Mem. Differential Equations Math. Phys. 26(2002), 139–148

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COMPARISON THEOREMS FOR DIFFERENTIAL EQUATIONS WITH SEVERAL DEVIATIONS. THE CASE OF PROPERTY B.

(Reported on February 4, 2002)

1. Introduction

In the present paper we consider the following differential equations

$$u^{(n)}(t)=\sum_{i=1}^{m}p_i(t)u(\tau_i(t)), \tag{1.1}$$

$$v^{(n)}(t)=\sum_{i=1}^{r}q_i(t)v(\sigma_i(t)), \tag{1.2}$$

where $n,\ m,\ r\in N$, $n\geq 3$, $p_i,\ q_j\in L_{\rm loc}(R_+;R_+)$, $\tau_i,\ \sigma_j\in C(R_+;R_+)$, $\lim\limits_{t\to+\infty}\tau_i(t)=\lim\limits_{t\to+\infty}\sigma_j(t)=+\infty$ $(i=1,\dots,m;\ j=1,\dots,r)$.

Definition 1.1. We say that the equation (1.1) has Property **B** if any of its proper solutions either is oscillatory or satisfies

$$|u^{(i)}(t)|\uparrow+\infty \quad \text{for} \quad t\uparrow+\infty \quad (i=0,\dots,n-1), \tag{1.3}$$

when n is odd, and either is oscillatory or satisfies either (1.3) or

$$|u^{(i)}(t)|\downarrow 0 \quad \text{for} \quad t\uparrow+\infty \quad (i=0,\dots,n-1),$$

when n is even.

Below we give comparison theorems allowing to deduce Property **B** of the equation (1.1) from Property **B** of the equation (1.2). The results obtained here generalize those of [1]. The result obtained in [1] is a generalization of a theorem of T. Chanturia (see [2], Theorem 1.5) even in the case of ordinary differential equations ($\tau_i(t)\equiv\sigma_j(t)\equiv t$, $i=1,\dots,m$, $j=1,\dots,r$). For analogous results concerning Property **A** see [3].

2. General Comparison Theorems

Let $\varphi\in C([t_0,+\infty),(0,+\infty))$. Below we use the following notation

$$p_{\tau_i,\varphi}(t)=\begin{cases}p_i(t), & \text{if} \ \ \varphi(t)\leq\tau_i(t),\\ 0 & \text{if} \ \ \varphi(t)>\tau_i(t),\ t\in[t_0,+\infty),\ (i=1,\dots,m),\end{cases} \tag{2.1}$$

$$q_{\sigma_i,\varphi}(t)=\begin{cases}q_i(t), & \text{if} \ \ \varphi(t)\leq\sigma_i(t),\\ 0 & \text{if} \ \ \varphi(t)>\sigma_i(t),\ t\in[t_0,+\infty),\ (i=1,\dots,r).\end{cases} \tag{2.2}$$

2000 *Mathematics Subject Classification.* 34K15.

Key words and phrases. Adanced differential equation, Property **B**, comparison theorem.

Theorem 2.1. *Let*

$$\tau_i(t) \le t, \quad \text{if} \quad t \in R_+ \quad (i = 1, \dots, m), \tag{2.3}$$

$$\int^{+\infty} \sum_{i=1}^{m} p_i(t) \tau_i^{n-1}(t) dt = +\infty, \tag{2.4}$$

and there exist natural numbers $k \in N$, m_j, $r_j \in N$ $(j = 1, \dots, k)$ *and nondecreasing functions* $\varphi_j \in C(R_+; (0, +\infty))$ $(j = 0, \dots, k-1)$ *such that*

$$1 \le m_1 < m_2 < \dots < m_k = m, \quad 1 \le r_1 < r_2 < \dots < r_k = r, \tag{2.5}$$

$$\lim_{t \to +\infty} \varphi_j(t) = +\infty \quad (j = 0, \dots, k-1), \tag{2.6}$$

the below inequality (2.7_{n-2}) *holds when* n *is even,*

$$\int_t^{+\infty} s^{n-l-1} \sum_{i=m_j+1}^{m_{j+1}} \tau_i^{l-1}(s) \Big(p_{\tau_i, \varphi_j}(s) + \frac{\tau_i(s)}{\varphi_j(s)} (p_i(s) - p_{\tau_i, \varphi_j}(s)) \Big) ds \ge$$

$$\ge \int_t^{+\infty} s^{n-l-1} \sum_{i=r_j+1}^{r_{j+1}} \sigma_i^{l-1}(s) \Big(q_i(s) - q_{\sigma_i, \varphi_j}(s) + \frac{\sigma_i(s)}{\varphi_j(s)} q_{\sigma_i, \varphi_j}(s) \Big) ds \tag{2.7$_l$}$$

