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OSCILLATION CRITERIA OF SOLUTIONS OF SECOND ORDER OF LINEAR DIFFERENCE EQUATIONS

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Consider the difference equation

$$\Delta^2 u(k) + \sum_{j=1}^{m} p_j(k)\, u(\tau_j(k)) = 0, \tag{1}$$

where $m \geq 1$ is a natural number, $p_j : N \to R_+$, $\tau_j : N \to N$, $(j = 1, \dots, m)$ are functions defined on the set of natural numbers $N = \{1, 2, \dots\}$, $\Delta u(k) = u(k+1) - u(k)$ and $\Delta^2 = \Delta \circ \Delta$. Everywhere below it is assumed that

$$\lim_{k \to +\infty} \tau_j(k) = +\infty \quad (j = 1, \dots, m),$$
$$\sup\big\{p_j(i) : i \geq k\big\} > 0 \quad \text{for} \quad k \in N \quad (j = 1, \dots, m).$$

For each $n \in N$ denote $N_n = \{n, n+1, \dots\}$.

Definition 1. For each $n \in N$ denote $n_0 = \min\big\{k \geq n : \bigcup\limits_{j=1}^{m} \tau_j(N_k) \subset N_n\big\}$. We will call a function $u : N_n \to R$ a proper solution of the equation (1) if it satisfies (1) on N_{n_0} and $\sup\{|u(i)| : i \geq k\} > 0$ for any $k \in N_n$.

Definition 2. We say that a proper solution $u : N_n \to R$ of the equation (1) is oscillatory if for any $k \in N_n$ there are $n_1, n_2 \in N_k$ such that $u(n_1)u(n_2) \leq 0$. Otherwise the solution is called nonoscillatory.

The problem of oscillation of solutions of the equation of the type (1) has been studied by several authors, see e.g. [1–6] and the references therein. Everywhere below it is assumed that the conditions

$$\sum_{k=1}^{+\infty} k \bigg(\sum_{j=1}^{m} p_j(k) \bigg) = +\infty, \tag{2}$$

and

$$\sum_{k=1}^{+\infty} \bigg(\sum_{j=1}^{m} \tau_j(k)\, p_j(k) \bigg) = +\infty \tag{3}$$

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are fulfilled.

Using the fixed point principle, one can easily show that the conditions (2) and (3) are necessary for oscillation of all solutions of the equation (1) [6].

The obtained results make those obtained in [6] more precise even in the case considered there when the conditions

$$\liminf_{k\to+\infty}\frac{\tau_j(k)}{k}>0\quad (j=1,\dots,m)$$

is fulfilled. Besides the paper covers also the cases where the latter inequality does not hold.

Lemma 1. *Let* $\tau_j : N\to N$ $(j=1,\dots,m)$ *and* (1) *be fulfilled. Then there exists a nondecreasing function* $\sigma : N\to N$ *such that*

$$\begin{aligned}
&1)\qquad \lim_{k\to+\infty}\sigma(k)=+\infty,\\
&2)\qquad \sigma(k)\le\min\{k,\tau_j(k): j=1,\dots,m\},\\
&3)\qquad \sigma(N_k)\supset\bigcup_{j=1}^{m}\tau_j(N_k)\ \ \textit{for any}\ \ k\in N.
\end{aligned}\tag{4}$$

Let $k_0\in N$. Denote by U_{k_0} the set of all proper solutions of (1) satisfying $u(k)>0$ for $k\in N_{k_0}$.

Theorem 1. *Let* $k_0\in N$, $U_{k_0}\ne\varnothing$ *and* σ *be any nondecreasing function satisfying* (4) (*such a function exists due to Lemma* 1). *Then there exists* $\lambda\in[0,1]$ *such that*

$$\limsup_{\varepsilon\to0+}\Big(\liminf_{k\to+\infty}\rho(k,\varepsilon,\lambda)\Big)\le1,$$

where

$$\begin{aligned}
\rho(k,\varepsilon,\lambda)=k^{-\lambda-h_{2\varepsilon}(\lambda)}\sum_{i=1}^{k-1}(\sigma(i))^{h_{1\varepsilon}(\lambda)+h_{2\varepsilon}(\lambda)}\times\\
\times\sum_{l=i}^{+\infty}\Big(\sum_{j=1}^{m}p_j(l)(\tau_j(l))^{\lambda-h_{1\varepsilon}(\lambda)}\Big),
\end{aligned}\tag{5}$$

$$h_{1\varepsilon}(\lambda)=\begin{cases}0 & \textit{for}\ \lambda=0,\\ \varepsilon & \textit{for}\ \lambda\in(0,1],\end{cases}\qquad h_{2\varepsilon}(\lambda)=\begin{cases}0 & \textit{for}\ \lambda=1,\\ \varepsilon & \textit{for}\ \lambda\in[0,1).\end{cases}\tag{6}$$

