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Singular Integrals and Potentials in Some Banach Function Spaces with Variable Exponent

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1. On some Banach function Spaces

Let (Ω, μ) be a measure space. Let $M(\Omega, \mu)$ be a space of measurable functions on Ω.

Definition 1. A normed linear space $X = (X(\Omega, \mu), \| \ \|_X)$ is called a Banach function space if the following conditions are satisfied:

i) The norm $\|f\|_X$ is defined for all $f \in M(\Omega, \mu)$.

ii) $\|f\|_X = 0$ if and only if, $f(x) = 0$ μ-a.e., on Ω.

iii) $\|f\|_X = \big\| |f| \big\|_X$ for all $f \in X$.

iv) For every $Q \subset \Omega$ with $\mu Q < \infty$ we have $\|\chi_Q\|_X < \infty$.

v) If $f_n \in M(\Omega, \mu)$, $n = 1, 2, \dots$ and $f_n \nearrow f$ μ-a.e., on Ω then

$$\|f_n\|_X \nearrow \|f\|_X.$$

vi) If f, $g \in M(\Omega, \mu)$ and $0 \le f(x) \le g(x)$ μ-a.e., on Ω then

$$\|f\|_X \le \|g\|_X.$$

vii) Given $Q \subset \Omega$ with $\mu Q < \infty$, there exists a constant c_Q such that for all $f \in X$,

$$\int\limits_Q |f(x)| d\mu \le c_Q \|f\|_X.$$

Every Banach function space is a Banach space. For definition and fundamental properties of Banach function space we refer to [1].

We shall deal with some special Banach function space.

Let Ω be a bounded open subset of R^n and $p(x)$ is a measurable function on Ω such that

$$1 < p_0 \le p(x) \le P < \infty, \quad x \in \overline{\Omega}, \tag{1}$$

and

$$|p(x) - p(y)| \le \frac{A}{\ln 1/(|x-y|)}, \quad |x-y| \le 1/2, \quad x,\, y \in \overline{\Omega}. \tag{2}$$

By $L^{p(\cdot)}(\Omega)$ we denote the space of measurable functions $f(x)$ on Ω such that

$$A_p(f) = \int\limits_\Omega |f(x)|^{p(x)} dx < \infty.$$

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This is a Banach function space with respect to the norm

$$\|f\|_{L^{p(\cdot)}} = \inf\{\lambda > 0 : A_p(f/\lambda) \leq 1\}$$

(see e.g., [2]).

In [3] the boundedness of maximal functions in $L^{p(\cdot)}$ spaces has been proved. Further in [4] the mapping properties of maximal operator and singular operator with fixed singularity in weighted $L^{p(\cdot)}$ spaces was studied.

On the base of $L^{p(\cdot)}$ we introduce some new Banach function space. Let us denote by

$$f^*(t) = \sup\Big\{s \geq 0 : m\{x \in \Omega : |f(x)| > s\} > t\Big\}$$

-the non-increasing rearrangment of function f. Here by m we denote the Lebesgue measure. It is clear that $f^*(t) = 0$ when $t > m\Omega$, since $m\Omega < \infty$.

Let a function $p(t)$ satisfy the condition (1.1) when $t \in [0, m\Omega]$.

Definition 2. The subset of all functions of $M(\Omega, m)$ for which

$$\|f\|_{\Lambda^{p(\cdot)}} = \|f^{**}\|_{L^{p(\cdot)}} < \infty$$

we call a space $\Lambda^{p(\cdot)}$.

Here

$$f^{**}(t) = 1/t \int\limits_0^t f^*(y)dy.$$

It is clear that $f^*(t) \leq f^{**}(t)$. According to the Theorem IV from [4] we conclude that there exists such constant $c > 0$ that

$$\|f^*\|_{L^{p(\cdot)}} \leq \|f^{**}\|_{L^{p(\cdot)}} \leq c\|f^*\|_{L^{p(\cdot)}}.$$

Note that $\|f^{**}\|_{L^{p(\cdot)}}$ is a norm. The triangle inequality follows from the inequality

$$(f+g)^{**}(t) \leq f^{**}(t) + g^{**}(t)$$

(See e.g., [5], Section 2).

