Oscillation of second-order linear difference equations with deviating arguments

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by

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ABSTRACT

Sufficient conditions which guarantee the oscillation of all solutions to the difference equation

$$\Delta^2 u(k) + \sum_{j=1}^{m} p_j(k) u\left(\tau_j(k)\right) = 0 \tag{1.1}$$

are established. Here $\Delta u(k) = u(k+1) - u(k)$, $\Delta^2 = \Delta \circ \Delta$ and the coefficients $p_j (j = 1, ..., m)$ are arbitrary sequences of nonnegative real numbers. It is to be emphasized that the deviations τ_j are subject to the restriction $lim\ inf_{k\to\infty} \frac{\tau_j(k)}{k} > 0$ $(j = 1, ..., m)$ only. In the case where $j = 1$ and $\tau_1(k) \equiv k$, a discrete analogue of the well known Hille's oscillation theorem is obtained.

Key words: Oscillation, difference equation, deviating arguments

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1 Introduction

Consider the equation

$$\Delta^2 u(k) + \sum_{j=1}^{m} p_j(k) u\left(\tau_j(k)\right) = 0, \tag{1.1}$$

where $m \geq 1$ is a natural number, $p_j : \mathbb{N} \to [0, +\infty), \tau_j : \mathbb{N} \to \mathbb{N}$ $(j = 1, ..., m)$ are functions defined on the set of natural numbers $\mathbb{N} = \{1, 2, ...\}$, i.e. sequences, $\Delta u(k) = u(k+1) - u(k)$ and $\Delta^2 = \Delta \circ \Delta$.

Throughout this paper, without further mentioning, we will suppose that

$$lim_{k\to\infty} \ \tau_j(k) = +\infty \qquad (j = 1, ..., m), \tag{1.2}$$

$$sup\,\{p_j(i) : \ i \geq k\} > 0 \ \text{ for } \ k \in \mathbb{N} \ (j = 1, ..., m). \tag{1.3}$$

For any $n \in \mathbb{N}$ we set $\mathbb{N}_n = \{n, n+1, ...\}$.

Definition 1.1. For $n \in \mathbb{N}$ put $n_0 = min\,\{\tau_j(k) : k \in \mathbb{N}_n, \ \ j = 1, ..., m\}$. A function $u : \mathbb{N}_{n_0} \to \mathbb{R}$ is said to be a proper solution of (1.1) if it satisfies (1.1) on $\mathbb{N}_n$ and

$$sup\,\{|u(i)| : \ \ i \geq k\} \ > 0 \quad \text{for} \quad k \in \mathbb{N}_{n_0}.$$

Definition 1.2. We say that a proper solution $u : \mathbb{N} \to \mathbb{R}$ of the equation (1.1) is oscillatory if for any $n \in \mathbb{N}$ there are $n_1, n_2 \in \mathbb{N}_n$ such that $u(n_1)u(n_2) \leq 0$. Otherwise the proper solution is called nonoscillatory.

The present paper is concerned with the problem of oscillation of all solutions of the equation (1.1) under the assumption that the deviations $\tau_j(k) - k$ $(j = 1, ..., m)$ are not necessarily constant and may be unbounded.

The overwhelming majority of the papers devoted to oscillatory properties of difference equations treat the case where the deviations are constant. In that case (or, more generally, in the case where the deviations are bounded), a definition of the order of difference equations (see, e.g., [4, p.163]) considers the equation (1.1) as a linear difference equation of the order

$$max\{2, \tau_j(k) - k : k \in \mathbb{N}, \ j = 1, ..., m\} - min\{0, \tau_j(k) - k : k \in \mathbb{N}, \ j = 1, ..., m\}.$$

In the investigation of oscillatory properties, for the most part, it is more convenient to look at the equation (1.1) as a discrete analogue of the second order ordinary differential equation with deviating arguments

$$u''(t) + \sum_{j=1}^{m} p_j(t) u(\tau_j(t)) = 0.$$

In the case where the deviations $\tau_j(k) - k$ are unbounded, only the second approach seems natural since in that case, according to the above mentioned definition, the equation (1.1) should be considered as an infinite order difference equation. For this reason we call the equation under consideration a second order linear difference equation with deviating arguments. Of the papers treating oscillatory properties of linear difference equations in the case of unbounded deviations, we cite [5, 10, 12].

Oscillatory properties of difference equations analogous to first order ordinary differential equations with constant deviations are set forth in Chapter 7 of the monograph [4] and the references cited therein. Of the works studying oscillatory properties of linear second order difference equations we mention [1, 3, 7, 8, 11] as being most relevant to the matter of the present paper.

In section 2 some auxiliary statements are proved. In section 3 criteria for oscillation of all solutions of (1.1) are established. They imply, as a corollary, a discrete analogue of Hille's oscillation theorem [6]. The latter result also generalizes Theorem 3.4 from [1].

In the sequel it will be assumed that the condition

$$\sum_{k=1}^{\infty} \sum_{j=1}^{m} p_j(k)\tau_j^*(k) = +\infty, \quad \tau_j^*(k) = min\,\{\tau_j(k), k\}\,(j = 1, ..., m) \quad (1.4)$$

is fulfilled. It can be shown using the Schauder-Tychonoff fixed point principle (see, e.g., [2, p.161-163]) that if (1.4) is violated, then (1.1) has a nonoscillatory solution. Hence (1.4) is necessary for oscillation of all solutions of (1.1). The present paper being devoted to this problem, (1.4) does not restrict the generality.