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ON THE INVERSE INEQUALITIES FOR TRIGONOMETRIC POLYNOMIAL APPROXIMATIONS IN WEIGHTED LORENTZ SPACES

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1. Introduction

The Weiersterass well-known theorem on the approximation of the continuous function by the trigonometric polynomials and its quantitative refinement represented by Jackson inequality

$$E_n(f) \leq C\omega(f, \frac{1}{n}) \qquad (1.1)$$

are one of the basement of the Approximation Theory.

In the inequality (1.1), for $2\pi-$periodic continuous function f, $E_n(f)$ denotes the best approximation of f by the trigonometric polynomials, i. e.

$$E_n(f) = \inf \max_{x\in[0,2\pi]} |f(x) - T_k(x)|\,,$$

where the infimum is taken over all trigonometric polynomials of order $k \leq n$, and

$$\omega(f, \delta) = \sup_{|h|\leq\delta} \max_{x\in[0,2\pi]} |f(x+h) - f(x)|$$

denotes the modulus of continuity of f. The analog of Jackson inequality is correct for the mean approximation and higher order modulus of continuity as well (see [1]).

Yet by 1912, S. Bernstein [2], obtained the reversed estimations of Jackson's inequality in the space of continuous functions for some specific cases. Later Quade [3], brothers A. and M. Timan [4], S. B. Stechkin [5], M. Timan [6], etc. proved the reversed type inequalities of Jackson's inequality, including in L^p, $1 < p < \infty$, spaces. These type inequalities played an important role in the investigation of properties of the conjugate functions [7], in the study of absolute convergent Fourier series [8], and in the related problems. In the weighted Lebesgue spaces the inverse inequalities for classical module of smoothness and best approximations were derived in the papers [6], [9]. In [10] this result is extended for reflexive Orlicz space. For the approximation in weighted Lebesgue and Orlicz spaces we refer to [11], [12], [13], [14].

The order of generalized modulus of smoothness, as it has been shown in [6] and [9], depends not only on the rate of the best approximation but also on the metric of the spaces. In the present paper we reveal that the similar influence in weighted Lorentz spaces is expressed not only by the "leading" parameter of the space, but also by the second parameter in the definition of Lorentz spaces.

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Let $\mathbf{T} = [-\pi, \pi)$ and $w : \mathbf{T} \to \mathbb{R}^1$ be an almost everywhere positive, integrable function. Let $f_w^*(t)$ be a nondecreasing rearrangement of $f : \mathbf{T} \to \mathbb{R}^1$ with respect to the Borel measure

$$w(e) = \int_e w(x)dx$$

i. e.

$$f_w^*(t) = \inf\{\tau \geq 0 : w\,(x \in \mathbf{T} : |f(x)| > \tau) \leq t\}.$$

Let $1 < p,\ s < \infty$ and let $L_w^{ps}(\mathbf{T})$ be a weighted Lorentz space, i. e. a set of all measurable functions for which

$$\|f\|_{L_w^{ps}} = \left(\int_{\mathbf{T}} (f^{**}(t))^s\, t^{\frac{s}{p}} \frac{dt}{t}\right)^{1/s} < \infty,$$

where

$$f^{**}(t) = \frac{1}{t}\int_0^t f_w^*(u)du.$$

In what follows, $E_n(f)_{L_w^{ps}}$ denotes the best approximation of $f \in L_w^{ps}(\mathbf{T})$ by trigonometric polynomials of order n, i. e.

$$E_n(f)_{L_w^{ps}} = \inf \|f - T_k\|_{L_w^{ps}},$$

where the infimum is taken over all trigonometric polynomials of order $k \leq n$.

Let for $f \in L_w^{ps}(\mathbf{T})$ define the generalized modulus of smoothness as

$$\Omega_l\,(f, \delta)_{L_w^{ps}} = \sup_{0<h<\delta} \left\| \prod_{i=1}^{l} \left(I - A_{h_i}\right) f \right\|_{L_w^{ps}}, \qquad \delta > 0,$$

where I is the identity operator and

$$\left(A_{h_i} f\right)(x) := \frac{1}{2h_i} \int_{x-h_i}^{x+h_i} f(u)du.$$

The weights w considered in the paper are from Muckenhoupt class A_p, i. e. they satisfy

$$\sup \frac{1}{|I|}\int_I w(x)dx \left(\frac{1}{|I|}\int_I w^{1-p'}(x)dx\right)^{p-1} < \infty, \qquad p' = \frac{p}{p-1},$$

where the supremum is taken over all intervals with length $\leq 2\pi$.

