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APPROXIMATION IN WEIGHTED LEBESGUE AND SMIRNOV SPACES WITH VARIABLE EXPONENTS

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Abstract. The paper deals with the approximation problems for periodic and analytic functions in weighted Lebesque spaces with variable exponents.

რეზიუმე. სტატიაში განხილულია პერიოდული და ანალიზური ფუნქციების აპროქსიმაციის პრობლემები წონიან ლებეგის სივრცეებში ცვალებადი მაჩვენებლით.

In this survey paper we present the basic theorems of approximation theory for weighted Lebesgue spaces with variable exponents in the one-dimensional periodic setting. These spaces have been studied intensively by many mathematicians (see e.g. the papers [28], [22], [26] and the surveys [7], [21] and [27]). The study of these spaces has been stimulated by various problems of elasticity, fluid mechanics, calculus of variation and differential equations with nonstandard growth conditions. Nowadays both the problem of denseness of nice functions in variable Sobolev spaces and the problem of boundedness of integral operators in Lebesgue spaces are solved (see for example the above mentioned surveys). The approximation problems in the spaces with nonstandard growth conditions are studied not so much. We refer to the pioneering paper of I. I. Sharapudinov [29], see also [30].

Let $\mathbf{T} := [0, 2\pi]$ and ω: $\mathbf{T} \to \mathbb{R}^1$be a weight function, i. e., almost everywhere positive and integrable function on **T.**

In the sequel the set of all measurable functions $p : \mathbf{T} \to (1, \infty)$, for which

$$\underline{p} := \operatorname*{ess\,inf}_{x\in\mathbf{T}} p(x) > 1 \quad \text{and} \quad \overline{p} := \operatorname*{ess\,sup}_{x\in\mathbf{T}} p(x) < \infty$$

is denoted by $\wp$.

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For a given $p \in \wp$ and weight ω, by $L_{\omega}^{p(\cdot)}(\mathbf{T})$ we denote the set of measurable functions f, for which

$$\int_{\mathbf{T}} |f(x)\,\omega(x)|^{p(x)}\,dx < \infty;$$

equipped with the norm

$$\|f\|_{L_{\omega}^{p(\cdot)}(\mathbf{T})} := \inf\left\{\lambda > 0 : \int_{\mathbf{T}} \left|\frac{f(x)\,\omega(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}$$

it is a Banach space.

It is known [19] that the set of trigonometric polynomials is dense in $L_{\omega}^{p(\cdot)}(\mathbf{T})$, if $[\omega(x)]^{p(x)}$ is integrable on $\mathbf{T}$.

Let $p(x)$ and $\omega(x)$ be the functions, such that the Hardy-Littlewood maximal operator

$$Mf(x) := \sup_{0<h<\pi} \frac{1}{2h} \int_{x-h}^{x+h} |f(t)|\,dt$$

is bounded in $L_{\omega}^{p(\cdot)}(\mathbf{T})$. By $\Re$ we denote the class of such pairs.

In particular, we consider the weight

$$\omega(x) := \prod_{i=1}^{n} \left(\sin\left|\frac{x - x_k}{2}\right|\right)^{\alpha_k},$$

where x_k, $k = 1, 2, \ldots$ are the distinct points on $\mathbf{T}$ and

$$-\frac{1}{p(x_k)} < \alpha_k < \frac{1}{p'(x_k)}, \quad k = 1, 2, \ldots, n.$$

for some $p \in \wp$. It is known [17] that if there exists a positive constant $c > 0$ such that

$$|p(x) - p(y)| \leq \frac{c}{\ln(1/|x-y|)}$$

for arbitrary x and y on $\mathbf{T}$ with the condition $|x - y| < 1/2$, then $(p, \omega) \in \Re$.

