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Nonlinear functional differential equations with Properties A and B

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Abstract

In this paper, an nth order functional differential equation is considered for which the generalized Emden–Fowler-type equation

$$u^{(n)}(t) + p(t)\big|u(t)\big|^{\mu(t)} \operatorname{sign} u(t) = 0, \quad t \geqslant 0, \tag{0.1}$$

can be considered as a nonlinear model. Here, we assume that $n \geqslant 2$, $p \in L_{\mathrm{loc}}(R_+; R)$, and $\mu \in C(R_+; (0,1])$ is a nondecreasing function. In case $\mu(t) \equiv \mathrm{const} > 0$, oscillatory properties of Eq. (0.1) have been extensively studied, where as if $\mu(t) \not\equiv \mathrm{const}$, to the extent of authors' knowledge, the analogous questions have not been examined. It turns out that the oscillatory properties of Eq. (0.1) substantially depend on the rate at which the function $\mu^+ - \mu(t)$ tends to zero as $t \to +\infty$, where $\mu^+ = \lim_{t\to+\infty} \mu(t)$. In this paper, new sufficient conditions for a general class of nonlinear functional differential equations to have Properties A and B are established, and these results apply to the special case of Eq. (0.1) as well.

Keywords: Functional differential equations; Higher order equations; Property A; Property B; Oscillation; Convergence to zero

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1. Introduction

Let $\tau \in C(R_+; R_+)$ with $\lim_{t\to+\infty} \tau(t) = +\infty$. Let $V(\tau)$ denote the set of continuous mappings $F : C(R_+; R) \to L_{\text{loc}}(R_+; R)$ satisfying the condition

$$F(x)(t) = F(y)(t) \quad \text{holds for any } t \in R_+ \text{ and } x, y \in C(R_+; R), \text{ provided that } x(s) = y(s) \text{ for } s \geqslant \tau(t).$$

This work is dedicated to the study of oscillatory properties of the functional differential equation

$$u^{(n)}(t) + F(u)(t) = 0, \tag{1.1}$$

where $n \geqslant 2$ and $F \in V(\tau)$. For any $t_0 \in R_+$, we let $H_{t_0,\tau}$ denote the set of all functions $u \in C(R_+; R)$ satisfying $u(t) \neq 0$ for $t \geqslant t_*$, where $t_* = \min\{t_0, \tau_*(t_0)\}$ and $\tau_*(t) = \inf\{\tau(s):\ s \geqslant t\}$. Throughout this work, where ever the notation $V(\tau)$ and $H_{t_0,\tau}$ occur, it will be understood that the function τ satisfies the conditions stated above, unless specified otherwise. It will always be assumed that either

$$F(u)(t)u(t) \geqslant 0 \quad \text{for } t \geqslant t_0 \text{ and } u \in H_{t_0,\tau}, \tag{1.2}$$

or

$$F(u)(t)u(t) \leqslant 0 \quad \text{for } t \geqslant t_0 \text{ and } u \in H_{t_0,\tau}, \tag{1.3}$$

holds.

Let $t_0 \in R_+$. A function $u : [t_0, +\infty) \to R$ is said to be a *proper solution* of Eq. (1.1) if it is locally continuous along with its derivatives of order up to and including $n - 1$, $\sup\{|u(s)|:\ s \in [t_0, +\infty)\} > 0$ for $t \geqslant t_0$, there exists a function $\bar{u} \in C(R_+; R)$ such that $\bar{u}(t) \equiv u(t)$ on $[t_0, +\infty)$, and the equality $\bar{u}^{(n)}(t) + F(\bar{u})(t) = 0$ holds for $t \in [t_0, +\infty)$. A proper solution of Eq. (1.1) is said to be *oscillatory* if it has a sequence of zeros tending to $+\infty$. Otherwise, the solution is said to be *nonoscillatory*.

Definition 1.1 [1]. We say that Eq. (1.1) has Property A if any proper solution u is oscillatory if n is even, and is either oscillatory or satisfies

$$\left|u^{(i)}(t)\right| \downarrow 0 \quad \text{as } t \uparrow +\infty \ (i = 0, \dots, n-1) \tag{1.4}$$

if n is odd.

Definition 1.2 [2]. We say that Eq. (1.1) has Property B if any proper solution u is either oscillatory, satisfies (1.4), or satisfies

$$\left|u^{(i)}(t)\right| \uparrow +\infty \quad \text{as } t \uparrow +\infty \ (i = 0, \dots, n-1) \tag{1.5}$$

if n is even, and is either oscillatory or satisfies (1.5) if n is odd.

The higher order nonlinear ordinary differential equation

$$u^{(n)}(t) + p(t)\left|u(t)\right|^{\lambda} \operatorname{sign} u(t) = 0, \tag{1.6}$$

where $p \in L_{\text{loc}}(R_+; R)$, $\lambda > 0$, and $\lambda \neq 1$, is a special case of Eq. (1.1). The problem of determining criteria for nonlinear differential equations of the second and higher orders to have each solution oscillatory or converge to zero (or be oscillatory, converge to zero, or diverge to ∞) has been of interest to researchers even before the now commonly used names of Properties A and B. It has its roots in the pioneering paper of Atkinson [3] for second-order equations, the work of Kiguradze [4], who gave sufficient conditions for this behavior in case n is even and $\lambda > 1$, and Ličko and Švec [5], who gave necessary and sufficient conditions for n both even and odd as well as both $0 < \lambda < 1$ and $\lambda > 1$. There have been a number of survey papers and monographs written on various aspects of oscillation of nonlinear differential equations, and we refer the reader to Kartsatos [6], Kiguradze and Chanturia [2], Ladde, Lakshmikantham, and Zhang [7], Györi and Ladas [8], Erbe, Kong, and Zhang [9], Agarwal, Grace, and O'Regan [10], and Koplatadze and Canturia [11]. The analogous problems for the equations of the type (1.1) in case where the operator F has either a nonlinear or a linear minorant are extensively studied in the monograph [12] and the paper [13].

In the present paper, oscillatory properties of the functional differential equation (1.1) are investigated, and this allows us to obtain results for

$$u^{(n)}(t) + p(t)\left|u(t)\right|^{\mu(t)} \operatorname{sign} u(t) = 0, \tag{1.7}$$

where $p \in L_{\text{loc}}(R_+; R)$ and $\mu \in C(R_+; (0,1])$ is nondecreasing. Clearly, this equation is a generalization of Eq. (1.6). If we let $\lambda = \lim_{t\to+\infty} \mu(t)$ and $\mu(t) \not\equiv \lambda$ for $t \in R_+$, then it turns out (see Remarks 4.1 and 7.1 below) that in certain cases, Eq. (1.7) may not have Property A (B), but the "limiting" equation does have this property.

2. Some auxiliary lemmas

In the sequel, $\tilde{C}^{n-1}_{\text{loc}}([t_0, +\infty))$ denotes the set of all functions $u : [t_0, +\infty) \to R$ that are absolutely continuous on any finite subinterval of $[t_0, +\infty)$ along with their derivatives of order up to and including $n - 1$.

Lemma 2.1 (Kiguradze [4]). *Let $u \in \tilde{C}^{n-1}_{\text{loc}}([t_0, +\infty))$ satisfy $u(t) > 0$ and $u^{(n)}(t) \leqslant 0$ ($u^{(n)}(t) \geqslant 0$) for $t \geqslant t_0$ and $u^{(n)}(t) \not\equiv 0$ in any neighborhood of $+\infty$. Then there exist $t_1 \geqslant t_0$ and $\ell \in \{0, \dots, n\}$ such that $\ell + n$ is odd (even) and*

$$\begin{aligned} &u^{(i)}(t) > 0 \quad \text{for } t \geqslant t_1 \ (i = 0, \dots, \ell - 1), \\ &(-1)^{i+l} u^{(i)}(t) > 0 \quad \text{for } t \geqslant t_1 \ (i = \ell, \dots, n - 1). \end{aligned} \tag{2.1$_\ell$}$$

Note. In case $\ell = 0$, we mean that the second inequality in (2.1_ℓ) holds, while if $\ell = n$, the first one holds.

Lemma 2.2. *Let $u \in \tilde{C}_{\text{loc}}([t_0, +\infty))$ and (2.1_ℓ) be satisfied for some $\ell \in \{1, \dots, n - 1\}$ with $\ell + n$ odd (even). Then*

$$\int^{+\infty} t^{n-\ell-1}\left|u^{(n)}(t)\right| dt < +\infty. \tag{2.2}$$

If, moreover,

$$\int^{+\infty} t^{n-\ell}\left|u^{(n)}(t)\right| dt = +\infty, \tag{2.3}$$

then there exists $t_* \geqslant t_0$ *such that*

$$\frac{u^{(i)}(t)}{t^{\ell-i}} \downarrow, \quad \frac{u^{(i)}(t)}{t^{\ell-i-1}} \uparrow +\infty \quad (i = 0, \ldots, \ell-1), \tag{2.4$_i$}$$

$$u(t) \geqslant \frac{t^{\ell-1}}{\ell!} u^{(\ell-1)}(t) \quad \textit{for } t \geqslant t_*, \tag{2.5}$$

and

$$u^{(\ell-1)}(t) \geqslant \frac{t}{(n-\ell)!} \int_t^{+\infty} s^{n-\ell-1}\left|u^{(n)}(s)\right| ds + \frac{1}{(n-\ell)!} \int_{t_*}^{t} s^{n-\ell}\left|u^{(n)}(s)\right| ds \quad \textit{for } t \geqslant t_*. \tag{2.6}$$

The proof of the lemma in the case where $u^{(n)}(t) \leqslant 0$ can be found in [14]. The case where $u^{(n)}(t) \geqslant 0$ can be proved analogously.

Remark 2.1. Inequality (2.6) was first proved in this form in [15].