2. Integral Transforms in R^n

Let us start by mapping property of singular integrals in $\Lambda^{p(\cdot)}$. The singular operators we take into account have the form

$$Kf(x) = \lim_{\varepsilon \to 0+} \int\limits_{\{y:|y|\geq\varepsilon\}} \frac{k(y)}{|y|^n} f(x-y)\, dy, \quad x \in \Omega,$$

where K is an odd function on R^n which is homogeneous of degree 0 and satisfies the following Dini condition on the unit sphere S^{n-1} on R^n

$$\int\limits_0 \frac{\omega(\delta)}{\delta}\, d\delta < \infty, \quad \text{where} \quad \omega(\delta) = \sup_{\substack{x,y\in S^{n-1} \\ |x-y|\leq\delta}} \Big|k(x) - k(y)\Big|.$$

Observe that this definition includes classical operators, such as the Hilbert transform ($n = 1, k(x) = x/|x|$) and Riesz transform ($n \geq 2, k(x) = (x_j)/(|x|)$, $j = 1, \ldots, n$).

Theorem 1. *Let* $1 \le p(t) < P < \infty$ *for* $t \in [0, m\Omega]$. *Let the conditions*

$$1 < p_0 \le p(t) < P < \infty$$

and

$$\Big|p(t_1) - p(t_2)\Big| \le \frac{A}{\ln 1/(|t_1 - t_2|)}, \quad |t_1 - t_2| \le 1/2,$$

be satisfied in a neighbourhood $[0, d]$ *of the origin,* $d > 0$.

Then K *is bounded in* $\Lambda^{p(\cdot)}$.

Theorem 2. *Let* $p(t)$ *satisfy the conditions of previous theorem. Suppose that*

$$-1/\big(p(0)\big) < \beta < 1/\big(q(0)\big). \tag{3}$$

Then the inequality

$$\|Kf\|^{p}_{\Lambda_\beta}(\cdot) \le c\|f\|_{\Lambda^{p(\cdot)}_\beta}$$

holds with a constant c *independent of* f.

Corollary 1. *Let* p *be as in Theorem* 1. *Then if the condition* (3) *is satisfied the operators* R_j $(j = 1, \ldots, n)$ *are bounded in* $\Lambda^{p(\cdot)}_\beta$.

In the sequel we discuss the boundedness in $\Lambda^{p(\cdot)}$ of Riesz potentials and application to the imbedding of certain spaces of differentiable functions.

Let us start by Riesz potential

$$I_\alpha f(x) = \int\limits_\Omega \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \quad x \in \Omega, \quad 0 < \alpha < n.$$

Theorem 3. *Let us suppose that* $p(t)$ *satisfy the requirments from the previous Theorem. Let* $s(x)$ *be a measurable function on* $[0, m\Omega]$ *such that* $1 \le s(x) < S < \infty$ *for all* $x \in [0, m\Omega]$, *and*

$$s(0) = p(0) \quad \textit{and} \quad |s(x) - p(x)| \le \frac{A}{\ln 1|x|}, \quad 0 < x < \delta, \quad \delta > 0.$$

Then I_α *acts boundedly from* $L^{p(\cdot)}$ *into* $L^{s(\cdot)}$.

Moreover, if

$$-1/p(0) < \beta < 1/(q(0)),$$

then the inequality

$$\|t^\beta I_\alpha\|_{\Lambda^{s(\cdot)}} \le c\|t^\beta f\|_{\Lambda^{p(\cdot)}}$$

holds with a constant c *independent of* f.

Theorem 4. *Let* $n \ge 2$ *and let* k *be any positive integer smaller than* n. *Suppose that* $p(x)$ *and* $s(x)$ *satisfy the conditions of Theorem* 1. *Then*

i) a positive constant c *exists such that*

$$\|u\|_{\Lambda^{s(\cdot)}} \le c\|D^k u\|_{\Lambda^{p(\cdot)}} \tag{4}$$

for all real-valued functions u *in* Ω *where the continuation by* 0 *outside* Ω *has weak derivatives up to order* k *over* R^n. *Here* D^k *stands for the vector of* k*-th order derivatives of* u.

If Ω *is convex, then a positive constant* c *exists such that*

$$\inf_{P \in \mathcal{P}_{k-1}} \|f - Q\|_{\Lambda^{s(\cdot)}} \le c\|D^k u\|_{\Lambda^{p(\cdot)}} \tag{5}$$

for all real valued functions u *in* Ω *having weak derivatives up to order* k *in* Ω. *Here* $\mathcal{Q}_{k-1}$ *denotes the set of all polynomials* Q *of degree* $\le k-1$.