Definition 1. For $f \in L_{\omega}^{p(\cdot)}(\mathbf{T})$ and $(p, \omega) \in \Re$ we define the k-th generalized modulus of continuity $\Omega_{p(\delta),\omega,k}(f, \cdot)$ in the space $L_{\omega}^{p(\cdot)}(\mathbf{T})$ as

$$\Omega_{p(\cdot),\omega,k}(f, \delta) := \sup_{0<h_i<\delta} \left\| \prod_{i=1}^{k} (E - \sigma_{h_i})\, f \right\|_{L_{\omega}^{p(\cdot)}(\mathbf{T})},$$

where

$$\sigma_{h_i} f(x) := \frac{1}{2h_i} \int_{x-h_i}^{x+h_i} f(t)\,dt.$$

Since $(p,\omega)\in\Re$, it is clear that $\Omega_{p(\cdot),\omega,k}(f,\cdot)$ is well defined.

For $f\in L^{p(\cdot)}_{\omega}(\mathbf{T})$ we set

$$E_n(f)_{L^{p(\cdot)}_{\omega}(\mathbf{T})} := \inf\|f-P_k\|_{L^{p(\cdot)}_{\omega}(\mathbf{T})}, \quad n=1,2,\ldots$$

where inf is taken over all trigonometric polynomials of degree not exceeding n.

The order of approximation by trigonometric polynomials have been studied by several authors. The elegant presentation of the corresponding results in the nonweighted and weighted (for some special weights) Lebesgue spaces L^p, $1\leq p\leq\infty$, can be found in [31], [4] and [25]. Problems of best approximation by trigonometric polynomials in weighted spaces with weights satisfying the Muckenhoupt $A_p(\mathbf{T})$ $-$condition were investigated in the papers [10], [23], [24]. In particular, using $L^p(\mathbf{T},\omega)$ version of the $k-$th modulus of continuity $\Omega_{p(\cdot),\omega,k}(f,\cdot)$ some direct and inverse theorems in weighted Lebesgue spaces were obtained in [10].

The order of polynomial approximation were also considered in the complex domain, in the weighted and nonweighted Smirnov spaces. In particular, Walsh and Russel [32] obtained such results when Γ is an analytic curve. For domains with sufficiently smooth boundary these problems were considered by S. Y. Alper [1]. The results obtained in this direction in nonweighted and weighted Smirnov spaces were later extended to the more general domains by several authors (see for example: [15], [2], [3] in nonweighted cases and [11], [8], [12], [13] and [14],in weighted cases).

In this paper we consider the above mentioned approximation problems in the weighted Lebesgue and Smirnov spaces with variable exponents.

Let $W^r_{p(\cdot)}(\mathbf{T},\omega)$ $(r=1,2,\ldots)$ be the linear space of functions for which $f^{(r-1)}$ is absolutely continuous on $\mathbf{T}$ and $f^{(r)}\in L^{p(\cdot)}_{\omega}(\mathbf{T})$. It becomes a Banach spaces with respect to the norm:

$$\|f\|_{W^r_{p(\cdot)}(\mathbf{T},\omega)} := \|f\|_{L^{p(\cdot)}_{\omega}(\mathbf{T})} + \|f^{(r)}\|_{L^{p(\cdot)}_{\omega}(\mathbf{T})}.$$

The weighted Lebesgue spaces with variable exponents may be also defined on a Jordan rectifiable curve Γ, see [16], [20]. For any measurable bounded exponent $p(z)\geq 1$ and any weight $\rho\geq 0$, such that

$$\nu\{t\in\Gamma:\ \rho(t)=0\}=0,$$

the space $L^{p(\cdot)}(\Gamma,\rho)$ is defined as the set of functions f, for which

$$I_p(f) := \int\limits_{\Gamma} |f(z)\rho(z)|^{p(z)}\,d\nu(z) < \infty,$$

where $\nu(z)$ is the arc-length measure on Γ. The norm in this space is defined as

$$\|f\|_{L^{p(\cdot)}(\Gamma,\rho)} := \inf\left\{\lambda > 0 : \int\limits_{\Gamma} \left|\frac{f(z)\,\rho(z)}{\lambda}\right|^{p(z)} d\nu(z) \leq 1\right\}.$$

If

$$1 \leq p(z) \leq p_1 < \infty$$

and

$$|\rho(z)|^{p(z)} \in L^1(\Gamma),$$

then the set $\mathbf{C}^{\infty}(\Gamma)$ (and the set of bounded rational functions on Γ) is dense in $L^{p(\cdot)}(\Gamma,\rho)$ [18].