3. On solutions of the type (2.1_ℓ)

In this section, sufficient conditions will be given in order for Eq. (1.1) to have no solutions of the type (2.1_ℓ), where $\ell \in \{1, \ldots, n-1\}$. Everywhere below, it is assumed that for sufficiently large t_0, we have

$$\left|F(u)(t)\right| \geqslant \sum_{i=1}^{m} \int_{\tau_i(t)}^{\sigma_i(t)} \left|u(s)\right|^{\mu_i(s)} d_s r_i(s,t) \quad \text{for } t \geqslant t_0 \text{ and } u \in H_{t_0,\tau}, \tag{3.1}$$

where

$$\begin{cases} \tau_i, \sigma_i \in C(R_+; R), \text{ and } \tau_i(t) \leqslant \sigma_i(t) \quad \text{for } t \in R_+, \\ \lim_{t\to+\infty} \tau_i(t) = +\infty \quad (i = 1, \ldots, m), \end{cases} \tag{3.2}$$

$$\mu_i \in C\big(R_+; (0,1]\big) \text{ are nondecreasing functions } (i = 1, \ldots, m), \tag{3.3}$$

and

$$r_i(s,t) \text{ are measurable in } t \text{ and nondecreasing in } s \ (i = 1, \ldots, m). \tag{3.4}$$

Also, for $i \in \{1, \ldots, m\}$, $j \in \{0, 1, \ldots, n-1\}$, and $\varphi \in C([t_0, +\infty); (0, +\infty))$, we let

$$\eta_{\varphi,\sigma_i}(t)=\begin{cases}1 & \text{for } \varphi(t)\leqslant\sigma_i(t),\\ \frac{(\sigma_i(t))^{\mu_i(\sigma_i(t))}}{(\varphi(t))^{\mu_i(\varphi(t))}} & \text{for } \varphi(t)>\sigma_i(t),\end{cases} \tag{3.5}$$

$$\rho_{ji}(t)=\int\limits_{\tau_i(t)}^{\sigma_i(t)} s^{j\mu_i(s)}\,d_s r_i(s,t)\quad (i=1,\dots,m), \tag{3.6}$$

and

$$\mu_*(t)=\min\bigl\{\mu_i(t):\ i=1,\dots,m\bigr\}, \tag{3.7}$$

where the functions τ_i, σ_i, μ_i, and r_i satisfy conditions (3.2)–(3.4).

Proposition 3.1. *Let $F\in V(\tau)$, conditions* (1.2) ((1.3)) *and* (3.1)–(3.4) *hold, $\ell\in\{1,\dots,n-1\}$, $\ell+n$ be odd (even), and*

$$\int\limits^{+\infty} t^{n-\ell-1}\sum_{i=1}^{m}\int\limits_{\tau_i(t)}^{\sigma_i(t)} s^{\ell\mu_i(s)}\,d_s r_i(s,t)\,dt=+\infty. \tag{3.8$_\ell$}$$

Moreover, assume there is a nondecreasing function $\varphi\in C(R_+;(0,+\infty))$ such that

$$\lim_{t\to+\infty}\varphi(t)=+\infty\quad\textit{and}\quad\varphi(t)\leqslant t\quad\textit{for } t\geqslant 1, \tag{3.9}$$

and

$$\begin{aligned}\limsup_{t\to+\infty}\sum_{i=1}^{m}\Biggl(&\bigl(\varphi(t)\bigr)^{\mu_*(\varphi(t))}\int\limits_t^{+\infty}\frac{s^{n-\ell-1}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}}\eta_{\varphi,\sigma_i}(s)\rho_{\ell,i}(s)\,ds\\ &+\bigl(\varphi(t)\bigr)^{\mu_*(\varphi(t))-\mu_i(\varphi(t))}\int\limits_{\varphi(t)}^{t}\frac{s^{n-\ell-1}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}}\eta_{\varphi,\sigma_i}(s)\rho_{\ell,i}(s)\bigl(\varphi(s)\bigr)^{\mu_i(\varphi(s))}\,ds\\ &+\bigl(\varphi(t)\bigr)^{\mu_*(\varphi(t))-\mu_i(\varphi(t))-1}\int\limits_0^{\varphi(t)}\frac{s^{n-\ell}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}}\bigl(\varphi(s)\bigr)^{\mu_i(\varphi(s))}\eta_{\varphi,\sigma_i}(s)\rho_{\ell,i}(s)\,ds\Biggr)\\ &>\delta\bigl(\mu_*^+\bigr)\ell!(n-\ell)!,\end{aligned} \tag{3.10$_\ell$}$$

where

$$\mu_*^+=\lim_{t\to+\infty}\mu_*(t)\quad\textit{and}\quad\delta(s)=\begin{cases}1 & \textit{if } s=1,\\ 0 & \textit{if } 0<s<1.\end{cases} \tag{3.11}$$

Then Eq. (1.1) *has no solution of the type* (2.1$_\ell$).

Proof. We will first show that (3.9) and (3.10$_\ell$) imply

$$\int\limits^{+\infty} t^{n-\ell}\sum_{i=1}^{m}\int\limits_{\tau_i(t)}^{\sigma_i(t)} s^{(\ell-1)\mu_i(s)}\,d_s r_i(s,t)\,dt=+\infty. \tag{3.12$_\ell$}$$

Indeed, if this is not the case, then in view of (3.3), (3.9), the inequality

$$\eta_{\varphi,\sigma_i}(t)\rho_{\ell,i}(t) \leqslant \big(\sigma_i(t)\big)^{\mu_i(\sigma_i(t))} \int\limits_{\tau_i(t)}^{\sigma_i(t)} s^{(\ell-1)\mu_i(s)}\, d_s r_i(s,t) \quad (i=1,\dots,m),$$

and the fact that $(\varphi(t))^{\mu_*(\varphi(t))}$ is nondecreasing, we have

$$\begin{aligned}
&\sum_{i=1}^{m}\big(\varphi(t)\big)^{\mu_*(\varphi(t))} \int\limits_{t}^{+\infty} \frac{s^{n-\ell-1}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}}\eta_{\varphi,\sigma_i}(s)\rho_{\ell,i}(s)\, ds \\
&\quad\leqslant \sum_{i=1}^{m}\int\limits_{t}^{+\infty} s^{n-\ell-1}\big(\varphi(s)\big)^{\mu_*(\varphi(s))} \int\limits_{\tau_i(s)}^{\sigma_i(s)} \xi^{(\ell-1)\mu_i(\xi)}\, d_\xi r_i(\xi,s)\, ds \\
&\quad\leqslant \sum_{i=1}^{m}\int\limits_{t}^{+\infty} s^{n-\ell}\left(\frac{\varphi(s)}{s}\right)^{\mu_*(\varphi(s))} \int\limits_{\tau_i(s)}^{\sigma_i(s)} \xi^{(\ell-1)\mu_i(\xi)}\, d_\xi r_i(\xi,s)\, ds \\
&\quad\leqslant \sum_{i=1}^{m}\int\limits_{t}^{+\infty} s^{n-\ell} \int\limits_{\tau_i(s)}^{\sigma_i(s)} \xi^{(\ell-1)\mu_i(\xi)}\, d_\xi r_i(\xi,s)\, ds \to 0 \quad \text{as } t\to+\infty, \qquad (3.13)
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^{m}\big(\varphi(t)\big)^{\mu_*(\varphi(t))-\mu_i(\varphi(t))} \int\limits_{\varphi(t)}^{t} \frac{s^{n-\ell-1}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}}\big(\varphi(s)\big)^{\mu_i(\varphi(s))}\eta_{\varphi,\sigma_i}(s)\rho_{\ell,i}(s)\, ds \\
&\quad\leqslant \sum_{i=1}^{m}\int\limits_{\varphi(t)}^{t} s^{n-\ell}\left(\frac{\varphi(s)}{s}\right)^{\mu_i(\varphi(s))} \int\limits_{\tau_i(s)}^{\sigma_i(s)} \xi^{(\ell-1)\mu_i(\xi)}\, d_\xi r_i(\xi,s)\, ds \\
&\quad\leqslant \sum_{i=1}^{m}\int\limits_{\varphi(t)}^{t} s^{n-\ell} \int\limits_{\tau_i(s)}^{\sigma_i(s)} \xi^{(\ell-1)\mu_i(\xi)}\, d_\xi r_i(\xi,s)\, ds \to 0 \quad \text{as } t\to+\infty, \qquad (3.14)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=1}^{m}\big(\varphi(t)\big)^{\mu_*(\varphi(t))-\mu_i(\varphi(t))-1} \int\limits_{0}^{\varphi(t)} \frac{s^{n-\ell}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}}\big(\varphi(s)\big)^{\mu_i(\varphi(s))}\eta_{\varphi,\sigma_i}(s)\rho_{\ell,i}(s)\, ds \\
&\quad\leqslant \sum_{i=1}^{m}\big(\varphi(t)\big)^{-1}\int\limits_{0}^{\varphi(t)} s^{n-\ell}\big(\varphi(s)\big)^{\mu_i(\varphi(s))} \int\limits_{\tau_i(s)}^{\sigma_i(s)} \xi^{(\ell-1)\mu_i(\xi)}\, d_\xi r_i(\xi,s)\, ds \\
&\quad\leqslant \sum_{i=1}^{m}\big(\varphi(t)\big)^{-1}\int\limits_{0}^{t_*} s^{n-\ell}\big(\varphi(s)\big)^{\mu_i(\varphi(s))} \int\limits_{\tau_i(s)}^{\sigma_i(s)} \xi^{(\ell-1)\mu_i(\xi)}\, d_\xi r_i(\xi,s)\, ds
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\bigl(\varphi(t)\bigr)^{-1}\int_{t_*}^{\varphi(t)} s^{n-\ell}\bigl(\varphi(s)\bigr)^{\mu_i(\varphi(s))}\int_{\tau_i(s)}^{\sigma_i(s)}\xi^{(\ell-1)\mu_i(\xi)}\,d_\xi r_i(\xi,s)\,ds\\
&\leqslant\sum_{i=1}^{m}\bigl(\varphi(t)\bigr)^{-1}\int_{0}^{t_*} s^{n-\ell}\bigl(\varphi(s)\bigr)^{\mu_i(\varphi(s))}\int_{\tau_i(s)}^{\sigma_i(s)}\xi^{(\ell-1)\mu_i(\xi)}\,d_\xi r_i(\xi,s)\,ds\\
&\quad+\sum_{i=1}^{m}\int_{t_*}^{+\infty} s^{n-\ell}\int_{\tau_i(s)}^{\sigma_i(s)}\xi^{(\ell-1)\mu_i(\xi)}\,d_\xi r_i(\xi,s)\,ds\\
&\leqslant\bigl(\varphi(t)\bigr)^{-1}\sum_{i=1}^{m}\int_{0}^{t_*} s^{n-\ell}\bigl(\varphi(s)\bigr)^{\mu_i(\varphi(s))}\int_{\tau_i(s)}^{\sigma_i(s)}\xi^{(\ell-1)\mu_i(\xi)}\,d_\xi r_i(\xi,s)\,ds+\varepsilon, \qquad (3.15)
\end{aligned}
$$

where $\varepsilon > 0$ is an arbitrary positive number and t_* is chosen so that

$$
\int_{t_*}^{+\infty} s^{n-\ell}\sum_{i=1}^{m}\int_{\tau_i(s)}^{\sigma_i(s)}\xi^{(\ell-1)\mu_i(\xi)}\,d_\xi r_i(\xi,s)\,ds<\varepsilon.
$$

Since ε is arbitrary, we have

$$
\sum_{i=1}^{m}\bigl(\varphi(t)\bigr)^{\mu_*(\varphi(t))-\mu_i(\varphi(t))-1}\int_{0}^{\varphi(t)}\frac{s^{n-\ell}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}}\bigl(\varphi(s)\bigr)^{\mu_i(\varphi(s))}\eta_{\varphi,\sigma_i}(s)\rho_{\ell j}(s)\,ds\to 0
\quad\text{as } t\to+\infty. \qquad (3.16)
$$

Now, (3.13), (3.14), and (3.16) contradict (3.10_ℓ), and this shows that (3.12_ℓ) holds.