$$\text{if} \quad t \ge t_0 \quad (j = 0, \dots, k-1),$$

and (2.7_1) *and* (2.7_{n-2}) *hold when* n *is odd, where* t_0 *is sufficiently large,* $m_0 = r_0 = 0$, *the functions* p_{τ_i, φ_j} *and* q_{σ_i, φ_j} *are defined by* (2.1) *and* (2.2), *respectively. Let, moreover, the equation* (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Theorem 2.2. *Let*

$$\tau_i(t) \ge t \quad \text{if} \quad t \in R_+ \ (i = 1, \dots, m), \tag{2.8}$$

and there exist natural numbers k, m_j, $r_j \in N$ $(j = 1, \dots, k)$ *and nondecreasing functions* $\varphi_j \in C(R_+; (0, +\infty))$ $(j = 0, \dots, k-1)$ *satisfying* (2.5) *and* (2.6) *such that the below inequality* (2.7_2) *and* (2.9) *hold,*

$$\int^{+\infty} t^{n-1} \sum_{i=1}^{m} p_i(t) dt = +\infty \tag{2.9}$$

when n *is even and* (2.4) *and* (2.7_1) *hold when* n *is odd, where the functions* p_{τ_i, φ_j} *and* q_{σ_i, φ_j} *are defined by* (2.1) *and* (2.2), *respectively. Let, moreover, the equation* (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Theorem 2.3. *Let the conditions* (2.3) *and* (2.4) *be fulfilled, and for sufficiently large* t_0 *there exist* $t_1 = t_1(t_0) \ge t_0$, *natural numbers* k, m_j, $r_j \in N$ $(j = 1, \dots, k)$ *and nondecreasing functions* $\varphi_j \in C(R_+; (0, +\infty))$ $(j = 0, \dots, k-1)$ *satisfying* (2.5) *and* (2.6) *such that the below inequality* (2.10_{n-2}) *holds when* n *is even,*

$$\int_{t_0}^{t} s^{n-l} \sum_{i=m_j+1}^{m_{j+1}} \tau_i^{l}(s) \Big((p_i(s) - p_{\tau_i, \varphi_j}(s)) + \frac{\varphi_j(s)}{\tau_i(s)} p_{\tau_i, \varphi_j}(s) \Big) ds \ge$$

$$\geq \int_{t_0}^{t} s^{n-l} \sum_{i=r_j+1}^{r_{j+1}} \sigma_i^l(s)\Big(q_{\sigma_i,\varphi_j}(s) + \frac{\varphi_j(s)}{\sigma_i(s)}(q_i(s) - q_{\sigma_i,\varphi_j}(s))\Big) ds \qquad (2.10_l)$$

$$\text{if } \ t \geq t_1 \quad (j = 0, \dots, k-1),$$

and (2.10_1) *and* (2.10_{n-2}) *hold when* n *is odd, where the functions* $p_{\tau_i,\varphi}$ *and* $q_{\sigma_i,\varphi}$*; are defined by* (2.1) *and* (2.2) *respectively. Let, moreover, the equation* (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Theorem 2.4. *Let* (2.8) *be fulfilled, and for sufficiently large* t_0 *there exist* $t_1 = t_1(t_0) \geq t_0$*, natural numbers* k, m_j, $r_j \in N$ $(j = 1, \dots, k)$ *and nondecreasing functions* $\varphi_j(t)$ $(j = 0, \dots, k-1)$ *satisfying* (2.5) *and* (2.6) *respectively, such that the inequalities* (2.9) *and* (2.10_2) *hold when* n *is even and* (2.4) *and* (2.10_1) *hold when* n *is odd. Let, moreover, the equation* (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

3. Effective Comparison Theorems

Everywhere below we assume that $\sigma_i(t)$ $(i = 1, \dots, r)$ are nondecreasing functions.

Theorem 3.1. *Let* $m = r$*, the conditions* (2.3) *and* (2.4) *be fulfilled, the below inequality* (3.1_{n-2}) *hold, when* n *is even*

$$\int_t^{+\infty} s^{n-l-1}\tau_i^{l-1}(s)\Big(p_{\tau_i,\sigma_i}(s) + \frac{\tau_i(s)}{\sigma_i(s)}(p_i(s) - p_{\tau_i,\sigma_i}(s))\Big) ds \geq$$

$$\geq \int_t^{+\infty} s^{n-l-1}\sigma_i^{l-1}(s) q_i(s) ds \quad \text{if} \ \ t \geq t_0 \ \ (i = 1, \dots, m) \qquad (3.1_l)$$

and the inequalities (3.1_1) *and* (3.1_{n-2}) *hold when* n *is odd, where* t_0 *is sufficiently large and the functions* p_{τ_i,σ_i} *are defined by* (2.1).