Theorem 2. *Let* σ *be any nondecreasing function satisfying* (4) (*such a function exists due to Lemma* 1), *and for any* $\lambda\in[0,1]$

$$\limsup_{\varepsilon\to0+}\Big(\liminf_{k\to+\infty}\rho(k,\varepsilon,\lambda)\Big)>1.$$

where the function ρ *is defined by* (5), (6). *Then any proper solution of the equation* (1) *is oscillatory.*

Theorem 3. *Let* $\alpha_j \in (0,+\infty)$ $(j=1,\dots,m)$ *and*

$$\liminf_{k\to+\infty} \frac{\tau_j(k)}{k^{\alpha_j}} > 0 \quad (j=1,\dots,m). \tag{7}$$

Then for all proper solutions of (1) *to be oscillatory it is sufficient that for any* $\lambda \in [0,1]$

$$\limsup_{\varepsilon\to0+}\Bigg(\liminf_{k\to+\infty} k^{-\lambda-h_{2\varepsilon}(\lambda)} \sum_{i=1}^{k-1} i^{\alpha(h_{1\varepsilon}(\lambda)+h_{2\varepsilon}(\lambda))}\times \\ \times \sum_{l=i}^{+\infty}\bigg(\sum_{j=1}^{m} p_j(l)\,(\tau_j(l))^{\lambda-h_{1\varepsilon}(\lambda)}\bigg)\Bigg) > 1,$$

where

$$\alpha = \min\{1,\alpha_1,\dots,\alpha_m\}. \tag{8}$$

Theorem 4. *Let the conditions* (7) *be fulfilled and for any* $\lambda\in[0,1]$

$$\limsup_{\varepsilon\to0+}\Bigg(\liminf_{k\to+\infty} k^{1-\lambda+\alpha h_{1\varepsilon}(\lambda)+(\alpha-1)h_{2\varepsilon}(\lambda)}\times \\ \times \sum_{i=k}^{+\infty}\bigg(\sum_{j=1}^{m} p_j(i)\,(\tau_j(i))^{\lambda-h_{1\varepsilon}(\lambda)}\bigg)\Bigg) > \lambda,$$

where the functions $h_{1\varepsilon}$, $h_{2\varepsilon}$ *and* α *are given by* (6) *and* (8). *Then any proper solution of* (1) *is oscillatory.*

Theorem 5. *Let the conditions* (7) *hold and for any* $\lambda\in[0,1]$

$$\limsup_{\varepsilon\to0+}\Bigg(\liminf_{k\to+\infty} k^{1+(\alpha-1)(h_{2\varepsilon}(\lambda)+h_{1\varepsilon}(\lambda))}\times \\ \times \sum_{i=k}^{+\infty}\bigg(\sum_{j=1}^{m} p_j(i)\,\Big(\frac{\tau_j(i)}{i}\Big)^{\lambda-h_{1\varepsilon}(\lambda)}\bigg)\Bigg) > \lambda(1-\lambda).$$

Then any proper solution of (1) *is oscillatory.*

Theorem 5′. *Let the condition* (7) *be fulfilled with* $\alpha_i \ge 1$ $(i=1,\dots,m)$. *Then for any proper solution of* (1) *to be oscillatory it is sufficient that for any* $\lambda\in[0,1]$

$$\limsup_{\varepsilon\to0+}\Bigg(\liminf_{k\to+\infty} k\sum_{i=k}^{+\infty}\bigg(\sum_{j=1}^{m} p_j(i)\Big(\frac{\tau_j(i)}{i}\Big)^{\lambda-h_{1\varepsilon}(\lambda)}\bigg)\Bigg) > \lambda(1-\lambda).$$

Theorem 5′ makes Theorem 3.2 of [1] more precise.

Corollary 1. *Let there exist* α_j $(j = 1, \dots, m)$ *such that* $\alpha_j \in (0, +\infty)$ *and*

$$\liminf_{i\to+\infty} \frac{\tau_j(i)}{i} = \alpha_j \quad (j = 1, \dots, m). \tag{9}$$

Then for any $\lambda \in [0, 1]$ *the condition*

$$\liminf_{k\to+\infty} k \sum_{i=k}^{+\infty} \Bigg(\sum_{j=1}^{m} p_j(i)\, \alpha_j^{\lambda} \Bigg) > \lambda(1-\lambda)$$

is sufficient for oscillation of all proper solution of (1).