In the case $1 < p_0 \leq p(z) \leq p_1 < \infty$, the space $L^{p(\cdot)}(\Gamma)$ coincides with the space of functions

$$\left\{f : \left|\int\limits_{\Gamma} f(z)\,g(z)\,d\nu(z)\right| < \infty \quad \text{for all} \quad g \in L^{p'(\cdot)}(\Gamma)\right\},$$

where

$$p'(z) := \frac{p(z)}{p(z)-1}.$$

Let $f \in E^1(G)$. Then f has a nontangential limit a. e. on Γ and the boundary function belongs to $L^1(\Gamma)$. This boundary function will be also denoted by f.

Definition 2. The set

$$E^r_{p(\cdot)}(G,\rho) := \left\{f \in E^1(G) : f^{(r)} \in L^{p(\cdot)}(\Gamma,\rho)\right\}$$

is called the $\rho-$ weighted Smirnov class of variable exponent $p(\cdot)$.

In case of $r = 0$ and $p(\cdot) = p$ we have the usual weighted Smirnov space $E_p(G,\rho)$. The space $E_{p(\cdot)}(G,\rho)$ becomes a Banach spaces with respect to the norm:

$$\|f\|_{E^r_{p(\cdot)}(G,\rho)} := \|f\|_{L^{p(\cdot)}(\Gamma,\rho)} + \left\|f^{(r)}\right\|_{L^{p(\cdot)}(\Gamma,\rho)}.$$

In this paper we present basic theorems of approximation theory in the spaces $W^r_{p(\cdot)}(\mathbf{T},\omega)$ and $E^r_{p(\cdot)}(G,\rho)$ $(r = 1, 2, \ldots)$, respectively.

New results

The following theorems hold.

Theorem 1. *Let $W^r_{p(\cdot)}(\mathbf{T},\omega)$ $(r = 1, 2, \ldots)$ be the space with the pair $(p,\omega) \in \Re$. Then for every $f \in W^r_{p(\cdot)}(\mathbf{T},\omega)$ the estimate*

$$E_n(f)_{L^{p(\cdot)}_{\omega}(\mathbf{T})} \leq c_r n^{-r} \Omega_{p(\cdot),\omega,k}\left(f^{(r)}, \frac{1}{n}\right), \quad k = 1, 2, \ldots,$$

holds with a constant $c_r > 0$ independent of n.

The inverse results are the following.

Theorem 2. *Let* $W_{p(\cdot)}^{r}(\mathbf{T},\omega)$ $(r=0,1,2,\ldots)$ *be the space with the pair* $(p,\omega)\in\Re,$. *Then for* $f\in W_{p(\cdot)}^{r}(\mathbf{T},\omega)$ *and for every natural number* n *the estimate*

$$\Omega_{p(\cdot),\omega,k}\Big(f,\frac{1}{n}\Big)\leq$$
$$\leq c\bigg\{\frac{1}{n^{2k}}\sum_{m=1}^{n}m^{2k-r-1}E_m\left(f^{(r)}\right)_{L_\omega^{p(\cdot)}(\mathbf{T})}+\sum_{m=n+1}^{\infty}k^{r-1}E_k\left(f^{(r)}\right)_{L_\omega^{p(\cdot)}(\mathbf{T})}\bigg\}$$

holds for a constant $c>0$ *independent of* n.