Suppose next that Eq. (1.1) has a proper nonoscillatory solution $u:[t_0,+\infty)\to(0,+\infty)$ satisfying (2.1_ℓ), where $\ell\in\{1,\dots,n-1\}$ with $\ell+n$ odd (even). In view of (2.1_ℓ), it is clear that there exists $c>0$ such that $u(t)\geqslant ct^{\ell-1}$ for $t\geqslant t_*$, where t_* is a sufficiently large number. Therefore, from (3.1) and (3.12_ℓ), we see that u satisfies the hypotheses of Lemma 2.2, that is, condition $(2.4_{\ell-1})$ is satisfied and

$$
\begin{aligned}
u^{(\ell-1)}\bigl(\varphi(t)\bigr)&\geqslant\frac{\varphi(t)}{(n-\ell)!}\int_{\varphi(t)}^{+\infty}s^{n-\ell-1}\bigl|u^{(n)}(s)\bigr|\,ds\\
&\quad+\frac{1}{(n-\ell)!}\int_{t_*}^{\varphi(t)}s^{n-\ell}\bigl|u^{(n)}(s)\bigr|\,ds\quad\text{for } t\geqslant t_*, \qquad (3.17)
\end{aligned}
$$

where t_* is sufficiently large. As it is noted below, $u^{(\ell-1)}/t\downarrow 0$ as $t\uparrow+\infty$. Hence the functions $(u^{(\ell-1)}(t)/t)^{\mu_i(t)}$ $(i=1,\dots,m)$ are nonincreasing for large t. Taking this fact into account, in view of (1.1), (3.1), (3.3), $(2.4_{\ell-1})$, and (2.5), we obtain

$$u^{(\ell-1)}\big(\varphi(t)\big) \geqslant \frac{1}{\ell!(n-\ell)!}\varphi(t) \int\limits_{\varphi(t)}^{+\infty} s^{n-\ell-1} \sum_{i=1}^{m} \frac{(u^{(\ell-1)}(\sigma_i(s)))^{\mu_i(\sigma_i(s))}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}}$$
$$\times \int\limits_{\tau_i(s)}^{\sigma_i(s)} \xi^{\ell\mu_i(\xi)}\, d_\xi r_i(\xi,s)\, ds$$
$$+ \frac{1}{\ell!(n-\ell)!} \int\limits_{t_*}^{\varphi(t)} s^{n-\ell} \sum_{i=1}^{m} \frac{(u^{(\ell-1)}(\sigma_i(s)))^{\mu_i(\sigma_i(s))}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}}$$
$$\times \int\limits_{\tau_i(s)}^{\sigma_i(s)} \xi^{\ell\mu_i(\xi)}\, d_\xi r_i(\xi,s)\, ds.$$

On the other hand, according to (2.1_ℓ) and (3.3), since the functions $(u^{(\ell-1)}(t))^{\mu_i(t)}$ ($i = 1,\dots,m$) are nondecreasing for large t due to second relation of (2.4_ℓ), for sufficiently large t, we have

$$\big(u^{(\ell-1)}\big(\sigma_i(t)\big)\big)^{\mu_i(\sigma_i(t))} \geqslant \big(u^{(\ell-1)}\big(\varphi(t)\big)\big)^{\mu_i(\varphi(t))} \tag{3.18}$$

provided $\varphi(t) \leqslant \sigma_i(t)$ ($i = 1,\dots,m$). Since the functions $(u^{(\ell-1)}(t)/t)^{\mu_i(t)}$ ($i = 1,\dots,m$) are nonincreasing, we have

$$\big(u^{(\ell-1)}\big(\sigma_i(t)\big)\big)^{\mu_i(\sigma_i(t))} \geqslant \frac{(\sigma_i(t))^{\mu_i(\sigma_i(t))}}{(\varphi(t))^{\mu_i(\varphi(t))}}\big(u^{(\ell-1)}\big(\varphi(t)\big)\big)^{\mu_i(\varphi(t))} \tag{3.19}$$

if $\varphi(t) > \sigma_i(t)$ ($i = 1,\dots,m$). From (3.18), (3.19), and (3.5), we obtain

$$\big(u^{(\ell-1)}\big(\sigma_i(t)\big)\big)^{\mu_i(\sigma_i(t))} \geqslant \eta_{\varphi,\sigma_i}(t)\big(u^{(\ell-1)}\big(\varphi(t)\big)\big)^{\mu_i(\varphi(t))} \quad (i = 1,\dots,m)$$

for sufficiently large t. Therefore, (3.17) together with (3.3) and (3.9) imply

$$u^{(\ell-1)}\big(\varphi(t)\big) \geqslant \frac{1}{\ell!(n-\ell)!}$$
$$\times \sum_{i=1}^{m}\Bigg[\varphi(t) \int\limits_{t}^{+\infty} \frac{s^{n-\ell-1}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}}\eta_{\varphi,\sigma_i}(s)\big(u^{(\ell-1)}\big(\varphi(t)\big)\big)^{\mu_i(\varphi(s))}\rho_{\ell,i}(s)\, ds$$
$$+ \varphi(t) \int\limits_{\varphi(t)}^{t} \frac{s^{n-\ell-1}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}}\eta_{\varphi,\sigma_i}(s)\big(u^{(\ell-1)}\big(\varphi(t)\big)\big)^{\mu_i(\varphi(s))}\rho_{\ell,i}(s)\, ds$$
$$+ \int\limits_{t_1}^{\varphi(t)} \frac{s^{n-\ell}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}}\eta_{\varphi,\sigma_i}(s)\big(u^{(\ell-1)}\big(\varphi(t)\big)\big)^{\mu_i(\varphi(s))}\rho_{\ell,i}(s)\, ds\Bigg]$$

for $t \geqslant t_1$, where $t_1 \geqslant t_*$ is sufficiently large and the functions η_{φ,σ_i} and $\rho_{\ell,i}$ ($i = 1,\dots,m$) are defined by (3.5) and (3.6), respectively. From the last inequality, taking into account the fact that $(u^{(\ell-1)}(t))^{\mu_*(t)}$ is nondecreasing and $(u^{(\ell-1)}(t)/t)^{\mu_*(t)}$ is nonincreasing, we obtain

$$u^{(\ell-1)}\big(\varphi(t)\big) \geqslant \frac{(u^{(\ell-1)}(\varphi(t)))^{\mu_*(\varphi(t))}}{\ell!(n-\ell)!} \sum_{i=1}^{m} \Bigg[\varphi(t) \int_t^{+\infty} \frac{s^{n-\ell-1}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}} \eta_{\varphi,\sigma_i}(s) \rho_{\ell,i}(s)\, ds$$
$$+ \big(\varphi(t)\big)^{1-\mu_i(\varphi(t))} \int_{\varphi(t)}^{t} \frac{s^{n-\ell-1}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}} \big(\varphi(s)\big)^{\mu_i(\varphi(s))} \eta_{\varphi,\sigma_i}(s) \rho_{\ell,i}(s)\, ds$$
$$+ \big(\varphi(t)\big)^{-\mu_i(\varphi(t))} \int_{t_1}^{\varphi(t)} \frac{s^{n-\ell}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}} \eta_{\varphi,\sigma_i}(s) \rho_{\ell,i}(s) \big(\varphi(s)\big)^{\mu_i(\varphi(s))}\, ds \Bigg] \tag{3.20}$$

for $t \geqslant t_1$, where the function $\mu_*(t)$ is defined by (3.7). On the other hand, from (2.1_ℓ), (3.1), (3.12_ℓ), and the first condition in (2.4_ℓ), we can easily derive that

$$\frac{u^{(\ell-1)}(t)}{t} \downarrow 0 \quad \text{if } t \uparrow +\infty.$$

Therefore, (3.3) and (3.7) imply that

$$\limsup_{t\to+\infty} \left(\frac{u^{(\ell-1)}(\varphi(t))}{\varphi(t)} \right)^{1-\mu_*(\varphi(t))} \leqslant \delta\big(\mu_*^+\big),$$

where δ and μ_*^+ are given in (3.11). So from (3.20), we have

$$\limsup_{t\to+\infty} \sum_{i=1}^{m} \Bigg(\big(\varphi(t)\big)^{\mu_*(\varphi(t))} \int_t^{+\infty} \frac{s^{n-\ell-1}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}} \eta_{\varphi,\sigma_i}(s) \rho_{l,i}(s)\, ds$$
$$+ \big(\varphi(t)\big)^{\mu_*(\varphi(t))-\mu_i(\varphi(t))} \int_{\varphi(t)}^{t} \frac{s^{n-\ell-1}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}} \big(\varphi(s)\big)^{\mu_i(\varphi(s))} \eta_{\varphi,\sigma_i}(s) \rho_{\ell,i}(s)\, ds$$
$$+ \big(\varphi(t)\big)^{\mu_*(\varphi(t))-\mu_i(\varphi(t))-1} \int_{t_1}^{\varphi(t)} \frac{s^{n-\ell}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}} \big(\varphi(s)\big)^{\mu_i(\varphi(s))} \eta_{\varphi,\sigma_i}(s) \rho_{\ell,i}(s)\, ds \Bigg)$$
$$\leqslant \delta\big(\mu_*^+\big)\ell!(n-\ell)!.$$

But this contradicts (3.10_ℓ), and completes the proof of the proposition. □

Remark 3.1. For a rather wide class of operators F, condition (3.12_ℓ) is also necessary for Eq. (1.1) not to have a solution of the type (2.1_ℓ) (see Lemma 4.1 in [12]).