Let, moreover, the equation (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Corollary 3.1. *Let the conditions* (2.3), (2.4) *be fulfilled,* $m = r$, $\sigma_i(t) \leq \tau_i(t)$ $(i = 1, \dots, m)$ *the below inequality* (3.2_{n-2}) *hold, when* n *is even*

$$\int_t^{+\infty} s^{n-l-1}\tau_i^{l-1}(s) p_i(s) ds \geq \int_t^{+\infty} s^{n-l-1}\sigma_i^{l-1}(s) q_i(s) ds \quad \text{if} \ \ t \geq t_0 \ (i = 1, \dots, m), \quad (3.2_l)$$

and the inequalities (3.2_1) *and* (3.2_{n-2}) *hold when* n *is odd, where* t_0 *is sufficiently large. Let, moreover, the equation* (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Corollary 3.2. *Let the conditions* (2.3), (2.4) *be fulfilled,* $m = r$, $\sigma_i(t) \geq \tau_i(t)$ $(i = 1, \dots, m)$ *the below inequality* (3.3_{n-2}) *hold, when* n *is even*

$$\int_t^{+\infty} \frac{s^{n-l-1}\tau_i^l(s)}{\sigma_i(s)} p_i(s) ds \geq \int_t^{+\infty} s^{n-l-1}\sigma_i^{l-1}(s) q_i(s) ds \quad \text{if} \ \ t \geq t_0 \ (i = 1, \dots, m), \quad (3.3_l)$$

and the inequalities (3.3_1) *and* (3.3_{n-2}) *hold when n is odd, where t_0 is sufficiently large. Let, moreover, the equation* (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Theorem 3.2. *Let $m = r$, the inequality* (2.8) *be fulfilled, the conditions* (2.9) *and* (3.1_2) *hold when n is even and the conditions* (2.4) *and* (3.1) *hold when n is odd. Let, moreover, the equation* (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Corollary 3.3. *Let $m = r$, the inequality* (2.8) *be fulfilled, $\sigma_i(t) \leq \tau_i(t)$ $(i = 1, \dots, m)$, the conditions* (2.9) *and* (3.2_2) *hold when n is even and the conditions* (2.4) *and* (3.2_2) *hold when n is odd. Let, moreover, the equation* (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Corollary 3.4. *Let $m = r$, the inequality* (3.8) *be fulfilled, $\sigma_i(t) \geq \tau_i(t)$ $(i = 1, \dots, m)$, the conditions* (2.9) *and* (3.3_2) *hold when n is odd and the conditions* (2.4) *and* (3.3_1) *hold when n is odd. Let, moreover, the equation* (1.2) *have Property B. Then the equation* (1.1) *also has Property B.*

Theorem 3.3. *Let $m = r$, the conditions* (2.3) *and* (2.4) *be fulfilled and for any sufficiently large t_0 there exist $t_1 = t_1(t_0) \geq t_0$ such that the below inequality* (3.4_{n-2}) *holds, when n is even*

$$\int_{t_0}^{t} s^{n-l}\tau_i^l(s)\Big((p_i(s) - p_{\tau_i,\sigma_i}(s)) + \frac{\sigma_i(s)}{\tau_i(s)}p_{\tau_i,\sigma_i}(s)\Big)ds \geq$$

$$\geq \int_{t_0}^{t} s^{n-l}\sigma_i^l(s)q_i(s)ds \quad for \quad t \geq t_1, \quad (i = 1, \dots, m), \tag{3.4$_l$}$$

and the inequalities (3.4_1) *and* (3.4_{n-2}) *hold when n is odd. Let, moreover, the equation* (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Corollary 3.5. *Let $m = r$, the conditions* (2.3), (2.4) *be fulfilled, $\sigma_i(t) \leq \tau_i(t)$ $(i = 1, \dots, m)$, and for any sufficiently large t_0 there exist $t_1 = t_1(t_0) \geq t_0$ such that the below inequality* (3.5_{n-2}) *holds, when n is even*

$$\int_{t_0}^{t} s^{n-l}\tau_i^{l-1}(s)\sigma_i(s)p_i(s)ds \geq \int_{t_0}^{t} s^{n-l}\sigma_i^l(s)q_i(s)ds, \quad if \ t \geq t_1 (i = 1, \dots, m), \tag{3.5$_l$}$$

and the inequalities (3.5_1) *and* (3.5_{n-2}) *hold when n is odd. Let, moreover, the equation* (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Corollary 3.6. *Let $m = r$, the conditions* (2.3), (2.4) *be fulfilled, $\sigma_i(t) \geq \tau_i(t)$ $(i = 1, \dots, m)$, and for any sufficiently large t_0 there exist $t_1 = t_1(t_0) \geq t_0$ such that the below inequality* (3.6_{n-2}) *holds, when n is even*