Theorem 3. *Let* $W_{p(\cdot)}^{r}(\mathbf{T},\omega)$ $(r=1,2,\ldots)$ *be the space with the pair* $(p,\omega)\in\Re$. *If* $f\in W_{p(\cdot)}^{r}(\mathbf{T},\omega)$ *then*

$$E_n\left(f^{(r)}\right)_{L_\omega^{p(\cdot)}(\mathbf{T})}\leq c\bigg\{n^rE_n(f)_{L_\omega^{p(\cdot)}(\mathbf{T})}+\sum_{k=n+1}^{\infty}k^{r-1}E_k(f)_{L_\omega^{p(\cdot)}(\mathbf{T})}\bigg\},$$

with a constant $c=c(r)$.

Theorem 4. *Let* $L_\omega^{p(\cdot)}(\mathbf{T},\omega)$ $(r=1,2,\ldots)$ *be the space with the pair* $(p,\omega)\in\Re$. *If for* $f\in L_\omega^{p(\cdot)}(\mathbf{T},\omega)$ *the inequality*

$$\sum_{m=1}^{\infty}m^{r-1}E_m(f)_{L_\omega^{p(\cdot)}(\mathbf{T})}<\infty$$

holds for some $r=1,2,\ldots,$ *then* $f\in W_{p(\cdot)}^{r}(\mathbf{T},\omega)$ *and*

$$\Omega_{p(\cdot),\omega,k}\Big(f^{(r)},\frac{1}{n}\Big)\leq$$
$$\leq c\bigg\{\frac{1}{n^{2k}}\sum_{m=0}^{n}(m+1)^{2k+r-1}E_m(f)_{L_\omega^{p(\cdot)}(\mathbf{T})}+\sum_{k=n+1}^{\infty}k^{r-1}E_k(f)_{L_\omega^{p(\cdot)}(\mathbf{T})}\bigg\}.$$

From Theorem 2, in case of $r=0$, we obtain the following Corollary.

Corollary 1. *Under the conditions of Theorem* 2, *if* $f\in L_\omega^{p(\cdot)}(\mathbf{T})$ *satisfies the inequality*

$$E_m(f)_{L_\omega^{p(\cdot)}(\mathbf{T})}=O\left(m^{-\alpha}\right),\quad m=1,2,\ldots,$$

for some $\alpha>0$, *then for any natural number* k *and* $\delta>0$,

$$\Omega_{p(\cdot),\omega,k}(f,\delta)=\begin{cases}O\left(\delta^{\alpha}\right), & k>\alpha/2,\\ O\left(\delta^{\alpha}\log\left(1/\delta\right)\right), & k=\alpha/2,\\ O\left(\delta^{2k}\right), & k<\alpha/2.\end{cases}$$

Hence if we define the generalized Lipschitz class Lip $\alpha(p(\cdot),\omega)$ for $\alpha > 0$ and $k := [\alpha/2] + 1$ as

$$\text{Lip }\ \alpha(p(\cdot),\omega) := \left\{ f \in L^{p(\cdot)}_{\omega}(\mathbf{T}) : \quad \Omega_{p(\cdot),\omega,k}(f,\delta) \le c\delta^{\alpha}, \delta > 0 \right\},$$

then from Corollary 1 we obtain the following

Corollary 2. *Under the conditions of Theorem* 2, *if* $f \in L^{p(\cdot)}_{\omega}(\mathbf{T})$ *satisfies the inequality*

$$E_m(f)_{L^{p(\cdot)}_{\omega}(\mathbf{T})} = O\left(m^{-\alpha}\right), \quad m = 1, 2, \ldots,$$

for some $\alpha > 0$, *then* $f \in \text{Lip }\alpha(p(\cdot),\omega)$.

Combining this Corollary with Theorem 1 we obtain the following constructive description of the classes Lip $\alpha(p(\cdot),\omega)$.