Proposition 3.2. *Let* $F \in V(\tau)$, *conditions* (1.2) ((1.3)) *and* (3.1)–(3.4) *hold, and let* $\ell \in \{1,\dots,n-1\}$ *with* $\ell + n$ *odd* (*even*). *Moreover, suppose there exists a nondecreasing function* $\varphi \in C(R_+;(0,+\infty))$ *such that conditions* (3.8_ℓ) *and* (3.9) *hold, and*

$$\limsup_{t\to+\infty}\sum_{i=1}^{m}\Biggl(\bigl(\varphi(t)\bigr)^{\mu_*(\varphi(t))}\int_t^{+\infty}s^{n-\ell-1}\eta_{\varphi,\tau_i}(s)\rho_{\ell-1,i}(s)\,ds$$
$$+\bigl(\varphi(t)\bigr)^{\mu_*(\varphi(t))-\mu_i(\varphi(t))}\int_{\varphi(t)}^{t}s^{n-\ell-1}\bigl(\varphi(s)\bigr)^{\mu_i(\varphi(s))}\eta_{\varphi,\tau_i}(s)\rho_{\ell-1,i}(s)\,ds$$
$$+\bigl(\varphi(t)\bigr)^{\mu_*(\varphi(t))-\mu_i(\varphi(t))-1}\int_0^{\varphi(t)}s^{n-\ell}\bigl(\varphi(s)\bigr)^{\mu_i(\varphi(s))}\eta_{\varphi,\tau_i}(s)\rho_{\ell-1,i}(s)\,ds\Biggr)$$
$$>\delta\bigl(\mu_*^+\bigr)\ell!(n-\ell)!. \tag{3.21$_\ell$}$$

Then Eq. (1.1) *has no solution of the type* (2.1_ℓ).

Proof. Similar to the proof of Proposition 3.1, we will use (3.21_ℓ) to show that (3.12_ℓ) holds. Assume that Eq. (1.1) has a proper solution satisfying (2.1_ℓ), where $\ell\in\{1,\dots,n-1\}$ with $\ell+n$ odd (even). The function u obviously satisfies the conditions of Lemma 2.2, so as in the proof of Proposition 3.1, (3.17) holds for t_* sufficiently large. From (2.5) and the first condition in $(2.4_{\ell-1})$, we obtain

$$u^{(\ell-1)}\bigl(\varphi(t)\bigr)\geqslant\frac{\varphi(t)}{\ell!(n-\ell)!}\int_{\varphi(t)}^{+\infty}s^{n-\ell-1}\sum_{i=1}^{m}\bigl(u^{(\ell-1)}\bigl(\tau_i(s)\bigr)\bigr)^{\mu_i(\tau_i(s))}$$
$$\times\int_{\tau_i(s)}^{\sigma_i(s)}\xi^{(\ell-1)\mu_i(\xi)}\,d_\xi r_i(\xi,s)\,ds$$
$$+\frac{1}{\ell!(n-\ell)!}\int_{t_*}^{\varphi(t)}s^{n-\ell}\sum_{i=1}^{m}\bigl(u^{(\ell-1)}\bigl(\tau_i(s)\bigr)\bigr)^{\mu_i(\tau_i(s))}$$
$$\times\int_{\tau_i(s)}^{\sigma_i(s)}\xi^{(\ell-1)\mu_i(\xi)}\,d_\xi r_i(\xi,s)\,dx. \tag{3.22}$$

If we then proceed as in the proof of Proposition 3.1 with the functions σ_i replaced by τ_i, we have

$$\limsup_{t\to+\infty}\sum_{i=1}^{m}\Biggl(\bigl(\varphi(t)\bigr)^{\mu_*(\varphi(t))}\int_t^{+\infty}s^{n-\ell-1}\eta_{\varphi,\tau_i}(s)\rho_{\ell-1,i}(s)\,ds$$
$$+\bigl(\varphi(t)\bigr)^{\mu_*(\varphi(t))-\mu_i(\varphi(t))}\int_{\varphi(t)}^{t}s^{n-\ell-1}\bigl(\varphi(s)\bigr)^{\mu_i(\varphi(s))}\eta_{\varphi,\tau_i}(s)\rho_{\ell-1,i}(s)\,ds$$

$$+\big(\varphi(t)\big)^{\mu_*(\varphi(t))-\mu_i(\varphi(t))-1}\int_{t_*}^{\varphi(t)} s^{n-\ell}\big(\varphi(s)\big)^{\mu_i(\varphi(s))}\eta_{\varphi,\tau_i}(s)\rho_{\ell-1,i}(s)\,ds\Bigg)$$

$$\leqslant \ell!(n-\ell)!\delta\big(\mu_*^+\big).$$

The last inequality contradicts (3.21_ℓ), and this completes the proof of the proposition. □

The previous two propositions were concerned with the case $\varphi(t)\leqslant t$. The next two are for the case $\varphi(t)\geqslant t$.

Proposition 3.3. *Let $F\in V(\tau)$, conditions* (1.2) ((1.3)) *and* (3.1)–(3.4) *hold, and let $\ell\in\{1,\dots,n-1\}$ with $\ell+n$ odd* (*even*). *In addition, suppose there exists a nondecreasing function $\varphi\in C(R_+;(0,+\infty))$ such that*

$$\varphi(t)\geqslant t \quad \textit{for } t\in R_+ \tag{3.23}$$

and

$$\limsup_{t\to+\infty}\Bigg\{\sum_{i=1}^{m}\Bigg(\big(\varphi(t)\big)^{\mu_*(\varphi(t))}\int_{\varphi(t)}^{+\infty}\frac{s^{n-\ell-1}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}}\eta_{\varphi,\sigma_i}(s)\rho_{\ell,i}(s)\,ds$$
$$+\big(\varphi(t)\big)^{\mu_*(\varphi(t))-1}\int_{t}^{\varphi(t)}\frac{s^{n-\ell}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}}\eta_{\varphi,\sigma_i}(s)\rho_{\ell,i}(s)\,ds$$
$$+\big(\varphi(t)\big)^{\mu_*(\varphi(t))-\mu_i(\varphi(t))-1}\int_{0}^{t}\frac{s^{n-\ell}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}}\big(\varphi(s)\big)^{\mu_i(\varphi(s))}\eta_{\varphi,\sigma_i}(s)\rho_{\ell,i}(s)\,ds\Bigg)\Bigg\}$$
$$>\ell!(n-\ell)!\delta\big(\mu_*^+\big). \tag{3.24_ℓ}$$

Then Eq. (1.1) *has no solution of the type* (2.1_ℓ).

Proof. Again following the line of proof used for Proposition 3.1, (3.24_ℓ) implies (3.8_ℓ) holds. Assume that Eq. (1.1) has a proper solution satisfying (2.1_ℓ), where $\ell\in\{1,\dots,n-1\}$ with $\ell+n$ odd (even). As before, we can show that (3.17) holds for t_* sufficiently large. On the other hand, in view of (1.1), (3.1), (2.5), ($2.4_{\ell-1}$), and (3.23), for large t, inequality (3.17) yields

$$u^{(\ell-1)}\big(\varphi(t)\big)\geqslant\frac{\varphi(t)}{\ell!(n-\ell)!}\int_{\varphi(t)}^{+\infty}s^{n-\ell-1}\sum_{i=1}^{m}\eta_{\varphi,\sigma_i}(s)\rho_{\ell,i}(s)\frac{(u^{(\ell-1)}(\varphi(s)))^{\mu_i(\varphi(s))}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}}\,ds$$
$$+\sum_{i=1}^{m}\Bigg[\int_{t}^{\varphi(t)}\frac{s^{n-\ell}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}}\eta_{\varphi,\sigma_i}(s)\rho_{\ell,i}(s)(u^{(\ell-1)})^{\mu_i(\varphi(s))}\,ds$$
$$+\int_{t_*}^{t}\frac{s^{n-\ell}}{(\sigma_i(s))^{\mu_i(\sigma_i(s))}}\eta_{\varphi,\sigma_i}(s)\rho_{\ell,i}(s)(u^{(\ell-1)})^{\mu_i(\varphi(s))}\,ds\Bigg]$$

for large t. If, in the first and second summands on the right-hand side of this inequality, we take into account the second condition in $(2.4_{\ell-1})$, and in the first summand, we use the first condition in $(2.4_{\ell-1})$, we will easily obtain an inequality opposite to (3.24_ℓ). This completes the proof of the proposition. □

Proposition 3.4 below is proved analogously to Propositions 3.1–3.3.

Proposition 3.4. *Let $F \in V(\tau)$, conditions* (1.2) ((1.3)) *and* (3.1)–(3.4) *hold, and let $\ell \in \{1, \dots, n-1\}$ with $\ell + n$ odd* (*even*). *Moreover, assume that there exists a nondecreasing function $\varphi \in C(R_+; (0, +\infty))$ such that* (3.23) *holds and*

$$\begin{aligned}\limsup_{t\to+\infty}\Bigg\{\sum_{i=1}^{m}\Bigg(&\big(\varphi(t)\big)^{\mu_*(\varphi(t))}\int_{\varphi(t)}^{+\infty} s^{n-\ell-1}\eta_{\varphi,\tau_i}(s)\rho_{\ell-1,i}(s)\,ds\\&+\big(\varphi(t)\big)^{\mu_*(\varphi(t))-1}\int_{t}^{\varphi(t)} s^{n-\ell}\eta_{\varphi,\tau_i}(s)\rho_{\ell-1,i}(s)\,ds\\&+\big(\varphi(t)\big)^{\mu_*(\varphi(t))-\mu_i(\varphi(t))-1}\int_{t_*}^{t} s^{n-\ell}\big(\varphi(s)\big)^{\mu_i(\varphi(s))}\eta_{\varphi,\tau_i}(s)\rho_{\ell,i}(s)\,ds\Bigg)\Bigg\}\\&> \ell!(n-\ell)!\delta\big(\mu_*^+\big).\end{aligned} \qquad (3.25_\ell)$$

Then Eq. (1.1) *has no solution of the type* (2.1_ℓ).

4. Functional differential equations with Property A

Based on the results obtained in Section 3, in this section we obtain sufficient conditions for Eq. (1.1) to have Property A.

Theorem 4.1. *Let $F \in V(\tau)$, conditions* (1.2) *and* (3.1) *hold, and there exists a nondecreasing function $\varphi \in C(R_+; [0, +\infty))$ satisfying* (3.9) *such that for any $\ell \in \{1, \dots, n-1\}$ with $\ell + n$ odd, conditions* (3.8_ℓ) *and* (3.10_ℓ) *are satisfied. If, in addition, n is odd, let*

$$\int^{+\infty} t^{n-1}\sum_{i=1}^{m}\Big(r_i\big(\sigma_i(t), t\big) - r_i\big(\tau_i(t), t\big)\Big)\,dt = +\infty. \qquad (4.1)$$

Then Eq. (1.1) *has Property* A.