$$\int_{t_0}^{t} s^{n-l}\tau_i^l(s)p_i(s)ds \geq \int_{t_0}^{t} s^{n-l}\sigma_i^l(s)q_i(s)ds \quad if \quad t \geq t_1 \quad (i = 1, \dots, m), \tag{3.6$_l$}$$

and the inequalities (3.6_1) *and* (3.6_{n-2}) *hold when n is odd. Let, moreover, the equation* (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Theorem 3.4. *Let $m = r$, the inequality (2.8) be fulfilled, and for any sufficiently large t_0 there exist $t_1 = t_1(t_0) \geq t_0$ such that the conditions (2.9) and (3.4_2) hold when n is even and the conditions (2.4) and (3.4_1) hold when n is odd. Let, moreover, the equation (1.2) have Property **B**. Then the equation (1.1) also has Property **B**.*

Corollary 3.7. *Let $m = r$, the inequality (2.8) be fulfilled, $\sigma_i(t) \leq \tau_i(t)$ $(i = 1, \ldots, m)$, the conditions (2.9) and (3.5_2) hold when n is even and the conditions (2.4) and (3.5_1) hold when n is odd. Let, moreover, the equation (1.2) have Property **B**. Then the equation (1.1) also has Property **B**.*

Corollary 3.8. *Let $m = r$, the inequality (2.8) be fulfilled, $\sigma_i(t) \geq \tau_i(t)$ $(i = 1, \ldots, m)$, the conditions (2.9) and (3.6_2) hold when n is even and the conditions (2.9) and (3.6_1) hold when n is odd. Let, moreover, the equation (1.2) have Property **B**. Then the equation (1.1) also has Property **B**.*

Below we use the following notation

$$\tau_*(t) = \min\{\tau_i(t) : i = 1, \ldots, m\}, \tau^*(t) = \max\{\tau_i(t) : i = 1, \ldots, m\},$$
$$\sigma_*(t) = \min\{\sigma_i(t) : i = 1, \ldots, r\}, \tau^*(t) = \max\{\sigma_i(t) : i = 1, \ldots, r\}.$$

Theorem 3.5. *Let $\tau^*(t) \leq t$ for $t \in R_+$, the condition (2.4) be fulfilled, the below inequality (3.7_{n-2}) hold, when n is even*

$$\int_t^{+\infty} s^{n-l-1} \sum_{i=1}^{m} \tau_i^{l-1}(s) \Big(p_{\tau_i,\sigma^*}(s) + \frac{\tau_i(s)}{\sigma^*(s)} (p_i(s) - p_{\tau_i,\sigma^*}(s)) \Big) ds \geq$$
$$\geq \int_t^{+\infty} s^{n-l-1} \sum_{i=1}^{r} \sigma_i(s)^{l-1} q_i(s) ds \quad \text{if} \quad t \geq t_0, \tag{3.7$_l$}$$

*and the inequalities (3.7_1) and (3.7_{n-2}) holds when n is odd, where t_0 is sufficiently large and the functions p_{τ_i,σ^*} are defined by (2.1). Let, moreover, the equation (1.2) have Property **B**. Then the equation (1.1) also has Property **B**.*

Corollary 3.9. *Let $\tau^*(t) \leq t$ for $t \in R_+$ and the condition (2.4) be fulfilled along with one of the following four conditions (t_0 is sufficiently large):*

1) *$\tau_*(t) \geq \sigma^*(t)$ for $t \in R_+$, the below inequality (3.8_{n-2}) holds, when n is even*

$$\int_t^{+\infty} s^{n-l-1} \sum_{i=1}^{m} \tau_i^{l-1}(s) p_i(s) ds \geq \int_t^{+\infty} s^{n-l-1} \sum_{i=1}^{r} \sigma_i^{l-1}(s) q_i(s) ds \quad \text{if} \quad t \geq t_0, \tag{3.8$_l$}$$

and the inequalities (3.8_1) and (3.8_{n-2}) hold when n is odd;

2) *$\tau_*(t) \leq \sigma^*(t)$ for $t \in R_+$, the below inequality (3.9_{n-2}) holds, when n is even*

$$\int_t^{+\infty} s^{n-l-1} \frac{\tau_*(s)}{\sigma^*(s)} \sum_{i=1}^{m} \tau_i^{l-1}(s) p_i(s) ds \geq \int_t^{+\infty} s^{n-l-1} \sum_{i=1}^{r} \sigma_i^{l-1}(s) q_i(s) ds \quad \text{if } t \geq t_0, \tag{3.9$_l$}$$

and the inequalities (3.9_1) and (3.9_{n-2}) hold when n is odd;

