ALEPH analysis is more recent and based on more statistics. However, it was pointed out in [19] that the correlations due to unfolding have been omitted in the original ALEPH analysis. Thus the publicly available covariance matrices [8] should be corrected. Fortunately, the above mentioned obstacles have little relevance to the problem under investigation. In fact, our aim is to investigate the impact of the specific formulation of the quark-hadron duality (as given in (18)) on the extracted value of α_s .

Inserting into the system of equations (32)-(33) the empirical vector spectral function reconstructed from the 1998 OPAL data¹⁰, we solve the system numerically. We use the N²LO and N³LO approximations to the Adler function combined for consistency with the three- and four-loop order $\overline{\text{MS}}$ running couplings respectively. To determine the experimental uncertainties on the extracted values of the parameters we use covariance matrices provided by OPAL (relevant formulas were derived in the appendix to [40]). For the duality point, we obtain stable result

$$\begin{aligned} s_d|_{\text{N}^2\text{LO}} &= (1.680 \pm 0.100_{\text{exp}}) \text{ GeV}^2, \\ s_d|_{\text{N}^3\text{LO}} &= (1.679 \pm 0.100_{\text{exp}}) \text{ GeV}^2, \end{aligned} \quad (40)$$

the central value in (40) decreases very slowly as the order in perturbation theory increases. We see that the numerical values for the duality point extracted from the ALEPH and OPAL data are close (cf. (38) and (40)).

⁹I would like to thank S. Menke and S. Peris for making the data available to me.

¹⁰We use cubic splines to interpolate the data.

For the strong coupling, from the OPAL data, we find the values

$$\begin{aligned}\alpha_s(m_\tau^2)|_{\text{N}^2\text{LO}} &= 0.296 \pm 0.025_{\text{exp}} \\ \alpha_s(m_\tau^2)|_{\text{N}^3\text{LO}} &= 0.290 \pm 0.023_{\text{exp}},\end{aligned}\quad (41)$$

the central values here are somewhat smaller as compared to the corresponding values extracted from the ALEPH data (cf. formulas (36) and (41)). However, the two determinations of the coupling constant are consistent within their mutual errors. It should be remarked that, in the case of the OPAL data, we have obtained larger experimental uncertainties on the numerical values of the parameters. For comparison, the original OPAL analysis of the same data, within CIPT, gave the value [9]

$$\alpha_s(m_\tau^2)|_{\text{N}^3\text{LO}} = 0.348 \pm 0.009_{\text{exp}} \pm 0.019_{\text{th}}. \quad (42)$$

Comparing the numbers in formulas (41) and (42), one sees that the DCIPT determination of α_s is significantly smaller, and the two determinations are not consistent within their mutual errors. Performing evolution of the α_s values (41) to the Z^0 -mass scale, we obtain

$$\begin{aligned}\alpha_s(M_z^2)|_{\text{N}^2\text{LO}} &= 0.1154 \pm 0.0034_{\text{exp}} \pm 0.0005_{\text{ev}}, \\ \alpha_s(M_z^2)|_{\text{N}^3\text{LO}} &= 0.1145 \pm 0.0033_{\text{exp}} \pm 0.0005_{\text{ev}}.\end{aligned}\quad (43)$$

4. CONCLUSION

We have extracted numerical values for the $\overline{\text{MS}}$ scheme strong coupling constant α_s from the τ -lepton decay data. The data provided by the 2005 ALEPH and 1998 OPAL experiments are employed. We examine in detail the dispersive approach to the τ -decay suggested in our earlier work [40]. The errors observed in some numerical results of [40] have been corrected. Accordingly some of the conclusions of [40] are changed.

The new framework is based on the approximations to the Adler function which have correct analytical properties. Thus the application of the FESR (6) is mathematically justified. Moreover, these approximations correctly reproduce the infrared and ultraviolet behavior of the exact Adler function. In contrast, in the standard approaches (FOPT, CIPT or APT) some of these properties of the Adler function are violated. The global quark-hadron duality is used in the limited region of the energy squared $s_d < s < m_\tau^2$ ($E_d = \sqrt{s_d} \approx 1.31$ GeV). In the low energy region, $0 < s < s_d$, the hadronic spectral function is reconstructed from the experimental data. This enabled us to reduce the effects of duality violations coming from the low energy region. In fact, one expects in this region sizeable non-perturbative corrections to the Adler function which cannot be reproduced within the OPE (for example, the non-perturbative contributions from the p -meson resonance).

Technically, the new method is based on the system of equations (32)-(33). The first equation follows from the concept of global quark-hadron duality employed on the limited interval of the energy squared, $s_d < s < m_\tau^2$. The second equation is a consequence of the OPE which imposes the restrictions on the ultraviolet behavior of the Adler function. The parameters α_s and s_d are simultaneously extracted from the data. We have examined numerical stability of the extracted values of the parameters order-by-order in perturbation theory. The new framework (DCIPT) and the standard (CIPT) are systematically compared. We have demonstrated that the DCIPT determinations of the strong coupling constant are more stable against perturbation theory corrections (see Table 3). The central value of the coupling constant definitely became smaller as compared to the CIPT result (cf. Eqs. (36) and (37)). The changes in the central values are not within the quoted experimental and theoretical errors. Using the error estimated within the DCIPT, $\sigma = \sqrt{\sigma_{\text{exp}}^2 + \sigma_{\text{th}}^2} \approx 0.0151$, we find that at N³LO the central values of $\alpha_s(m_\tau^2)$ in formulas (36) and (37) differ from each other in about 2.7 standard deviation. However, assuming the error estimated within CIPT, $\sigma \approx 0.0107$, one finds even large difference, 3.8σ ¹¹. A shortcoming of the new procedure is the increased experimental error on the extracted values of α_s . This is a direct consequence of the reduction of the duality region.

Having included into analysis the fourth order coefficient d_4 and the geometric estimate $d_5 = 378$, we have observed excellent agreement between the lattice and τ -decay determinations of the strong coupling constant. At N⁴LO, the central value for α_s (see Table 4) coincides with the central value quoted in [6] (see formula (2)). For this reason, we believe that the DCIPT provides better approximation as compared to CIPT.

For comparison purposes, we have extracted the strong coupling constant from the 2005 ALEPH V+A data by using the RSI method of work [41], extending the result of [41] up to N³LO (see Table 6). Good agreement between the RSI and DCIPT determinations of α_s has been observed.

The duality point s_d is found to be surprisingly stable with respect to higher order QCD corrections: $s_d = 1.71 \pm 0.05_{\text{exp}} \pm 0.00_{\text{th}}$ GeV² (see Tables 2 and 5). In Table 5, we have performed a more accurate test of stability of the numerical results, choosing consistently the orders of the approximations to the β - and Adler functions.

To examine the stability of the numerical results with respect to the change in the input data, we have also analyzed the 1998 OPAL data for the non-strange vector spectral function. The extracted values for the parameters from the ALEPH and OPAL data are found to be consistent.

¹¹Due to the large experimental error within the DCIPT, $\sigma(\text{DCIPT})/\sigma(\text{CIPT}) \approx 1.4$.

The procedure suggested here can obviously be extended for analyzing the non-strange τ -data from the axial-vector (A) and vector plus axial-vector (V+A) channels.

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