Proof. Let Eq. (1.1) have a proper nonoscillatory solution $u : [t_0, +\infty) \to (0, +\infty)$ (the case $u(t) < 0$ is similar). Then by (1.1), (1.2), and Lemma 2.1, there exists $\ell \in \{0, \dots, n-1\}$ such that $\ell + n$ is odd and condition (2.1_ℓ) holds. In view of (3.9), (3.10_ℓ), and Proposition 3.1, we have $\ell \notin \{1, \dots, n-1\}$. Therefore, n is odd and $\ell = 0$. We claim that (1.4) holds. If this is not the case, then there exist $c > 0$, $t_* > t_0$, and $t_1 > t_*$ such that

$u(t) \geqslant c$ for $t \geqslant t_*$ and $\tau_i(t) \geqslant t_*$ for $t \geqslant t_1$ $(i = 1, \dots, m)$. Therefore, in view of (2.1_ℓ) and (3.1), Eq. (1.1) yields

$$\sum_{i=1}^{n-1} (n-i-1)! t_1^i \big|u^{(i)}(t_1)\big| \geqslant c \int_{t_1}^{t} s^{n-1} \sum_{i=1}^{m} \big(r_i\big(\sigma_i(s), s\big) - r_i\big(\tau_i(s), s\big)\big)\, ds$$

for $t \geqslant t_1$, which contradicts (4.1). Thus, (1.4) holds, and so Eq. (1.1) has Property A. □

Corollary 4.1. *Let $F \in V(\tau)$, condition* (1.2) *hold, and*

$$\big|F(u)(t)\big| \geqslant \sum_{i=1}^{m} p_i(t) \int_{\alpha_i t}^{\beta_i t} \big|u(s)\big|^{\mu_i(s)}\, ds \quad \text{for } t \geqslant t_0 \text{ and } u \in H_{t_0,\tau}, \tag{4.2}$$

where

$$\alpha_i, \beta_i \in (0, +\infty), \quad \alpha_i < \beta_i,\ \beta_i \geqslant 1,\ p_i \in L_{\text{loc}}(R_+; R_+), \tag{4.3}$$

$$\mu_i(t) = \mu_i - \frac{d_i}{\ln t}, \quad 0 < \mu_i \leqslant 1,\ d_i \geqslant 0. \tag{4.4}$$

Moreover, for any $\ell \in \{1, \dots, n-1\}$ with $\ell + n$ odd, let

$$\limsup_{t\to+\infty} \sum_{i=1}^{m} \frac{(\beta_i^{\ell\mu_i+1} - \alpha_i^{\ell\mu_i+1}) e^{-d_i(\ell-1)}}{\beta_i^{\mu_i}(1+\ell\mu_i)} \Bigg(t^{\mu_0} \int_{t}^{+\infty} s^{n-\ell+\mu_i(\ell-1)} p_i(s)\, ds + t^{\mu_0-\mu_i-1} \int_{0}^{t} s^{n+1-\ell(\mu_i-1)} p_i(s)\, ds \Bigg) > e^{d_0} \ell!(n-\ell)!\delta(\mu_0) \tag{4.5_ℓ}$$

hold, where $\mu_0 = \min\{\mu_i\colon i = 1, \dots, m\}$ and $d_0 = \max\{d_i\colon i = 1, \dots, m\}$. Then Eq. (1.1) *has Property* A.

Proof. To prove the corollary, it suffices to note that (4.2)–(4.4) imply the conditions of Theorem 4.1 are satisfied with $\tau_i(t) = \alpha_i t$, $\sigma_i(t) = \beta_i t$, $r_i(s,t) = p_i(t)s$ $(i = 1, \dots, m)$, and $\varphi(t) \equiv t$.

If, in Corollary 4.1, the functions $p_i(t)$ $(i = 1, \dots, m)$ are in a sense "close" to each other, then the conditions (4.5_ℓ) can be replaced by one condition. In fact, we have the following result. □

Corollary 4.1′. *Let $F \in V(\tau)$, conditions* (1.2) *and* (4.2)–(4.4) *hold with $\mu_i = 1$ $(i = 1, \dots, m)$, and there exist $\tilde{p} \in L_{\text{loc}}(R_+; R_+)$ such that*

$$\liminf_{t\to+\infty} t \int_{t}^{+\infty} s^{n-1}\big(p_i(s) - \tilde{p}(s)\big)\, ds \geqslant 0, \tag{4.6}$$

and

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t s^n\big(p_i(s)-\tilde p(s)\big)\,ds\geqslant 0\quad (i=1,\dots,m)\tag{4.7}$$

hold. Then, for Eq. (1.1) *to have Property* A*, it is sufficient that*

$$\limsup_{t\to+\infty}\left(t\int_t^{+\infty}s^{n-1}\tilde p(s)\,ds+\frac{1}{t}\int_0^t s^n\tilde p(s)\,ds\right)$$
$$> e^{d_0}\max\left\{(1+\ell)!(n-\ell)!\left(\sum_{i=1}^m\frac{e^{-d_i(\ell-1)}(\beta_i^{\ell+1}-\alpha_i^{\ell+1})}{\beta_i}\right)^{-1}:\ \ell\in\{1,\dots,n-1\},\ \ell+n \text{ is odd}\right\},\tag{4.8}$$

where $d_0=\max\{d_i:\ i=1,\dots,m\}$.

Proof. Since $\mu_0=\min\{\mu_i:\ i=1,\dots,m\}=1$, $\delta(\mu_0)=1$, conditions (4.6)–(4.8) imply (4.5_ℓ) holds for any $\ell\in\{1,\dots,n-1\}$ with $\ell+n$ odd. Therefore, the hypotheses of Corollary 4.1 are satisfied, and the conclusion follows. □

Corollary 4.1″. *Let* $F\in V(\tau)$*, conditions* (4.2)–(4.4) *hold with* $\mu_i=1$ $(i=1,\dots,m)$*, and let there exist a function* $\tilde p\in L_{\text{loc}}(R_+;R_+)$ *such that*

$$p_i(t)=\tilde p(t)+o(t^{n+1})\quad (i=1,\dots,m).\tag{4.9}$$

Then condition (4.8) *is sufficient for Eq.* (1.1) *to have Property* A.

Proof. To prove the corollary, it suffices to note that condition (4.9) implies that (4.6) and (4.7) hold, so the hypotheses of Corollary 4.1′ are satisfied. □

Theorem 4.2. *Let* $F\in V(\tau)$ *and conditions* (1.2), (3.1)–(3.4)*, and* (4.1) *hold. Assume that there is a nondecreasing function* $\varphi\in C(R_+;(0,+\infty))$ *such that* (3.9) *holds, and for any* $\ell\in\{1,\dots,n-1\}$ *with* $\ell+n$ *odd, condition* (3.21_ℓ) *holds. Then Eq.* (1.1) *has Property* A.

Corollary 4.2. *Let* $F\in V(\tau)$*, conditions* (1.2) *and* (4.2)–(4.4) *hold with* $\alpha_i\leqslant 1$ $(i=1,\dots,m)$*, and for any* $\ell\in\{1,\dots,n-1\}$ *with* $\ell+n$ *odd, let*

$$\limsup_{t\to+\infty}\sum_{i=1}^m\frac{\alpha_i^{\mu_i}(\beta_i^{1+(\ell-1)\mu_i}-\alpha_i^{1+(\ell-1)\mu_i})e^{-(\ell-1)d_i}}{1+(\ell-1)\mu_i}$$
$$\times\left(t^{\mu_0}\int_t^{+\infty}s^{n-\ell+(\ell-1)\mu_i}p_i(s)\,ds+t^{\mu_0-\mu_i-1}\int_0^t s^{n+\ell(\mu_i-1)+1}p_i(s)\,ds\right)$$
$$> e^{d_0}\ell!(n-\ell)!\delta(\mu_0),\tag{4.10}$$

where $\mu_0 = \min\{\mu_i\colon\ i = 1, \dots, m\}$. *Then Eq.* (1.1) *has Property* A.

Proof. The conditions of the corollary imply that the hypotheses of Theorem 4.2 are satisfied with $\tau_i(t) = \alpha_i t$, $\sigma_i(t) = \beta_i t$, $r_i(s,t) = p_i(t)$, and $\varphi(t) \equiv t$. □

Corollary 4.2′. *Let* $F \in V(\tau)$, *conditions* (1.2) *and* (4.2)–(4.4) *hold with* $\alpha_i \leqslant 1$, $\mu_i = 1$ $(i = 1, \dots, m)$, *and suppose there exists* $\tilde{p} \in L_{\mathrm{loc}}(R_+; R_+)$ *such that for any* $\ell \in \{1, \dots, n-1\}$ *with* $\ell + n$ *odd, we have*

$$\liminf_{t\to+\infty} \sum_{i=1}^{m} \alpha_i \big(\beta_i^{\ell} - \alpha_i^{\ell}\big) e^{-(\ell-1)d_i} \left(t \int_t^{+\infty} s^{n-1}\big(p_i(s) - \tilde{p}(s)\big)\, ds + t^{-1} \int_0^t s^{n+1}\big(p_i(s) - \tilde{p}(s)\big)\, ds \right) \geqslant 0. \tag{4.11}$$

Then for Eq. (1.1) *to have Property* A, *it is sufficient that*

$$\limsup_{t\to+\infty} \left(t \int_t^{+\infty} s^{n-1} \tilde{p}(s)\, ds + \frac{1}{t} \int_0^t s^{n+1} \tilde{p}(s)\, ds \right) > \max\left\{ e^{d_0} \ell \ell!(n-\ell)! \left(\sum_{i=1}^{m} \alpha_i \big(\beta_i^{\ell} - \alpha_i^{\ell}\big) e^{-(\ell-1)d_i} \right)^{-1} : \ \ell \in \{1, \dots, n-1\},\ \ell + n \text{ is odd} \right\}. \tag{4.12}$$

Proof. It suffices to note that the conditions (4.11) and (4.12) imply (4.10). □

Corollary 4.2″. *Let* $F \in V(\tau)$, *conditions* (1.2) *and* (4.2)–(4.4) *hold with* $\alpha_1 \leqslant 1$, $\mu_i = 1$ $(i = 1, \dots, m)$, *and there exist* $\tilde{p} \in L_{\mathrm{loc}}(R_+; R_+)$ *such that* (4.9) *holds. Then the inequality* (4.12) *is sufficient for Eq.* (1.1) *to have Property* A.

The proof follows from the observation that (4.9) implies (4.11).

Theorem 4.3. *Let* $F \in V(\tau)$ *and conditions* (1.2), (3.1)–(3.4), *and* (4.1) *hold. In addition, suppose there is a nondecreasing function* $\varphi \in C(R_+; (0, +\infty))$ *such that* (3.23) *is satisfied, and for any* $\ell \in \{1, \dots, n-1\}$ *with* $\ell + n$ *odd, conditions* (3.24$_\ell$) *and* (3.8$_\ell$) *hold. Then Eq.* (1.1) *has Property* A.