Corollary 2. *Let the condition* (9) *be fulfilled and there exist* $c_j \in (0, +\infty)$ $(j = 1, \dots, m)$ *and a function* $p : N \to [0, +\infty)$ *such that* $p_j(k) \ge c_j p(k)$ $(j = 1, \dots, m)$. *Then the condition*

$$\liminf_{k\to+\infty} k \sum_{i=k}^{+\infty} p(i) > \\ > \max\Bigg\{ \lambda(1-\lambda) \Bigg(\sum_{j=1}^{m} c_j \alpha_j^{\lambda} \Bigg)^{-1} : \lambda \in [0, 1] \Bigg\} \tag{10}$$

is sufficient for oscillation of all proper solutions of (1).

It should be noted that for any $m \in N$ the inequality (10) can not be changed by the nonstrict one. Otherwise Corollary 2, in general, will not be valid.

Corollary 3. *Let the condition* (7) *be fulfilled, there exist a nonincreasing function* $\widetilde{p} \in C(R_+; R_+)$ *and a nondecreasing function* $\widetilde{\tau} \in C(R_+; R_+)$ *such that* $\lim\limits_{t\to+\infty} \widetilde{\tau}(k) = +\infty$ *and*

$$p_j(k) \ge c_j\, \widetilde{p}(k), \quad \tau_j(k) \ge d_j\, \widetilde{\tau}(k) \quad (j = 1, \dots, m), \tag{11}$$

where $c_j, d_j \in (0, +\infty)$. *Let, moreover, for any* $\lambda \in [0, 1]$ *the condition*

$$\limsup_{\varepsilon\to0+} \Bigg(\liminf_{k\to+\infty} k^{1+(\alpha-1)(h_{1\varepsilon}(\lambda)+h_{2\varepsilon}(\lambda))} \int_{k-1}^{+\infty} \widetilde{p}(1+\xi) \big(\widetilde{\tau}(\xi)\big)^{\lambda - h_{1\varepsilon}(\lambda)} d\xi \Bigg) > \\ > \lambda(1-\lambda) \Bigg(\sum_{j=1}^{m} c_j\, d_j^{\lambda} \Bigg)^{-1}$$

be fulfilled, where α *is given by* (8). *Then any proper solution of* (1) *is oscillatory.*

Corollary 4. *Let* $c_j, d_j, \alpha \in (0, +\infty)$ $(j = 1, \dots, m)$ *and*

$$p_j(i) \ge \frac{c_j}{i^2}, \quad \tau_j(i) \ge d_j\, i^{1+\alpha} \quad (j = 1, \dots, m).$$

Then any proper solution of (1) *is oscillatory.*

Corollary 5. *Let the conditions* (7) *be fulfilled and there exist nondecreasing functions* $\widetilde{\tau}, \widetilde{p} \in C(R_+; R_+)$ *such that the conditions* (11) *are fulfilled, where* $c_j, d_j \in (0, +\infty)$ $(j = 1, \dots, m)$*. Let, moreover, for any* $\lambda \in [0, 1]$ *the condition*

$$\limsup_{\varepsilon \to 0+} \left(\liminf_{k \to +\infty} k^{1+(\alpha-1)(h_{1\varepsilon}(\lambda)+h_{2\varepsilon}(\lambda))} \int_k^{+\infty} \widetilde{p}(s)\, \widetilde{\tau}^{\,\lambda - h_{1\varepsilon}(\lambda)}(s) ds \right) >$$

$$> \lambda(1-\lambda) \left(\sum_{j=1}^{m} c_j\, d_j^{\lambda} \right)^{-1}$$

be fulfilled. Then any proper solution of (1) *is oscillatory.*

Corollary 6. *Let* $c_j, d_j, \alpha \in (0, +\infty)$ $(j = 1, \dots, m)$ *and*

$$p_j(i) \geq \frac{c_j}{i^{\beta}}, \quad \tau_j(i) \geq d_j\, i^{1-\alpha} \quad (j = 1, \dots, m),$$

where $\beta < 2 - \alpha$*,* $\alpha \in (0, 1)$*. Then any proper solution of* (1) *is oscillatory.*

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