3) *$\tau^*(t) \geq \sigma^*(t)$ for $t \in R_+$, the below inequality (3.10_{n-2}) holds, when n is even*

$$\int_t^{+\infty} \frac{s^{n-l-1}}{\tau^*(s)} \sum_{i=1}^{m} \tau_i^{l}(s) p_i(s) ds \geq \int_t^{+\infty} s^{n-l-1} \sum_{i=1}^{r} \sigma_i^{l-1}(s) q_i(s) ds, \quad \text{if} \quad t \geq t_0, \tag{3.10$_l$}$$

and the inequalities (3.10_1) and (3.10_{n-2}) hold when n is odd;

4) $\tau^*(t) \le \sigma^*(t)$ *for* $t \in R)_+$, *the below inequality* (3.11_{n-2}) *holds, when* n *is even*

$$\int_t^{+\infty} \frac{s^{n-l-1}}{\sigma^*(s)} \sum_{i=1}^{m} \tau_i^l(s) p_i(s) ds \ge \int_t^{+\infty} s^{n-l-1} \sum_{i=1}^{r} \sigma_i^{l-1}(s) q_i(s) ds, \quad \text{if } t \ge t_0, \qquad (3.11_l)$$

and the inequalities (3.11_1) *and* (3.11) *hold when* n *is odd.*

Let, moreover, the equation (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Theorem 3.6. *Let* $\tau^*(t) \le t$ *for* $t \in R_+$, *the condition* (2.4) *be fulfilled, the below inequality* (3.12_{n-2}) *hold, when* n *is even*

$$\int_t^{+\infty} s^{n-l-1} \sum_{i=1}^{m} \tau_i^{l-1}(s) \Big(p_{\tau_i,\sigma_*}(s) + \frac{\tau_i(s)}{\sigma_*(s)} (p_(i)(s) - p_{\tau_i,\sigma_*}(s)) \Big) ds \ge$$

$$\ge \int_t^{+\infty} \frac{s^{n-l-1}}{\sigma_*(s)} \sum_{i=1}^{r} \sigma_i^l(s) q_i(s) ds \quad \text{if} \quad t \ge t_0, \qquad (3.12_l)$$

and the inequalities (3.12_1) *and* (3.12_{n-2}) *holds when* n *is odd, where* t_0 *is sufficiently large and the functions* p_{τ_i,σ_*} *are defined by* (2.1). *Let, moreover, the equation* (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Corollary 3.10. *Let* $\tau^*(t) \ge t$ *for* $t \in R_+$ *and the condition* (2.4) *be fulfilled along with one of the following four conditions* (t_0 *is sufficiently large*):

1) $\tau_*(t) \ge \sigma_*(t)$ *for* $t \in R_+$, *the below inequality* (3.13_{n-2}) *holds, when* n *is even*

$$\int_t^{+\infty} s^{n-l-1} \sum_{i=1}^{m} \tau_i^{l-1}(s) p_i(s) ds \ge \int_t^{+\infty} \frac{s^{n-l-1}}{\sigma_*(s)} \sum_{i=1}^{r} \sigma_i^l(s) q_i(s) ds \quad \text{if} \quad t \ge t_0, \qquad (3.13_l)$$

and the inequalities (3.13_1) *and* (3.13_{n-2}) *hold when* n *is odd;*

2) $\tau^*(t) \le \sigma_*(t)$ *for* $t \in R_+$, *the below inequality* (3.14_{n-2}) *holds, when* n *is even*

$$\int_t^{+\infty} \frac{s^{n-l-1} \tau_*(s)}{\sigma_*(s)} \sum_{i=1}^{m} \tau_i^{l-1}(s) p_i(s) ds \ge \int_t^{+\infty} \frac{s^{n-l-1}}{\sigma_*(s)} \sum_{i=1}^{r} \sigma_i^l(s) q_i(s) ds \quad \text{if} \quad t \ge t_0, \qquad (3.14_l)$$

and the inequalities (3.14_1) *and* (3.14_{n-2}) *hold when* n *is odd;*

3) $\tau^*(t) \ge \sigma_*(t)$ *for* $t \in R_+$, *the below inequality* (3.15_{n-2}) *holds, when* n *is even*

$$\int_t^{+\infty} \frac{s^{n-l-1}}{\tau_0^*(s)} \sum_{i=1}^{m} \tau_i^l(s) p_i(s) ds \ge \int_t^{+\infty} \frac{s^{n-l-1}}{\sigma_*(s)} \sum_{i=1}^{r} \sigma_i^l(s) q_i(s) ds \quad \text{if} \quad t \ge t_0, \qquad (3.15_l)$$

and the inequalities (3.15_1) *and* (3.15_{n-2}) *hold when* n *is odd;*

4) $\tau^*(t) \le \sigma_*(t)$ *for* $t \in R_+$, *the below inequality* (3.16_{n-2}) *holds, when* n *is even*