Corollary 4.3. *Let* $F \in V(\tau)$, (1.2) *and* (4.2)–(4.4) *hold, and for any* $\ell \in \{1, \dots, n-1\}$ *with* $\ell + n$ *odd, let*

$$\limsup_{t\to+\infty} \sum_{i=1}^{m} \frac{\beta_i^{1+\ell\mu_i} - \alpha_i^{1+\ell\mu_i}}{e^{\ell d_i}(1 + \ell\mu_i)} \left(t^{\mu_0} \int_{\beta^* t}^{+\infty} s^{n-\ell(1-\mu_i)-\mu_i} p_i(s)\, ds \right.$$

$$+ (\beta^*)^{-\mu_i} e^{d_i} \int_t^{\beta^* t} s^{n-\ell(1-\mu_i)+1-\mu_i} p_i(s)\, ds$$

$$+ \beta^{*\mu_0-\mu_i-1} e^{d_i} t^{\mu_0-\mu-1} \int_0^t s^{n-\ell(1-\mu_i)+1} p_i(s)\, ds \Bigg)$$

$$> e^{d_0} \ell!(n+\ell)!\delta(\mu_0), \tag{4.13$_\ell$}$$

where $\beta^* = \max\{\beta_i\colon i = 1, \ldots, m\}$*. Then Eq.* (1.1) *has Property* A.

Proof. To prove the corollary, it suffices to note that (4.13_ℓ) implies that the hypotheses of Theorem 4.3 hold with $\tau_i(t) = \alpha_i t$, $\sigma_i(t) = \beta_i t$, $r_i(s,t) = p_i(t)s$ $(i = 1, \ldots, m)$, and $\varphi(t) = \beta^* t$. □

Corollary 4.3′. *Let* $F \in V(\tau)$*, conditions* (1.2) *and* (4.2)–(4.4) *hold with* $\mu_i = 1$ *and* $d_i = d_0 \geqslant 0$ $(i = 1, \ldots, m)$*. If there exists* $\tilde{p} \in L_{\mathrm{loc}}(R_+; R_+)$ *such that* (4.9) *holds and*

$$\limsup_{t\to+\infty} \left(\beta^* t \int_{\beta^* t}^{+\infty} s^{n-1} \tilde{p}(s)\, ds + e^{d_0} \int_t^{\beta^* t} s^n \tilde{p}(s)\, ds + \frac{1}{t} \int_0^t s^{n+1} \tilde{p}(s)\, ds \right)$$

$$> \max\left\{ (\ell+1)!(n-l)!\beta^* e^{\ell d_0} \left(\sum_{i=1}^m \left(\beta_i^{\ell+1} - \alpha_i^{\ell+1}\right) \right)^{-1} : \right. \tag{4.14}$$

$$\left. \ell \in \{1, \ldots, n-1\},\ \ell + n \text{ is odd} \right\},$$

where $\beta^* = \max\{\beta_i\colon i = 1, \ldots, m\}$*, then Eq.* (1.1) *has Property* A.

Proof. Since $\mu_i = 1$ $(i = 1, \ldots, m)$, we have $\mu_0 = 1$ and $\delta(1) = 1$. Now (4.14) implies (4.13_ℓ) for any $\ell \in \{1, \ldots, n-1\}$ with $\ell + n$ odd. Thus, the hypotheses of Corollary 4.3 are satisfied, which proves this corollary. □

Theorem 4.4. *Let* $F \in V(\tau)$ *and conditions* (1.2) *and* (4.2)–(4.4) *hold. Suppose there is a nondecreasing function* $\varphi \in C(R_+; (0, +\infty))$ *such that* (3.23) *is satisfied, and for any* $\ell \in \{1, \ldots, n-1\}$ *with* $l + n$ *odd, condition* (3.25_ℓ) *holds. Then Eq.* (1.1) *has Property* A.

Corollary 4.4. *Let* $F \in V(\tau)$*, conditions* (1.2) *and* (4.2)–(4.4) *hold, and for any* $\ell \in \{1, \ldots, n-1\}$ *with* $\ell + n$ *odd, let*

$$\limsup_{t\to+\infty} \sum_{i=1}^m \frac{(\beta_i^{1+(\ell-1)\mu_i} - \alpha_i^{1+(\ell-1)\mu_i})(\alpha_i/\beta^*)^{\mu_i}}{(1+(\ell-1)\mu_i)e^{(\ell-1)d_i+d_0}}$$

$$\times \Bigg((\beta^* t)^{\mu_0} \int_{\beta^* t}^{+\infty} s^{n-\ell+(\ell-1)\mu_i} p_i(s)\, ds + (\beta^* t)^{\mu_0-1} \int_t^{\beta^* t} s^{n-\ell+(\ell-1)\mu_i+1} p_i(s)\, ds$$

$$+ e^{d_i-d_0}(\beta^* t)^{\mu_0-\mu_i-1}\int_0^t s^{n-\ell(1-\mu_i)+1}p_i(s)\,ds\Bigg)$$

$$> \ell!(n-\ell)!\delta(\mu_0), \tag{4.15$_\ell$}$$

where $\mu_0 = \min\{\mu_i\colon\ i = 1,\dots,m\}$ *and* $d_0 = \max\{d_i\colon\ i = 1,\dots,m\}$. *Then Eq.* (1.1) *has Property* A.

Proof. The corollary follows from Theorem 4.4 since (4.15_ℓ) implies (3.25_ℓ) with the inequality (3.1) replaced by (4.2), $\varphi(t) \equiv \beta^* t$, and $\beta^* = \max\{\beta_i\colon\ i = 1,\dots,m\}$. □

Corollary 4.4′. *Let* $F \in V(\tau)$, *conditions* (1.2) *and* (4.2)–(4.4) *hold with* $\mu_i = 1$ *and* $d_i = d_0 \geqslant 0$ $(i = 1,\dots,m)$, *and suppose there exists* $\tilde{p} \in L_{\mathrm{loc}}(R_+; R_+)$ *such that* (4.11) *holds and*

$$\limsup_{t\to+\infty}\Bigg(\beta^* t\int_{\beta^* t}^{+\infty} s^{n-1}\tilde{p}(s)\,ds + \int_t^{\beta^* t} s^n\tilde{p}(s)\,ds + (\beta^* t)^{-1}\int_0^t s^{n+1}\tilde{p}(s)\,ds\Bigg)$$

$$> \max\left\{\ell\ell!(n-\ell)!\left(\sum_{i=1}^m \frac{(\beta_i^\ell-\alpha_i^\ell)\alpha_i}{\beta^* e^{d_0\ell}}\right)^{-1}\colon\ \ell\in\{1,\dots,n-1\},\ \ell+n \text{ is odd}\right\}, \tag{4.16}$$

where $\beta^* = \max\{\beta_i\colon\ i = 1,\dots,m\}$. *Then Eq.* (1.1) *has Property* A.

Proof. It suffices to note that the conditions (4.11) and (4.16) imply (4.10). □

Remark 4.1. The results given in this section essentially depend on the rate at which the functions $\mu_i^+ - \mu_i(t)$ tend to zero as $t \to +\infty$, where $\mu_i^+ = \lim_{t\to+\infty}\mu_i(t)$ $(i = 1,\dots,m)$. It may happen that the "limiting" equation has Property A while the original one does not (by "limiting" equation, we mean the equation obtained when the functions $\mu_i(t)$ are replaced by their limits μ_i).

To illustrate the situation described in Remark 4.1, we will give two examples. In the first example, the "limiting" equation is linear, while in the second one, it is essentially nonlinear.

Example 4.1. Consider the equation

$$u^{(n)}(t) + \sum_{i=1}^m p_i(t)\int_{\alpha_i t}^t \big|u(s)\big|^{\mu_i(s)}\operatorname{sign} u(s)\,ds = 0, \tag{4.17}$$

where $0 < \alpha_i < 1$, $p_i \in L_{\mathrm{loc}}(R_+; R_+)$, $\mu_i : R_+ \to (0,1)$ are nondecreasing functions, and $\lim_{t\to+\infty} \mu_i(t) = 1$ $(i = 1,\dots,m)$. The "limiting" equation for (4.17) has the form

$$u^{(n)}(t) + \sum_{i=1}^{m} p_i(t) \int_{\alpha_i t}^{t} u(s)\,ds = 0. \tag{4.18}$$

It is known (see [14]) that if

$$\limsup_{t\to+\infty} \sum_{i=1}^{m} \bigl(1-\alpha_i^n\bigr)\left(t \int_{t}^{+\infty} s^{n-1} p_i(s)\,ds + \frac{1}{t}\int_{0}^{t} s^{n+1} p_i(s)\,ds \right) > n!, \tag{4.19}$$

then Eq. (4.18) has Property A. Now choose $c_i > 0$ and $d_i > 0$ $(i = 1,\dots,m)$ such that

$$2\sum_{i=1}^{m} \bigl(1-\alpha_i^n\bigr) c_i > n! \tag{4.20}$$

and

$$\max\left\{ \frac{(1+\lambda)\lambda(\lambda-1)\cdots(\lambda-n+1)}{\sum_{i=1}^{m} c_i e^{-d_i\lambda}(1-\alpha_i^{1+\lambda})} \colon \lambda \in (n-2, n-1) \right\} \geqslant 1, \tag{4.21}$$

and let $p_i(t) = c_i/t^{n+1}$. According to (4.20), it is clear that (4.19) holds, that is, Eq. (4.18) has Property A. On the other hand, in view of (4.21), it is also clear that there exists $\lambda_0 \in (n-2, n-1)$ such that

$$\lambda_0(1+\lambda_0)(\lambda_0-1)\cdots(\lambda_0-n+1) = \sum_{i=1}^{m} c_i e^{-d_i\lambda_0}\bigl(1-\alpha_i^{1+\lambda_0}\bigr).$$

Therefore, t^{λ_0} is a solution of Eq. (4.17) with $p_i(t) = c_i/t^{n+1}$ and $\mu_i(t) = 1 - d_i/\ln t$ $(i = 1,\dots,m)$, that is, Eq. (4.17) does not have Property A. If in Eq. (4.17), we have $\mu_i(t) = 1 - d_i/t^{\gamma_i}$, where $d_i > 0$ and $\gamma_i > 0$ $(i = 1,\dots,m)$, then condition (4.19) is sufficient for both Eqs. (4.17) and (4.18) to have Property A. The above example shows that it can happen that the "limiting" equation has Property A while the original quasilinear equation may or may not have Property A. Whether the original Eq. (4.17) has Property A depends on the rate at which the functions $1 - \mu_i(t)$ tend to zero as $t \to +\infty$.