Theorem 5. *Let* $L^{p(\cdot)}_{\omega}(\mathbf{T})$ *be the space with the pair* $(p,\omega) \in \Re$. *Then for* $\alpha > 0$ *the following assertions are equivalent:*
(i) $f \in \text{Lip }\alpha(p(\cdot),\omega)$;
(ii) $E_m(f)_{L^{p(\cdot)}_{\omega}} = O(m^{-\alpha})$, $\quad m = 1, 2, \ldots.$

Let G be a finite domain in the complex plane bounded by a rectifiable Jordan curve L, and let D be the unit disk, $\mathbf{T} := \partial D$, $G^{-} := ext\, L$, $D^{-} := ext\ \mathbf{T}$. We denote by φ the conformal mapping of G^{-} onto D^{-} normalized by the conditions

$$\varphi(\infty) = \infty \quad \text{and} \quad \lim_{z\to\infty} \varphi(z)/z > 0.$$

Let ψ be the inverse mapping to φ.

We assume that Γ is a smooth Jordan curve and $\theta(s)$, the angle between the tangent and the positive real axis expressed as a function of the arc length s has modulus of continuity $\Omega(\theta,s)$ satisfying the Dini condition

$$\int_0^{\delta} \frac{\Omega(\theta,s)}{s} ds < \infty, \quad \delta > 0. \tag{1}$$

For $f \in L^{p(\cdot)}(\Gamma,\rho)$ we define the function

$$f_0(w) := (f \circ \psi)(w), \qquad w \in T.$$

It is clear that under the condition (1), if $f \in L^{p(\cdot)}(\Gamma,\rho)$, then $f_0(w) \in L^{p(\cdot)}(T,\rho_0)$ with $\rho_0 := \rho \circ \psi$. It is also easy to show that under the condition (1) the conditions $(p,\rho) \in \Re$ and $(p_0,\rho_0) \in \Re$ are equivalent.

We define the $k-$th modulus of smoothess of the function $f \in L^{p(\cdot)}(\Gamma,\rho)$ by

$$\Omega^k_{p(\cdot),\Gamma,\rho}(f,\delta) := \Omega^k_{p(\cdot),\rho_0}\left(f_0^{+},\delta\right), \qquad \delta > 0,$$

where the function

$$f_0^+(w) := \frac{1}{2\pi i}\int\limits_T \frac{f_0(\tau)}{\tau - w}d\tau, \qquad w \in D,$$

has nontangential boundary values a. e. on $\mathbf{T}$.

For $f \in E_{p(\cdot)}(G, \rho)$ we set

$$E_n(f)_{L^{p(\cdot)}(\Gamma,\rho)} := \inf \|f - P_k\|_{L^{p(\cdot)}(\Gamma,\rho)}, \quad n = 1, 2, \ldots,$$

where inf are taken over all algebraic polynomials of degree not exceeding n.

The $E^r_{p(\cdot)}(G, \rho)$ versions of the above presented results are formulated as follows.

Theorem 6. *Let G be a simply connected domain with the boundary, satisfying condition* (1). *Let also $E^r_{p(\cdot)}(G, \rho)$ be the space with the pair $(p_0, \rho_0) \in \Re$. If $f \in E^r_{p(\cdot)}(G, \rho)$, then*

$$E_n(f)_{L^{p(\cdot)}(\Gamma,\rho)} \le c_r n^{-r} \Omega^k_{p(\cdot),\Gamma,\rho}\Big(f^{(r)}, \frac{1}{n}\Big), \quad k = 1, 2, \ldots$$

with a constant $c_r > 0$.

Theorem 7. *Let G be a simply connected domain with the boundary, satisfying condition* (1). *Let also $E^r_{p(\cdot)}(G, \rho)$ $(r = 0, 1, 2, \ldots)$ be the spaces with the pair $(p, \rho) \in \Re$. If $f \in E^r_{p(\cdot)}(G, \rho)$, then*

$$\Omega^k_{p(\cdot),\Gamma,\rho}\left(f, \frac{1}{n}\right) \le c\left\{\frac{1}{n^{2k}}\sum_{m=1}^{n} m^{2k-r-1} E_m\left(f^{(r)}\right)_{L^{p(\cdot)}(\Gamma,\rho)} + \right.$$
$$\left. + \sum_{m=n+1}^{\infty} m^{r-1} E_m\left(f^{(r)}\right)_{L^{p(\cdot)}(\Gamma,\rho)}\right\}.$$