When $k=1$ *inequality* (5) *holds, in particular, with* $Q=1/(m\Omega)\int\limits_{\Omega}u(x)dx$*-the mean value of* u *over* Ω*.*

Now we are going to discuss the mapping properties of Poisson integral and conjugate Poisson integrals in $\Lambda_\beta^{p(\cdot)}$ spaces. Let us consider the Poisson integral

$$u_f(x,y)=\int\limits_{\Omega} f(u)\frac{y}{(|x-u|^2+y^2)^{(n+1)/2}}\,du,\quad x,y\in\Omega,$$

and the system of conjugate Poisson integrals

$$v_f^j(x,y)=\int\limits_{\Omega} f(u)\frac{x_j-y_j}{(|x-u|^2+y^2)^{(n+1)/2}}\,du,\quad x,y\in\Omega\;\; j=1,2,\dots,n.$$

Since $m\Omega<\infty$ for $f\in L^{p(\cdot)}(\Omega)$ we have that $f\in L^{p_0}(\Omega)$. Thus we conclude that

$$Tf(x)=\sup_{y>0}|u_f(x,y)|\le cMf(x) \tag{6}$$

and

$$v_f^j(x,y)=u_{R_j}(x,y) \tag{7}$$

(see [6], chapters 6 and 2).

From (6) thanks to the known estimate (see [7]) we have

$$\Big(\sup_y|u_f(x,y)|\Big)^*(t)\le c(Mf)^*(t)\le c_1 1/t\int\limits_0^t f^*(y)dy. \tag{8}$$

Theorem 5. *Let* $p(t)$ *and* β *satisfy the conditions of Theorem* 1*. Then* T *is bounded in* $\Lambda_\beta^{p(\cdot)}$*.*

Now consider the operator

$$\widetilde{T}_j f(x)=\sup_y\left|v_f^j(x,y)\right|.$$

Theorem 6. *Let a function* $p(t)$ *and a number* β *satisfy the conditions of Theorem* 1*. Then the operators* $\widetilde{T}_j$ *are bounded in* $\Lambda_\beta^{p(\cdot)}$*.*

3. Cauchy Singular Integrals on Lyapunov curves and curves of bounded rotation

In this section we deal with the Cauchy singular integral

$$S_\Gamma f(t)=\int\limits_\Gamma \frac{f(\tau)}{\tau-t}\,d\tau$$

where Γ is a finite rectifiable Jordan curve on which as a parameter the arc-length is chosen starting any fixed point. The equation of the curve in the case is $t=t(s)$, $0\le s\le l$, where l is its length.

Γ is called the Lyapunov curve if $t'(s)\in \operatorname{Lip}\alpha$, $0<\alpha\le 1$. When $t'(s)$ is a function of bounded variation, then Γ is called as a curve of bounded rotation.

Our goal is to study the mapping property of S_Γ when Γ is a Lyapunov curve or a curve of bounded rotation without cusps.

We assume that a function $p(s)$ is defined on $[0, l]$. In the sequel $f(t(s))$ will denoted by $f_0(s)$.

Theorem 7. *Let Γ be a Lyapunov curve.*
Let

$$1 \le p(s) \le P < \infty \quad \textit{for} \quad s \in [0, l].$$

Suppose that the conditions

$$1 < p_0 \le p(s) \le P < \infty$$

and

$$|p(s_1) - p(s_2)| \le \frac{A}{\ln 1/(|s_1 - s_2|)}, \quad s_1, s_2 \in [0, l]$$

are satisfied in some neighbourhood of the origin.
Then S_Γ is bounded in $\Lambda^{p(s)}$.

Theorem 8. *Let Γ be a curve of bounded rotation without cusps. Let $p(s)$ satisfy the condition of Theorem* 1 *supposing that m denotes the arc-length measure on Γ. Then S_Γ is bounded in $\Lambda^{p(s)}$.*

Note that for the constant p the boundedness of S_Γ on Lyapunov curve and on curve of bounded rotation without cusps has been proved in [8] and [9] respectively.

Theorem 9. *Let Γ be a Lyapunov curve or a curve of bounded rotation without cusps. Let a weight*

$$w(s) = |t(s) - t(0)|^\beta$$

where

$$-1/(p(0)) < \beta < 1/(q(0)).$$

Then Cauchy singular operator S_Γ acts boundedly in Λ^p_w.

Basing on the recent results on the singular integrals from [10] we conclude the validity of

Theorem 10. *Let Γ be a Lyapunov curve or a curve of bounded rotation without cusps. If the function $p(s)$ satisfies the conditions* (1) *and* (2) *on $\overline{\Omega} = [0, \ell]$, then S_Γ is bounded in $L^{p(s)}$.*

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