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ON THE SOLVABILITY OF DIVERGENCE EQUATION IN THE
THEORY OF INCOMPRESSIBLE FLUIDS

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The main goal of the present note is to investigate the solvability of the divergence equation

$$\operatorname{div} u = f, \tag{1}$$

which is of a great importance in the theory of incompressible fluids.

For a given f with the zero mean value and belonging to the generalized weighted Lebesgue space $L_{\rho}^{p(\cdot)}$, we seek for a solution of (1) in a weighted Sobolev space with variable exponent.

The generalized Lebesgue spaces $L^{p(\cdot)}$ with variable exponent and integral operators in these spaces have won a great interest not so long ago. During the last decade, especially in the last years, one can see growing interest in these and in the corresponding Sobolev type spaces. The progress in studying the $L^{p(\cdot)}$ and $W^{m,p(\cdot)}$ spaces themselves and the operator theory in these spaces is noticeable. As is known, this interest arose, apart from the mathematical curiosity, by possible applications to models with the so-called non-local growth in the fluid mechanics, elasticity theory and differential equations (see, e.g., [1] and references therein).

The weighted estimates with power weights have been proved in [2] and [3] for the maximal functions and singular integrals on bounded domains. The feature of these results is that they have the form of criteria.

1. Function spaces: weighted Lebesgue and Sobolev spaces with variable exponent.

Let $\Omega \subset R^n$, $n \geq 2$, be a bounded domain with the Lipschitz boundary. Let $x_0 \in \Omega$.

We introduce some important Banach function spaces for treatment of the problem we discuss about.

Let $p : \Omega \rightarrow R^1$ be a measurable function satisfying the following chain of inequalities:

$$1 < \underline{p} \leq p(x) \leq \bar{p} < \infty. \tag{2}$$

By $L^{p(\cdot)}$ we denote a Banach function space of all measurable functions φ for which

$$\|\varphi\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{\varphi(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\} < \infty. \tag{3}$$

The spaces $L^{p(\cdot)}(\Omega)$ are special cases of the generalized Orlicz spaces originated by Nakano [4] and developed by Musielak and Orlicz [5].

Let ρ be a measurable, almost everywhere non-negative function. The weighted Lebesgue spaces with variable exponent is the space of all measurable functions f for which

$$\|f\|_{L_{\rho}^{p(\cdot)}} \stackrel{\text{def}}{=} \|f\rho\|_{L^{p(\cdot)}} < \infty.$$

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In the sequel we will consider the power weights $\rho(x) = |x - x_0|^\alpha$. The space $L_\rho^{p(\cdot)}$ is a Banach function space.

Let $u : \Omega \rightarrow R^n$, i.e., $u = (u_1, u_2, \dots, u_n)$, $u_k = u_k(x_1, x_2, \dots, x_n)$, $k = 1, 2, \dots, n$. All functions u_k are assumed to be measurable.

By the weighted Sobolev space with variable exponent we mean the space of all measurable vector-functions u for which

$$\|u\|_{W_\rho^{1,p(\cdot)}} = \|u\rho\|_{L^{p(\cdot)}} + \|\nabla u\rho\|_{L^{p(\cdot)}} < \infty.$$

Here ∇u denotes the $n \times n$ matrix

$$\left[\frac{\partial u_i}{\partial x_j} \right]_{i,j=1,\dots,n}.$$

Along with the condition (1) it will be assumed that the function $p(x)$ satisfies the weak Hölder condition

$$|p(x) - p(y)| \leq \frac{c}{-\ln|x-y|}, \quad |x-y| < \frac{1}{2}, \quad (4)$$

with a constant c which does not depend on $x, x \in \Omega$.

We note that the class of functions $p(x)$ satisfying the conditions (2) and (4) is considerably wide. It contains also the non-Hölder functions.

Here we present an example of the function which satisfies the weak Hölder condition, but in fact it is not a Hölder function:

$$p(x) = a(x) + \frac{b(x)}{\left(\ln \frac{1}{|x|}\right)^\beta}, \quad x \in \Omega,$$

where a and b are the Hölder functions, and $a(x) \geq 1$, $b(x) \geq 1$ and $\gamma \geq 1$.