$$\int_t^{+\infty} \frac{s^{n-l-1}}{\sigma_*(s)} \sum_{i=1}^{m} \tau_i^l(s) p_i(s) ds \ge \int_t^{+\infty} \frac{s^{n-l-1}}{\sigma_*(s)} \sum_{i=1}^{r} \sigma_i^l(s) q_i(s) ds \quad \text{if} \quad t \ge t_0, \qquad (3.16_l)$$

and the inequalities (3.16_1) *and* (3.16_{n-2}) *hold when* n *is odd.*

Let, moreover, the equation (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Theorem 3.7. *Let* $\tau_*(t) \geq t$ *for* n, *the conditions* (2.9) *and* (3.7_2) *be fulfilled when* n *is even and the conditions* (2.4) *and* (3.7_1) *holds when* n *is odd. Let, moreover, the equation* (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Corollary 3.11. *Let* $\tau_*(t) \geq t$ *for* $t \in R_+$ *and one of the following four conditions be fulfilled:*

1) $\tau_*(t) \leq \sigma^*(t)$, *the conditions* (2.9) *and* (3.9_2) *hold when* n *is even and the conditions* (2.4) *and* (3.9_1) *hold when* n *is odd;*

2) $\tau_*(t) \geq \sigma^*(t)$, *the conditions* (2.9) *and* (3.8_2) *hold when* n *is even and the conditions* (2.4) *and* (3.8_1) *hold when* n *is odd;*

3) $\tau^*(t) \leq \sigma^*(t)$, *the conditions* (2.9) *and* (3.11_2) *hold when* n *is even and the conditions* (2.4) *and* (3.11_1) *hold when* n *is odd;*

4) $\tau^*(t) \geq \sigma^*(t)$, *the conditions* (2.9) *and* (3.10_2) *hold when* n *is even and the conditions* (2.4) *and* (3.10_1) *hold when* n *is odd.*

Let, moreover, the equation (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Theorem 3.8. *Let* $\tau_*(t) \geq t$, *the conditions* (2.9) *and* (3.12_2) *be fulfilled when* n *is even and the conditions* (2.4) *and* (3.12_1) *hold when* n *is odd. Let, moreover, the equation* (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Corollary 3.12. *Let* $\tau_*(t) \geq t$ *for* $t \in R_+$ *and one of the following four conditions be fulfilled:*

1) $\tau_*(t) \leq \sigma_*(t)$, *the conditions* (2.9) *and* (3.14_2) *hold when* n *is even and the conditions* (2.4) *and* (3.14_1) *hold when* n *is odd;*

2) $\tau_*(t) \geq \sigma_*(t)$, *the conditions* (2.9) *and* (3.13_2) *hold when* n *is even and the conditions* (2.4) *and* (3.13_1) *hold when* n *is odd;*

3) $\tau^*(t) \geq \sigma_*(t)$, *the conditions* (2.9) *and* (3.15_2) *hold when* n *is even and the conditions* (2.4) *and* (3.15_1) *hold when* n *is odd;*

4) $\tau^*(t) \leq \sigma_*(t)$, *the conditions* (2.9) *and* (3.16_2) *hold when* n *is even and the conditions* (2.4) *and* (3.16_1) *hold when* n *is odd.*

Let, moreover, the equation (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Theorem 3.9. *Let* $\tau^*(t) \leq t$, *the condition* (2.4) *be fulfilled and for any sufficiently large* t_0 *there exist* $t_1 = t_1(t_0) \geq t_0$ *such that the below inequality* (3.17_{n-2}) *holds, when* n *is even*

$$\int\limits_{t_0}^{t} s^{n-l} \sum_{i=1}^{m} \tau_i^l(s)\Big((p_i(s) - p_{\tau_i,\sigma^*}(s))) + \frac{\sigma^*(s)}{\tau_i(s)} p_{\tau_i,\sigma^*}(s)\Big)ds \geq$$

$$\geq \int\limits_{t_0}^{t} s^{n-l}\sigma^*(s) \sum_{i=1}^{r} \sigma_i^{l-1}(s) q_i(s) ds \quad \text{if} \quad t \geq t_1, \tag{3.17_l}$$

and the inequalities (3.17_1) *and* (3.17_{n-2}) *hold when* n *is odd. Let, moreover, the equation* (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Corollary 3.13. *Let $\tau^*(t) \le t$, the condition* (2.4) *be fulfilled and for any sufficiently large t_0 there exist $t_1 = t_1(t_0) \ge t_0$ such that one of the following four conditions is fulfilled:*