Example 4.2. Consider the essentially nonlinear equation

$$u^{(n)}(t) + p(t) \int_{\alpha t}^{\beta t} \bigl|u(s)\bigr|^{\mu(s)} \operatorname{sign} u(s)\,ds = 0, \tag{4.22}$$

where $p \in L_{\mathrm{loc}}(R_+; R_+)$, the function $\mu \in C(R_+; (0,1))$ is nondecreasing, $\lim_{t\to+\infty} \mu(t) = \mu_0 < 1$, and $0 < \alpha < \beta < +\infty$. The "limiting" equation for (4.22) is

$$u^{(n)}(t) + p(t) \int_{\alpha t}^{\beta t} \bigl|u(s)\bigr|^{\mu_0} \operatorname{sign} u(s)\,ds = 0. \tag{4.23}$$

It is known (see [12, Corollary 4.1]) that

$$\int^{+\infty} t^{1+\mu_0(n-1)} p(t)\,dt = +\infty \tag{4.24}$$

is necessary and sufficient for Eq. (4.23) to have Property A. Now consider Eq. (4.22) with

$$p(t) = \frac{1}{t^{2+\mu_0(n-1)} \ln t} \quad \text{and} \quad \mu(t) = \mu_0 - \frac{1}{\ln\ln t}. \tag{4.25}$$

From the first equality in (4.25), it is clear that (4.24) holds, and so Eq. (4.23) has Property A. On the other hand, in view of (4.25), we see that

$$\int^{+\infty} p(t) \int_{\alpha t}^{\beta t} s^{(n-1)\mu(s)}\,ds < +\infty.$$

Therefore, by Lemma 4.1 in [12], Eq. (4.22) has a solution $u : [t_0, +\infty) \to R$ satisfying $\lim_{t\to+\infty} u^{(n-1)}(t) = c_0 \neq 0$. Hence, Eq. (4.22) does not have Property A (see Definition 1.1), that is, in the case of essentially nonlinear equations, the original equation may not have Property A while the "limiting" does.

5. Differential equations with Volterra-type minorant

Everywhere in this section, it is assumed that the inequality (3.1) holds and

$$\sigma_i(t) \leqslant t \quad \text{for } t \in R_+ \ (i = 1, \dots, m). \tag{5.1}$$

If (5.1) holds, then the formulation of the results given in Section 4 become substantially simpler.

Theorem 5.1. *Let $F \in V(\tau)$, conditions* (1.2), (3.1)–(3.4), *and* (5.1) *hold, and suppose there is a nondecreasing function $\varphi \in C(R_+; (0, +\infty))$ such that either* (3.8_{n-1}), (3.9), *and* (3.10_{n-1}) *hold, or* (3.8_{n-1}), (3.9), *and* (3.21_{n-1}) *hold. Then Eq.* (1.1) *has Property* A.

Proof. Taking into account (3.6) and (5.1), we easily see that

$$\rho_{n-1,i}(t) \leqslant t^j \rho_{n-1-j,i}(t) \quad (i = 1, \dots, m,\ j = 1, \dots, n-2).$$

Therefore, in view of (3.10_{n-1}) and (3.8_{n-1}) ((3.21_{n-1}) and (3.8_{n-1})) conditions (3.10_ℓ) and (3.8_ℓ) ((3.10_ℓ) and (3.8_ℓ)) hold for any $\ell \in \{1, \dots, n-1\}$ with $\ell + n$ odd. On the other hand, (3.8_{n-1}) and (5.1) clearly imply (4.1) holds. The hypotheses of Theorem 4.1 (Theorem 4.2) are satisfied, and so the conclusion follows. □

Corollary 5.1. *Let $F \in V(\tau)$ and conditions* (1.2) *and* (4.2)–(4.4) *hold, where*

$$\beta_i \leqslant 1 \quad (i = 1, \dots, m). \tag{5.2}$$

Then, condition (4.5_{n-1}) *is sufficient for Eq.* (1.1) *have Property* A.

Proof. By (5.2) and (4.5_{n-1}), conditions (4.5_ℓ) are obviously satisfied for any $\ell \in \{1,\dots,n-1\}$ with $\ell+n$ odd, that is, the hypotheses of Corollary 4.1 are satisfied. □

Corollary 5.1′. *Let $F \in V(\tau)$ and conditions* (1.2), (4.2)–(4.4), *and* (5.2) *hold, where $\mu_i = 1$ $(i = 1,\dots,m)$. In addition, suppose there exists $\tilde p \in L_{\mathrm{loc}}(R_+; R_+)$ such that $p_i(t) = \tilde p(t) + o(t^{n+1})$ $(i = 1,\dots,m)$. Then the condition*

$$\limsup_{t\to+\infty}\left(t\int_t^{+\infty} s^{n-1}\tilde p(s)\,ds + \frac{1}{t}\int_0^t s^n\tilde p(s)\,ds\right) > n!\left(\sum_{i=1}^m \frac{d_i^{n-1}(\beta_i^n-\alpha_i^n)}{\beta_i}\right)^{-1}$$

is sufficient for Eq. (1.1) *to have Property* A.

Using Theorems 4.3 and 4.4, we have the following result that is analogous to Theorem 5.1.

Theorem 5.2. *Let $F \in V(\tau)$, conditions* (2.1_ℓ), (3.1)–(3.4), *and* (5.1) *hold, and suppose there is a nondecreasing function $\varphi \in C(R_+;(0,+\infty))$ such that either conditions* (3.23) *and* (3.24_{n-1}), *or conditions* (3.23) *and* (3.25_{n-1}) *hold. Then Eq.* (1.1) *has Property* A.

6. Differential equations with deviating arguments

Throughout this section, it is assumed that, instead of (3.1), the inequality

$$\bigl|F(u)(t)\bigr| \geqslant \sum_{i=1}^m p_i(t)\bigl|u\bigl(\delta_i(t)\bigr)\bigr|^{\mu_i(\delta_i(t))} \quad \text{for } t \geqslant t_0 \text{ and } u \in H_{t_0,\tau} \tag{6.1}$$

holds for large $t_0 \in R_+$. Here we ask that

$$\begin{aligned} &p_i \in L_{\mathrm{loc}}(R_+;R_+), \quad \mu_i \in C(R_+;(0,1]) \text{ are nondecreasing},\\ &\delta_i \in C\bigl(R_+;(0,+\infty)\bigr), \quad \lim_{t\to+\infty}\delta_i(t) = +\infty \ (i=1,\dots,m). \end{aligned} \tag{6.2}$$

Theorem 6.1. *Let $F \in V(\tau)$, conditions* (2.1_ℓ), (6.1), *and* (6.2) *hold,*

$$\delta_i(t) \leqslant t \quad \text{for } t \in R_+ \ (i=1,\dots,m) \tag{6.3}$$

and

$$\begin{aligned} \limsup_{t\to+\infty}\sum_{i=1}^m\Biggl(&\bigl(\delta_*(t)\bigr)^{\mu_*(\delta_*(t))}\int_t^{+\infty} p_i(s)\bigl(\delta_i(s)\bigr)^{(n-2)\mu_i(\delta_i(s))}\,ds\\ &\times\bigl(\delta_*(t)\bigr)^{\mu_*(\delta_*(t))-\mu_i(\delta_*(t))}\int_{\delta_*(t)}^t p_i(s)\bigl(\delta_i(s)\bigr)^{(n-2)\mu_i(\delta_i(s))}\bigl(\delta_*(s)\bigr)^{\mu_i(\delta_*(t))}\,ds\\ &+\bigl(\delta_*(t)\bigr)^{\mu_*(\delta_*(t))-\mu_i(\delta_*(t))-1}\int_0^{\delta_*(t)} sp_i(s)\bigl(\delta_*(s)\bigr)^{\mu_i(\delta_*(t))}\bigl(\delta_i(s)\bigr)^{(n-2)\mu_i(\delta_i(s))}\,ds\Biggr) \end{aligned}$$

$$> \delta\big(\mu_*^+\big)(n-1)!, \tag{6.4}$$

where $\delta_*(t) = \inf_{s \geqslant t}\{\min \delta_i(s)\colon\ i = 1, \ldots, m\}$. *Then Eq.* (1.1) *has Property* A.

Proof. In view of (6.1), inequality (3.1) clearly holds with

$$\tau_i(t) = \delta_i(t) - 1, \quad \sigma_i(t) = \delta_i(t), \quad r_i(s,t) = p_i(t)e\big(s - \delta_i(t)\big) \quad (i = 1, \ldots, m),$$

where

$$e(t) = \begin{cases} 0 & \text{for } t \in (-\infty, 0), \\ 1 & \text{for } t \in [0, +\infty). \end{cases}$$

Therefore, taking into account (6.2)–(6.4), we can easily check that the conditions of Theorem 5.1 are satisfied with $\varphi(t) = \delta_*(t)$. □

Corollary 6.1. *Let* $F \in V(\tau)$ *and conditions* (1.2) *and* (6.1) *hold, where*

$$\alpha_i(t) = \alpha_i t, \quad \alpha_i(0,1], \quad \mu_i(t) = 1 - \frac{d_i}{\ln t}, \quad \text{and} \quad d_i \geqslant 0 \quad (i = 1, \ldots, m). \tag{6.5}$$

Then

$$\limsup_{t \to +\infty} \sum_{i=1}^{m} e^{-(n-2)d_i} \Bigg(\alpha_* t \int_t^{+\infty} s^{n-2} p_i(s)\,ds + \alpha_* \int_{\alpha_* t}^{t} s^{n-1} p_i(s)\,ds + \frac{1}{t} \int_0^{\alpha_* t} s^n p_i(s)\,ds \Bigg) > (n-1)! e^{d_0},$$

where $\alpha_* = \min\{\alpha_i\colon\ i = 1, \ldots, m\}$ *and* $d_0 = \max\{d_i\colon\ i = 1, \ldots, m\}$, *is a sufficient for Eq.* (1.1) *to have Property* A.

Proof. To prove the corollary, it suffices to note that (6.5) and (6.6) imply (6.4) with $\delta_*(t) = \alpha_* t$. □

Corollary 6.1′. *Let* $F \in V(\tau)$, *conditions* (2.1_ℓ), (6.1), *and* (6.5) *hold, and suppose there is a function* $\tilde p \in L_{\mathrm{loc}}(R_+; R_+)$ *such that*

$$p_i(t) = \tilde p(t) + o(t_n) \quad (i = 1, \ldots, m) \tag{6.6}$$

and

$$\limsup_{t \to +\infty} \Bigg(\alpha_* t \int_t^{+\infty} s^{n-2} \tilde p(s)\,ds + \alpha_* \int_{\alpha_* t}^{t} s^{n-1} \tilde p(s)\,ds + \frac{1}{t} \int_0^{\alpha_* t} s^n \tilde p(s)\,ds \Bigg)$$
$$> (n-1)! e^{d_0} \Bigg(\sum_{i=1}^{m} e^{-(n-2)d_i} \Bigg)^{-1}. \tag{6.7}$$

Then Eq. (1.1) *has Property* A.