2. Some Auxiliary Results.

A kernel k on $R^n \times R^n$ is a locally integrable complex-valued function k defined on some distance off the diagonal. The kernel k satisfies the standard estimates if and only if there exist $\delta > 0$ and $c > 0$ such that for all $x, y \in R^n$, $x \neq y$ and $z \in R^n$ with $|x-z| < \frac{1}{2}|x-y|$ the inequalities

$$\begin{aligned} |k(x, y)| &\leq c|x-y|^{-n}, \\ |k(x, y) - k(z, y)| &\leq c|x-z|^\delta|x-y|^{-n-\delta}, \\ |k(y, x) - k(y, z)| &\leq c|x-z|^\delta|x-y|^{-n-\delta} \end{aligned}$$

hold. In this case we call k a standard kernel.

A linear and continuous operator $K : C_0^\infty(R^n) \rightarrow D'(R^n)$, where D' is the space of distributions, is said to be associated to a kernel k if

$$(Kf, g) = \int_{R^n} \int_{R^n} k(x, y)f(y)g(x) dx dy,$$

whenever $f, g \in C_0^\infty(R^n)$ with $\text{supp } f \cap \text{supp } g = \emptyset$. K is called a singular integral operator if K is associated to a standard kernel. If, in addition, K extends to a bounded, linear operator in $L^2(R^n)$, then K is called a Calderon-Zygmund operator.

Along with Calderon-Zygmund operators, we consider the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x, r) \cap \Omega|} \int_{B(x, r) \cap \Omega} |f(y)| dy.$$

Theorem A ([2]). *Let p satisfy the conditions (2) and (4). The necessary and sufficient condition for the boundedness of M in $L_\rho^{p(\cdot)}(\Omega)$, $\rho(x) = |x - x_0|^\alpha$, is*

$$-\frac{1}{p(x_0)} < \alpha < \frac{1}{q(x_0)}. \quad (5)$$

Theorem B ([3]). *Let p satisfy the conditions (2) and (4). Then the condition (5) is sufficient for the boundedness in $L_\rho^{p(\cdot)}$ of the operator K . Moreover, the condition (5) is necessary and sufficient for the boundedness of the finite Hilbert transformation.*

3. Extension of Korn's Inequality.

Let $D(u)$ be a symmetric part of ∇u , i.e.,

$$D(u) = \frac{1}{2} \left[\nabla u + \nabla u^\top \right].$$

Theorem 1. *Under the assumption of Theorem A, for all $f \in \mathring{W}_\rho^{1,p}(\Omega)$ the inequality*

$$\|\nabla u\|_{L_\rho^{p(\cdot)}} \leq c \|Du\|_{L^{p(\cdot)}} \quad (6)$$

holds, where c is a constant independent of u .

Note that the initial form of the inequality (5) has been proved by A. Korn in connection with a priori estimates of the solutions of nonlinear equations in the theory of elasticity.

4. The Divergence Equation.

The problem of solvability of the equation (1) has been studied by many authors in standard Lebesgue and Sobolev spaces. In this case the theory is based on Bogovski's [6,7] explicit representation formula. Using the same technique, along with the results on the boundedness of maximal functions and singular integrals in unweighted generalized Lebesgue spaces, L. Diening and M. Růžička [1] investigated the solvability of the equation (1) in Sobolev spaces with variable exponent.

Define

$$\mathring{L}_\rho^{p(\cdot)} = \left\{ f \in L_\rho^{p(\cdot)}(\Omega) : \int_\Omega f(x) dx = 0 \right\}.$$

Theorem 2. *Let p satisfy the conditions (2) and (4). Assume that*

$$-\frac{1}{p(x_0)} < \alpha < \frac{1}{q(x_0)}.$$

Then for every $f \in \mathring{L}_\rho^{p(\cdot)}$ the divergence equation (1) is solvable in $W_\rho^{1,p(\cdot)}$, and the estimate

$$\|\nabla u\|_{L_\rho^{p(\cdot)}} \leq c \|f\|_{L_\rho^{p(\cdot)}},$$

i.e.,

$$\|u\|_{W|_{rho^1,p}} \leq c \|f\|_{L_\rho^{p(\cdot)}}$$

holds.

Theorem 3. *In case $\alpha = \frac{1}{q(x_0)}$, Theorem 2 is invalid.*

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