Whenever $w \in A_p$, $1 < p,\ s < \infty$, the Hardy-Littlewood maximal function of any $f \in L_w^{ps}(\mathbf{T})$, and therefore the average $A_{h_i} f$ belong to $L_w^{ps}(\mathbf{T})$ [15]. Thus $\Omega_l(f, \delta)_{L_w^{ps}}$ makes a sense for any $w \in A_p(\mathbf{T})$.

In the sequel constants (often different constants in the same series of inequalities) will generally be denoted by c.

2. Main Results

In the present paper we prove the following results.

Theorem 1. *Let $1 < p < \infty$, and $1 < s \leq 2$ or $p > 2$ and $s \geq 2$. Let $w \in A_p(\mathbf{T})$. Then there exists a positive constant c such that*

$$\Omega_l\left(f, \frac{1}{n}\right)_{L_w^{ps}} \leq \frac{c}{n^{2l}} \left(\sum_{k=1}^{n} k^{2l\gamma-1} E_{k-1}^{\gamma}(f)_{L_w^{ps}}\right)^{1/\gamma} \tag{2.1}$$

for arbitrary $f \in L_w^{ps}(\mathbf{T})$ and natural n, where $\gamma = \min(s, 2)$.

Theorem 2. *Let $1 < p < \infty$, and $1 < s \le 2$ or $p > 2$ and $s \ge 2$, and let $w \in A_p(\mathbf{T})$ and $f \in L_w^{ps}(\mathbf{T})$. Assume that*

$$\sum_{k=1}^{\infty} k^{r\gamma-1} E_k^{\gamma}(f)_{L_w^{ps}} < \infty \tag{2.2}$$

for some natural number r and $\gamma = \min(s,2)$. Then there exists the absolutely continuous $(r-1)$th order derivative $f^{(r-1)}(x)$ such that $f^{(r)} \in L_w^{ps}$ and

$$E_n(f^{(r)})_{L_w^{ps}} \le c\left\{ n^r E_n(f)_{L_w^{ps}} + \left(\sum_{k=n+1}^{\infty} k^{r\gamma-1} E_k^{\gamma}(f)_{L_w^{ps}} \right)^{1/\gamma} \right\} \tag{2.3}$$

for arbitrary natural n, where $\gamma = \min(s,2)$ and the constant c does not depend on f and n.

Theorem 3. *Let the conditions of Theorem 2 be fulfilled. Then there exists a positive constant c such that*

$$\Omega_l\Big(f, \frac{1}{n}\Big)_{L_w^{ps}} \le \frac{c}{n^{2l}} \left(\sum_{k=1}^{n} k^{(r+2l)\gamma-1} E_{k-1}^{\gamma}(f)_{L_w^{ps}} \right)^{1/\gamma} + \tag{2.4}$$
$$+ c\left(\sum_{k=n+1}^{\infty} k^{r\gamma-1} E_k^{\gamma}(f)_{L_w^{ps}} \right)^{1/\gamma}$$

for arbitrary $f \in L_w^{ps}(\mathbf{T})$ and natural n, where $\gamma = \min(s,2)$.

Corollary. *Let for some integer $r \ge 1$ and $l \ge 1$*

$$E_n(f)_{L_w^{ps}} = O\Big(\frac{1}{n^{r+2l}}\Big).$$

Then

$$\Omega_l\Big(f^{(r)}, \frac{1}{n}\Big)_{L_w^{ps}} = O\Big(\frac{\ln n}{n^{2l}}\Big)^{1/\gamma}. \tag{2.5}$$

Let $\{\alpha_n\}$ be a monotonic sequence of positive numbers convergent to zero. Let $\Phi_w^{ps}(\alpha_n)$ be the set of functions $f \in L_w^{ps}$ for which

$$E_n(f)_{L_w^{ps}} \sim \alpha_n.$$

The sharpness of (2.5) when s, $p > 2$ shows the following.

Theorem 4. *There exists a constant $c > 0$ such that, for each $\alpha_n \downarrow 0$, there exists $f_0 \in \Phi_w^{ps}(\alpha_n)$ satisfying the inequality*

$$\Omega\Big(f_0, \frac{1}{n}\Big)_{L_w^{ps}} \ge \frac{c}{n^2} \left(\sum_{k=1}^{n} k^3 \alpha_{k-1}^2 \right)^{1/2} \tag{2.6}$$

for arbitrary natural n.

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