Proof. To prove the corollary, just note that (6.6) and (6.7) imply (6.6) with $\alpha_* = \min\{\alpha_i\colon\ i = 1, \dots, m\}$ and $d_0 = \max\{d_i\colon\ i = 1, \dots, m\}$. □

Remark 6.1. From the results obtained in the previous sections, it is clear that if (6.1) holds, it is possible to obtain results that do not require condition (6.3). We restricted our attention to the situation requiring (6.3) only for the sake of simplicity. In addition, by choosing the functions φ and μ_i appropriately, it would be possible to deduce, from our general theorems above, a variety of other conditions for Eq. (1.1) to have Property A.

Remark 6.2. In case $\mu_i(t) \equiv 1$ $(i = 1, \dots, m)$, i.e., the operator F has a linear minorant, the above results imply the results in [14].

7. Functional differential equations with Property B

Using Propositions 3.1–3.4, in this section we give sufficient conditions for Eq. (1.1) to have Property B similar to the results we obtained above for Property A.

Theorem 7.1. *Let $F \in V(\tau)$, conditions* (1.3), (3.1)–(3.4), *and* (3.8_{n-1}) *hold, and suppose there is a nondecreasing function $\varphi \in C(R_+; (0, +\infty))$ such that* (3.9) *holds, and for any $\ell \in \{1, \dots, n-2\}$ with $\ell + n$ even, conditions* (3.8_ℓ) *and* (3.10_ℓ) *hold. Moreover, if n is even, let condition* (4.1) *hold. Then Eq.* (1.1) *has Property B.*

Proof. Let Eq. (1.1) have a proper nonoscillatory solution $u : [t_0, +\infty) \to (0, +\infty)$ (the case $u < 0$ is similar). Then (1.1), (1.3), and Lemma 2.1 imply the existence of $\ell \in \{0, \dots, n\}$ such that $\ell + n$ is even and condition (2.1_ℓ) holds. In view of (3.9), (3.8_ℓ), (3.10_ℓ), and Proposition 3.1, we have $\ell \notin \{1, \dots, n-2\}$. Since $\ell + n$ is even, either $\ell = n$, or n is even and $\ell = 0$. In the latter case, as was shown in the proof of Theorem 4.1, using (4.1), we can easily show that (1.4) holds. On the other hand, if $\ell = n$, then by (2.1_n), there exist $c > 1$ and $t_* > t_0$ such that $u(t) \geqslant ct^{n-1}$ for $t \geqslant t_*$. Therefore, by (2.1_n), (3.1), and (3.8_{n-1}), Eq. (1.1) yields

$$u^{(n-1)}(t) \geqslant u^{(n-1)}(t_1) + \int_{t_1}^{t} \sum_{i=1}^{m} \int_{\tau_i(s)}^{\sigma_i(s)} \xi^{(n-1)\mu_i(\xi)}\, d_\xi r_i(\xi, s)\, ds \to +\infty$$

as $t \to +\infty$, where $t_1 > t_*$ is sufficiently large. Thus, if n is even and $\ell = 0$, then condition (1.4) holds, while if $\ell = n$, then condition (1.5) holds. This means that Eq. (1.1) has Property B, and the theorem is proved. □

Theorem 7.2. *Let $F \in V(\tau)$, conditions* (1.3), (3.1)–(3.4), (3.8_{n-1}), *and* (5.1) *hold, and there exist a nondecreasing function $\varphi \in C(R_+; (0, +\infty))$ satisfying conditions* (3.9) *and* (3.10_{n-2}), *if n is even, and satisfying conditions* (3.9), (3.10_{n-2}), *and* (3.10_1) *if n is odd. Then Eq.* (1.1) *has Property* B.

Proof. In view of (5.1) and (3.10_{n-2}), conditions (3.8_ℓ) are obviously satisfied, where $\ell \in \{2, \dots, n-2\}$ and $\ell + n$ is even. On the other hand, (3.8_{n-1}) and (5.1) imply (3.8_ℓ) holds with $\ell \in \{0, \dots, n-2\}$. Therefore, the hypotheses of Theorem 7.1. hold, and this completes the proof of the theorem. □

Corollary 7.1. *Let $F \in V(\tau)$ and conditions* (1.3) *and* (4.2)–(4.4) *hold with $\beta_i \leqslant 1$. Let* (4.13_{n-2}) *hold if n is even, and let conditions* (4.13_1) *and* (4.13_{n-2}) *hold if n is odd. Then Eq.* (1.1) *has Property* B.

Remark 7.1. It is clear that Remark 4.1 is valid in the case of Property B as well.

8. Generalized ordinary differential equations of Emden–Fowler type

Here, we give sufficient conditions for Eq. (1.7) to have Property A or B. The results of this section are consequences of those of previous sections, but we present them here because the conditions have quite a simple form in this case.

Theorem 8.1. *Let $p \in L_{\mathrm{loc}}(R_+; R_+)$, the function $\mu \in C(R_+; (0,1))$ be nondecreasing, and*

$$\limsup_{t \to +\infty} \left(t^{\mu(t)} \int_t^{+\infty} s^{(n-2)\mu(s)} p(s)\,ds + \frac{1}{t} \int_0^t s^{1+(n-1)\mu(s)} p(s)\,ds \right) > \delta(\mu^+)(n-1)!, \tag{8.1}$$

where $\mu^+ = \lim_{t \to +\infty} \mu(t)$. Then Eq. (1.7) *has Property* A.

Corollary 8.1. *Let $p \in L_{\mathrm{loc}}(R_+; R_+)$, $\mu(t) = 1 - d/\ln t$, $d > 0$, and*

$$\limsup_{t \to +\infty} \left(t \int_t^{+\infty} s^{n-2} p(s)\,ds + \frac{1}{t} \int_0^t s^n p(s)\,ds \right) > e^{(n-1)d}(n-1)!.$$

Then Eq. (1.7) *has Property* A.

Theorem 8.2. *Let $p \in L_{\mathrm{loc}}(R_+; (-\infty, 0])$, the function $\mu \in C(R_+; (0,1))$ be nondecreasing, and let*

$$\limsup_{t \to +\infty} \left(t^{\mu(t)} \int_t^{+\infty} s^{1+(n-3)\mu(s)} \left|p(s)\right| ds + \frac{1}{t} \int_0^t s^{2+(n-2)\mu(s)} \left|p(s)\right| ds \right) > \delta(\mu^+)2(n-2)! \tag{8.2}$$

hold if n is even, and let (8.2) *and*

$$\limsup_{t\to+\infty}\left(t^{\mu(t)}\int_t^{+\infty} s^{n-2}\left|p(s)\right|ds+\frac{1}{t}\int_0^t s^{n-1+\mu(s)}\left|p(s)\right|ds\right)$$
$$>\delta(\mu^+)(n-1)! \tag{8.3}$$

hold if n is odd, where $\mu^+=\lim_{t\to+\infty}\mu(t)$. Then Eq. (1.7) *has Property* B.

Corollary 8.2. *Let $p\in L_{\rm loc}(R_+;(-\infty,0])$, $\mu(t)=1-d/\ln t$, $d>0$, and*

$$\limsup_{t\to+\infty}\left(t\int_t^{+\infty} s^{n-2}\left|p(s)\right|ds+\frac{1}{t}\int_0^t s^{n}\left|p(s)\right|ds\right)>e^{(n-2)d}2(n-2)! \tag{8.4}$$

hold if n is even, and let (8.4) *and*

$$\limsup_{t\to+\infty}\left(t\int_t^{+\infty} s^{n-2}\left|p(s)\right|ds+\frac{1}{t}\int_0^t s^{n}\left|p(s)\right|ds\right)>e^{d}(n-1)!$$

hold if n is odd. Then Eq. (1.7) *has Property* B.

In Theorems 8.1 and 8.2 we put some additional conditions on the functions p and μ, then the conditions (8.1), (8.2), and (8.3) can be made simpler. In this respect, below we give some examples.

Example 8.1. Let the function $\mu\in C(R_+;(0,1))$ be nondecreasing,

$$\lim_{t\to+\infty}\mu(t)=1,\qquad \limsup_{t\to+\infty} t^{\mu(t)-1}=\gamma^*>0, \tag{8.5}$$

$p\in L_{\rm loc}(R_+;R_+)$ and for sufficiently large t

$$P(t)\geqslant\frac{c}{t^{1+(n-1)\mu(t)}},$$

with $c>0$. Then in order that Eq. (1.7) to have Property A, it is sufficient that

$$c(1+\gamma^*)>(n-1)!. \tag{8.6}$$

Example 8.2. Let the function $\mu\in C(R_+;(0,1))$ be nondecreasing,

$$\lim_{t\to+\infty}\mu(t)=1,\qquad \liminf_{t\to+\infty} t^{\mu(t)-1}=\gamma_*>0, \tag{8.7}$$

$p\in L_{\rm loc}(R_+;R_+)$ and for sufficiently large t $p(t)\geqslant c/t^n$ with $c>0$. Then the condition $c\gamma_*^{n-2}(\gamma^*+\gamma_*)>(n-1)!$ is sufficient in order that Eq. (1.7) to have Property A.

Example 8.3. Let the function $\mu\in C(R_+;(0,1))$ be nondecreasing, the condition (8.5) be fulfilled, $p\in C(R_+;(-\infty,0])$ and for sufficiently large t

$$\left|p(t)\right|\geqslant\frac{c}{t^{2+(n-2)\mu(t)}}$$

with $c>0$. Then in order that Eq. (1.7) to have Property B, it is sufficient that the condition (8.6) hold.

Example 8.4. Let the function $\mu \in C(R_+; (0,1))$ be nondecreasing, the condition (8.7) be fulfilled, $p \in L_{\text{loc}}(R_+; (-\infty, 0])$ and for sufficiently large t, $|p(t)| \geqslant c/t^n$, with $c > 0$. Then in order that Eq. (1.7) to have Property B, it is sufficient that

$$c\gamma_*^{n-3}(\gamma^* + \gamma_*) > 2(n-2)! \tag{8.8}$$

if n is even and (8.8) along with $c(\gamma^* + \gamma_*) > (n-1)!$ if n is odd.

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