1) *$\tau_*(t) \le \sigma^*(t)$ for $t \in R_+$, the below inequality* (3.18_{n-2}) *holds, when n is even*

$$\int_{t_0}^{t} s^{n-l}\tau_*(s)\sum_{i=1}^{m}\tau_i^{l-1}(s)p_i(s)ds \ge \int_{t_0}^{t} s^{n-l}\sigma^*(s)\sum_{i=1}^{r}\sigma_i^{l-1}(s)q_i(s)ds \quad \text{if} \quad t \ge t_1, \tag{3.18_l}$$

and the inequalities (3.18_1) *and* (3.18_{n-2}) *hold when n is odd;*

2) *$\tau_*(t) \ge \sigma^*(t)$ for $t \in R_+$, the below inequality* (3.19_{n-2}) *holds, when n is even*

$$\int_{t_0}^{t} s^{n-2}\sigma^*(s)\sum_{i=1}^{m}\tau_i^{l-1}(s)p_i(s)ds \ge \int_{t_0}^{t} s^{n-l}\sigma^*(s)\sum_{i=1}^{r}\sigma_i^{l-1}(s)q_i(s)ds \quad \text{if} \quad t \ge t_1, \tag{3.19_l}$$

and the inequalities (3.19_1) *and* (3.19_{n-2}) *hold when n is odd;*

3) *$\tau^*(t) \le \sigma^*(t)$ for $t \in R_+$, the below inequality* (3.20_{n-2}) *holds, when n is even*

$$\int_{t_0}^{t} s^{n-l}\sum_{i=1}^{m}\tau_i^{l}(s)p_i(s)ds \ge \int_{t_0}^{t} s^{n-l}\sigma^*(s)\sum_{i=1}^{r}\sigma_i^{l-1}(s)q_i(s)ds \quad \text{if} \quad t \ge t_1, \tag{3.20_l}$$

and the inequalities (3.20_1) *and* (3.20_{n-2}) *hold when n is odd;*

4) *$\tau^*(t) \ge \sigma^*(t)$ for $t \in R_+$, the below inequality* (3.21_{n-2}) *holds, when n is even*

$$\int_{t_0}^{t} s^{n-l}\frac{\sigma^*(s)}{\tau^*(s)}\sum_{i=1}^{m}\tau_i^{l}(s)p_i(s)ds \ge \int_{t_0}^{t} s^{n-l}\sigma^*(s)\sum_{i=1}^{r}\sigma_i^{l-1}(s)q_i(s)ds \quad \text{if} \quad t \ge t_1, \tag{3.21_l}$$

and the inequalities (3.21_1) *and* (3.21_{n-2}) *hold when n is odd.*

Let, moreover, the equation (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Theorem 3.10. *Let $\tau^*(t) \le t$, for $t \in R_+$ the condition* (2.4) *be fulfilled and for any sufficiently large t_0 there exist $t = t_1(t_0) \ge t_0$ such that the below inequality* (3.22_{n-2}) *holds, when n is even*

$$\int_{t_0}^{t} s^{n-l}\sum_{i=1}^{m}\tau_i^{l}(s)\Big((p_i(s) - p_{\tau_i,\sigma_*}(s) + \frac{\sigma_*(s)}{\tau_i(s)}p_{\tau_i,\sigma_*(s)}\Big)ds \ge$$

$$\ge \int_{t_0}^{t} s^{n-l}\sum_{i=1}^{r}\sigma_i^{l}(s)q_i(s)ds \quad \text{if} \quad t \ge t_1, \tag{3.22_l}$$

and the inequalities (3.22_1) *and* (3.22_{n-2}) *hold when n is odd. Let, moreover, the equation* (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Corollary 3.14. *Let $\tau^*(t) \le t$, for $t \in R_+$ the condition* (2.4) *be fulfilled and for any sufficiently large t_0 there exist $t_1 = t_1(t_0) \ge t_0$ such that one of the following four conditions is fulfilled:*

1) $\tau_*(t) \le \sigma_*(t)$ *for* $t \in R_+$, *the below inequality* (3.23_{n-2}) *holds, when* n *is even*

$$\int_{t_0}^{t} s^{n-l}\tau_*(s)\sum_{i=1}^{m}\tau_i^{l-1}(s)p_i(s)ds \ge \int_{t_0}^{t} s^{n-l}\sum_{i=1}^{r}\sigma_i^l(s)q_i(s)ds \quad \text{if} \quad t \ge t_1, \quad (3.23_l)$$

and the inequalities (3.23_1) *and* (3.23_{n-2}) *hold when* n *is odd;*

2) $\tau_*(t) \ge \sigma_*(t)$ *for* $t \in R_+$, *the below inequality* (3.24_{n-2}) *holds, when* n *is even*

