Proc. A. Razmadze Math. Inst. 132 (2003), 148-149

V. Kokilashvili

Two-Weighted Estimates for Fourier Multipliers in Lebesgue Spaces

(Reported on 01.05.2003)

The main goal of this report is to present two-weighted estimates for (L^p, L^q) multipliers of Fourier transforms in Lebesgue spaces with power weights.

One-weighted multiplier theorems of Marcinkiewicz, Mikhlin and Hörmander type in Lebesgue spaces with Muckenhoupt A_p weights are given in [1], [2]. General (L^p, L^q) $(1 < p \le q < \infty)$ Fourier multipliers in unweighted case have been studied in [3]–[5]. The Fourier multiplier theorems in weighted Triebel-Lizorkin spaces (including two-weighted inequalities) are established in [6]–[8].

Let w be a locally integrable almost everywhere positive function on R^n. By $L^p_w(\mathbb{R}^n)$ $(1 < p < \infty)$ we denote the set of measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ for which

$$\|f\|_{L^p_w(\mathbb{R}^n)} = \bigg(\int\limits_R^n |f(x)|^p w(x)\,dx\bigg)^{1/p} < \infty.$$

Let $S(\mathbb{R}^n)$ be the Schwartz space of rapidly decreasing functions. For $\varphi \in S(\mathbb{R}^n)$ the Fourier transform $\widehat{\varphi}$ is defined by

$$\widehat{\varphi}(\lambda) = (2\pi)^{-\frac{n}{2}} \int\limits_{R^n} \varphi(x) \exp\big\{-i\lambda x\big\}\,dx.$$

Let $S'(\mathbb{R}^n)$ be the space of tempered distributions, i.e., the space of linear bounded functionals on $S(\mathbb{R}^n)$. In the sequel, the Fourier transforms in the framework of the theory of S'-distributions will be considered.

Let X and Y be two function spaces on $\mathbb{R}^n$ with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Assume that $S(\mathbb{R}^n)$ is dense in both X and Y spaces.

Definition 1. A distribution $m \in S'$ is called an (X, Y) multiplier if for the operator $\mathcal{K}$ defined by the Fourier transform equation

$$\widehat{\mathcal{K}f} = m\widehat{f}, \quad f \in S,$$

there exists a constant c such that

$$\|\mathcal{K}f\|_Y \le c\|f\|_X$$

for all $f \in S(\mathbb{R}^n)$. In this case we write $m \in \mathcal{M}(X, Y)$.

Let Q_m $m = (m_1, m_2, \dots, m_n)$ be a set in R^n given by the inequalities

$$2^{m_j} < |\lambda_j| \le 2^{m_j+1}, \quad j = 1, 2, \dots, n, \quad m_j = 0, \pm 1, \pm 2, \dots.$$

2000 *Mathematics Subject Classification:* 42A45, 42B15, 42B20.

Key words and phrases. Fourier multiplier, two-weighted estimates, Lebesgue space.

For the given p and q, $1 < p \le q < \infty$ and given α_j, $0 < \alpha_j < 1$ $(i = 1, 2, \dots, n)$ in the sequel we consider the pairs of weights (v, w), where

$$w(x) = \prod_{j=1}^{n} |x_j|^{\beta_j} \tag{1}$$

and

$$v(x) = \prod_{j=1}^{n} |x_j|^{\gamma_j} \tag{2}$$

We assume that,

$$q \le \frac{p}{1 - \alpha_j p}, \quad \alpha_j p - 1 < \beta_j < p - 1 \quad \text{and} \quad \frac{\gamma_j + 1}{q} = \frac{\beta_j + 1}{p} - \alpha_j. \tag{3}$$

Theorem. *Let $1 < p < q < \infty$ and m be a function represented in each set Q_m as*

$$m(\lambda) = \int_{-\infty}^{\lambda_1} \cdots \int_{-\infty}^{\lambda_n} \prod_{j=1}^{n} (\lambda_j - t_j)^{-\alpha_j} \, d\mu(t_1, \dots, t_n), \quad 0 < \alpha_j < 1.$$

where μ_n are finite measures for which

$$\sup_m \operatorname{var} \mu_m = M < \infty.$$

Then $m \in \mathcal{M}(L_w^p, L_v^q)$.

Corollary. *Let $1 < p < q < \infty$ and let $0 < \alpha_j < 1$. Let m be continuous outside the coordinate planes and have there continuous derivatives*

$$\frac{\partial^k m}{\partial \lambda_1^{k_1} \partial \lambda_2^{k_2} \cdots \partial \lambda_n^{k_n}}, \quad 0 \le k_1 + \cdots + k_n = k \le n, \quad k_j = 0, 1.$$

Moreover, assume that

$$\left| \lambda_1^{k_1 + \alpha_1} \lambda_2^{k_2 + \alpha_2} \cdots \lambda_n^{k_n + \alpha_n} \left(\frac{\partial^k m}{\partial \lambda_1^{k_1} \cdots \partial \lambda_n^{k_n}} \right) \right| \le \mathcal{M}.$$

Then $m \in \mathcal{M}(L_w^p, L_v^q)$, where w and v are defined by (1), (2) *and* (3).

References

1. D. S. Kurtz, Littlewood–Paley and multiplier theorems in weighted L^p spaces. *Trans. Amer. Math. Soc.* **259**(1980), 235–263.
2. D. S. Kurtz and R. L. Wheeden, Results on weighted norm inequalities for multipliers. *Trans. Amer. Math. Soc.* **255**(1979), 343–362.
3. P. I. Lizorkin, L^p, L^q-multipliers of Fourier integrals. (Russian) *Dokl. Akad. Nauk SSSR,* **152**(1963), No. 4, 808–811; English transl:. *Soviet Math. Dokl.* **4**, 1420–1424.
4. P. I. Lizorkin, On multipliers of Fourier integrals in the spaces $L_{p,\theta}$. (Russian) *Trudy Mat. Inst. Steklov* **89**(1967), 231–248; English transl.: *Proc. Steklov Inst. Math.* No. 89, 269–290.
5. P. I. Lizorkin, On the theory of multipliers. (Russian) *Trudy Mat. Inst. Steklov* **173**(1986), 149–163.
6. V. M. Kokilashvili, On weighted Lizorkin-Triebel spaces. Singular integrals, multipliers, imbedding theorems. (Russian) *Trudy Mat. Inst. Steklov* **161**(1983); English transl.: *Proc. Steklov Inst. Math.* No. 3, 135–162.

7. V. M. Kokilashvili and P. I. Lizorkin, Two-weight estimates for multipliers, and imbedding theorems. (Russian) *Dokl. Akad. Nauk SSSR,* **336**(1994), No. 4; English transl.: *Russian Acad. Sci. Dokl. Math.* **49**, No. 3, 515–519.
8. D. E. Edmunds, V. Kokilashvili, and A. Meskhi, On Fourier multipliers in weighted Triebel–Lizorkin spaces. *J. Ineq. & Appl.* **7**(2002), No. 4, 555–591.

Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, Aleksidze St., Tbilisi 0193
Georgia