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ON THE CONVERGENCE OF SEQUENCES OF FUNCTIONS IN HARDY CLASSES WITH A VARIABLE EXPONENT

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In [1], we introduced the Hardy class of analytic functions with a variable exponent and announced some results on the properties of functions from that class. Continuing our investigation in this direction, we have obtained the results concerning the convergence of sequences of functions in these classes. These results are the natural generalizations of the well-known facts in the case of Hardy and Smirnov classes.

1^0. Let $U = \{z : |z| < 1\}$ be a circle bounded by the unit circunference γ, and $p = p(t)$, $t \in \gamma$ be a bounded, positive, measurable function defined on *it*.

We say that the analytic in the circle U function Φ belongs to the Hardy class $H^{p(\cdot)}$ if

$$\sup_{r<1} \int_0^{2\pi} \big|\Phi(re^{i\mu})\big|^{p(\mu)} d\mu = C < \infty,$$

where $p(\mu) = p(e^{i\mu})$, $\mu \in [0, 2\pi)$.

For $p(\mu) = p = \text{const} > 0$, the $H^{p(\cdot)}$ class coincides with the Hardy class H^p.

Suppose

$$\underline{p} = \inf_{t\in\gamma} p(t), \quad \overline{p} = \sup_{t\in\gamma} p(t).$$

If $\underline{p} > 0$, then it is obvious that $H^{\overline{p}} \subset H^{p(\cdot)} \subset H^{\underline{p}}$, and hence if $f \in \Phi \in H^{p(\cdot)}$, then for almost all $t \in \gamma$ there exists an angular boundary value $\Phi^+(t)$, and $\Phi^+ \in L^{p(\cdot)}(\gamma)$.

By $L^{p(\cdot)}(\gamma)$ we denote the class of measurable on γ functions f for which

$$\int_\gamma \big|f(t)\big|^{p(t)} |dt| < \infty.$$

2^0. When considering various problems of analysis concerning the classes of analytic functions and their applications, Tumarkin's theorem ([2], p.268) on the convergence of sequences of functions from Smirnov class $E^p(D)$ which is a generalization of Hardy classes for other than a circle domains D, turns frequently out very useful. For a circle this theorem is formulated as follows.

Theorem A. *If the sequence $\{f_n(\zeta)\}$ of boundary values of the functions $f_n(z)$ of the class H^p converges in measure on the set of positive measure e $(m(e) > 0,\ e \subset \gamma)$, and*

$$\int_\gamma \big|f_n(\zeta)\big|^p |d\zeta| < C,$$

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where C does not depend on n, then the sequence $\{f_n(z)\}$ converges uniformly in the circle U to the function $f(z)$ of the class H^p, and the sequence $\{f_n(\zeta)\}$ converges in measure on the set e to the function $f(\zeta)$, i.e., to the angular boundary values of $f(z)$.

3^0. Imposing supplementary conditions on the function $p(t)$, we can generalize Theorem A to the classes $H^{p(\cdot)}$. In our case, such a supplementary condition is either the Log-Hölder condition (which is encountered in the investigations of different operators in the classes $L^{p(\cdot)}$), or the Hölder condition.

We say that the function $p(t)$ belongs to the class $\mathcal{P}(\gamma)$ if there exists the constant A, such that for any $t_1, t_2 \in \gamma$, $|t_1 - t_2| < \frac{1}{2}$ the inequality

$$\big|p(t_1) - p(t_2)\big| < \frac{A}{|\ln|t_1 - t_2||}\,.$$

is fulfilled.

4^0.

Theorem 1. *If the sequence $\{f_n(\zeta)\}$ of boundary values of the functions $f_n(z)$ of the class $H^{p(\cdot)}$, $p \in \mathcal{P}(\gamma)$, $\underline{p} > 1$ converges in measure on the circumference γ, and*

$$\int\limits_{\gamma} |f_n(\zeta)\big|^{p(\zeta)}|d\zeta| = \int\limits_{0}^{2\pi} |f_n(e^{i\mu})\big|^{p(e^{i\mu})}d\mu < C,$$

where C does not depend on n, then $\{f_n(z)\}$ converges uniformly in the circle U to the function of the class $H^{p(\cdot)}$, while the sequence $\{f_n(\zeta)\}$ converges in measure on γ to the function $f(\zeta)$, i.e., to the angular boundary values of the function $f(z)$.

This theorem is an analogue of Theorem A for a particular case when the set e coincides with the circumference γ. Note that, it is desirable to have $\underline{p} > 0$ rather than $\underline{p} > 1$.

5^0. If we replace the condition $p \in \mathcal{P}(\gamma)$ by $p \in H(\gamma)$, i.e., by the condition that there exist the positive constants M and α $(0 < \alpha \le 1)$, such that

$$\big|p(t_1) - p(t_2)\big| < M|t_1 - t_2|^{\alpha},$$

then we will be able to establish more complete analogue of Theorem A.

Theorem 2. *If the sequence $\{f_n(\zeta)\}$ of boundary values of the functions $f_n(z)$ from $H^{p(\cdot)}$, $p \in H(\gamma)$, $\underline{p} > 0$, converges in measure on the set of positive measure e, $e \subset \gamma$ and $\int_\gamma \big|f_n(\zeta)\big|^{p(\zeta)}|d\zeta| < C$, where C does not depend on n, then the sequence $\{f_n(z)\}$ converges uniformly in U to the function $f(z)$ of the class $H^{p(\cdot)}$, and the sequence $\{f_n(\zeta)\}$ converges in measure on the set e to the function $f(\zeta)$.*

6^0. Theorem 1 is proved by using the results cited in [1], while in proving Theorem 2 we use besides the facts presented in [1], the following statement.

Theorem 3. *If $f \in H^{p(\cdot)}$, $\underline{p} > 0$, $p \in H(\gamma)$, then there exist the constants K_1 and K_2 depending only on $p(t)$, such that*

$$\int\limits_{0}^{2\pi} \big|f(re^{i\mu})\big|^{p(\mu)}d\mu \le K_1\bigg(\int\limits_{0}^{2\pi} \big|f(e^{i\mu})\big|^{p(\mu)}d\mu + 2\pi\bigg) \exp\bigg(K_2\int\limits_{0}^{2\pi} |f(e^{i\mu})|^{\underline{p}}\, d\mu\bigg).$$

7^0. Theorems 1 and 2 allow us to establish one characteristic property of functions from $H^{p(\cdot)}$ which is analogous to the properties of functions from H^p ([2], p.138).

Theorem 4. *Let $p \in H(\gamma)$, $\underline{p} > 0$. The necessary and sufficient condition for the measurable function $\varphi(\zeta) = \varphi(e^{i\mu})$ given on the set $E \subset \gamma$, $mE > 0$, to coincide almost everywhere on E with the boundary value $f(e^{i\mu})$ of some function $f(z)$ of the class $H^{p(\cdot)}$ is the existence of the polynomials $\mathcal{P}_n(z)$, such that*

(1) $\{\mathcal{P}_n(\zeta)\}$ *converges almost everywhere on E to $\varphi(\zeta)$;*

(2) $\overline{\lim_{n\to\infty}} \int\limits_0^{2\pi} \big|\mathcal{P}_n(e^{i\mu})\big|^{p(\mu)} d\mu < \infty.$

This statement remains valid if we replace the assumptions of the theorem by the conditions $p \in \mathcal{P}(\gamma)$, $\underline{p} > 1$ and $E = \gamma$.

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