$$\int_{t_0}^{t} s^{n-l}\sigma_*(s)\sum_{i=1}^{m}\tau_i^{l-1}(s)p_i(s)ds \ge \int_{t_0}^{t} s^{n-l}\sum_{i=1}^{r}\sigma_i^l(s)q_i(s)ds \quad \text{if} \quad t \ge t_1, \quad (3.24_l)$$

and the inequalities (3.24_1) *and* (3.24_{n-2}) *hold when* n *is odd;*

3) $\tau^*(t) \le \sigma_*(t)$ *for* $t \in R_+$, *the below inequality* (3.25_{n-2}) *holds*

$$\int_{t_0}^{t} s^{n-l}\sum_{i=1}^{m}\tau_i^{l}(s)p_i(s)ds \ge \int_{t_0}^{t} s^{n-l}\sum_{i=1}^{r}\sigma_i^l(s)q_i(s)ds \quad \text{if} \quad t \ge t_1, \quad (3.25_l)$$

when n *is even and the inequalities* (3.25_1) *and* (3.25_{n-2}) *hold when* n *is odd;*

4) $\tau^*(t) \ge \sigma_*(t)$ *for* $t \in R_+$, *the below inequality* (3.26_{n-2}) *holds, when* n *is even*

$$\int_{t_0}^{t} s^{n-l}\frac{\sigma_*(s)}{\tau^*(s)}\sum_{i=1}^{m}\tau_i^{l}(s)p_i(s)ds \ge \int_{t_0}^{t} s^{n-l}\sum_{i=1}^{r}\sigma_i^l(s)q_i(s)ds \quad \text{if} \quad t \ge t_1, \quad (3.26_l)$$

and the inequalities (3.26_1) *and* (3.26_{n-2}) *hold when* n *is odd.*

Let, moreover, the equation (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Theorem 3.11. *Let* $\tau_*(t) \ge t$ *and for any sufficiently large* t_0 *there exist* $t_1 = t_1(t_0) \ge t_0$ *such that the conditions* (2.9) *and* (3.17_2) *are fulfilled when* n *is even and the conditions* (2.4) *and* (3.17_1) *hold when* n *is odd. Let, moreover, the equation* (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Corollary 3.15. *Let* $\tau_*(t) \ge t$ *and for any sufficiently large* t_0 *there exist* $t_1 = t_1(t_0) \ge t_0$ *such that one of the following four conditions is fulfilled:*

1) $\tau_*(t) \le \sigma^*(t)$, *the conditions* (2.9) *and* (3.18_2) *hold when* n *is even and the conditions* (2.4) *and* (3.18_1) *hold when* n *is odd;*

2) $\tau_*(t) \ge \sigma^*(t)$, *the conditions* (2.9) *and* (3.19_2) *hold when* n *is even and the conditions* (2.4) *and* (3.19_1) *hold when* n *is odd;*

3) $\tau^*(t) \le \sigma^*(t)$, *the conditions* (2.9) *and* (3.20_2) *hold when* n *is even and the conditions* (2.4) *and* (3.20_1) *hold when* n *is odd;*

4) $\tau^*(t) \ge \sigma^*(t)$, *the conditions* (2.9) *and* (3.21_2) *hold when* n *is even and the inequalities* (2.4) *and* (3.21_1) *hold when* n *is odd.*

Let, moreover, the equation (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Theorem 3.12. *Let* $\tau_*(t) \ge t$ *and for any sufficiently large* t_0 *there exist* $t_1 = t_1(t_0) \ge t_0$ *such that the conditions* (2.9) *and* (3.22_2) *are fulfilled when* n *is even and the conditions* (2.4) *and* (3.22_1) *hold when* n *is odd. Let, moreover, the equation* (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

Corollary 3.16. *Let $\tau_*(t) \geq t$ and for any sufficiently large t_0 there exist $t_1 = t_1(t_0) \geq t_0$ such that one of the following four conditions is fulfilled:*

1) $\tau_*(t) \leq \sigma_*(t)$, *the conditions* (2.9) *and* (3.23_2) *hold when n is even and the conditions* (2.4) *and* (3.23_1) *hold when n is odd;*

2) $\tau_*(t) \geq \sigma_*(t)$, *the conditions* (2.9) *and* (3.24_2) *hold when n is even and the inequalities* (2.4) *and* (3.24_1) *hold when n is odd;*

3) $\tau^*(t) \leq \sigma_*(t)$, *the conditions* (2.9) *and* (3.25_2) *hold when n is even and the inequalities* (2.4) *and* (3.25_1) *hold when n is odd;*

4) $\tau^*(t) \geq \sigma_*(t)$, *the conditions* (2.9) *and* (3.26_2) *hold when n is even and the conditions* (2.4) *and* (3.26_1) *hold when n is odd.*

Let, moreover, the equation (1.2) *have Property* **B**. *Then the equation* (1.1) *also has Property* **B**.

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