Theorem 8. *Let G be a simply connected domain with the boundary, satisfying condition* (1). *Let also $E^r_{p(\cdot)}(G, \rho)$ $(r = 1, 2, \ldots)$be the space with the pair $(p, \rho) \in \Re$.If $f \in E^r_{p(\cdot)}(G, \rho)$ then*

$$E_n\left(f^{(r)}\right)_{L^{p(\cdot)}(\Gamma,\rho)} \le$$
$$\le c_r\Big\{n^r E_n(f)_{L^{p(\cdot)}(\Gamma,\rho)} + \sum_{k=n+1}^{\infty} k^{r-1} E_k(f)_{L^{p(\cdot)}(\Gamma,\rho)}\Big\},$$

with a constant $c_r > 0$.

Theorem 9. *Let G be a simply connected domain with the boundary, satisfying condition* (1). *Let also $E_{p(\cdot)}(G,\rho)$ $(r=1,2,\dots)$ be the space with the pair $(p,\rho)\in\Re$. If for $f\in E_{p(\cdot)}(G,\rho)$ holds*

$$\sum_{m=1}^{\infty} m^{r-1} E_m(f)_{L^{p(\cdot)}(\Gamma,\rho)} < \infty$$

for some $r=1,2,\dots$, then $f\in E^r_{p(\cdot)}(G,\rho)$ and

$$\Omega^k_{p(\cdot),\Gamma,\rho}\Big(f^{(r)},\frac{1}{n}\Big) \le$$
$$\le c\bigg\{\frac{1}{n^{2k}}\sum_{m=0}^{n}(m+1)^{2k+r-1}E_m(f)_{L^{p(\cdot)}(\Gamma,\rho)} +$$
$$+\sum_{m=n+1}^{\infty} m^{r-1}E_m(f)_{L^{p(\cdot)}(\Gamma,\rho)}\bigg\}$$

with a constant $c>0$.

From Theorem 7 in case of $r=0$ we obtain the following result.

Corollary 3. *Let G be a simply connected domain with the boundary, satisfying condition* (1). *Let also $E_{p(\cdot)}(G,\rho)$ be the space with the pair $(p,\rho)\in\Re$. If $f\in E_{p(\cdot)}(G,\rho)$ satisfies the inequality*

$$E_m(f)_{L^{p(\cdot)}(\Gamma,\rho)} = O\left(m^{-\alpha}\right), \quad m=1,2,\dots,$$

for some $\alpha>0$, then for any natural number k and $\delta>0$,

$$\Omega^k_{p(\cdot),\Gamma,\rho}(f,\delta) = \begin{cases} O(\delta^{\alpha}), & k>\alpha/2,\\ O(\delta^{\alpha}\log(1/\delta)), & k=\alpha/2,\\ O\left(\delta^{2k}\right) & k<\alpha/2.\end{cases}$$

If we define the generalized Lipschitz class $\mathrm{Lip}_{\Gamma}\,\alpha(p(\cdot),\rho)$ for some $\alpha>0$ and $k:=[\alpha/2]+1$ as

$$\mathrm{Lip}\ _{\Gamma}\alpha(p(\cdot),\rho) := \Big\{f\in E_{p(\cdot)}(G,\rho): \quad \Omega^k_{p(\cdot),\Gamma,\rho}(f,\delta)\le c\delta^{\alpha}, \ \ \delta>0\Big\},$$

then, taking Corollary 3 into account, we have the following result.

Corollary 4. *Under the conditions of Corollary 3, if $f\in E_{p(\cdot)}(G,\rho)$ satisfies the inequality*

$$E_m(f)_{E_{p(\cdot)}(G,\rho)} = O\left(m^{-\alpha}\right), \quad m=1,2,\dots,$$

for some $\alpha>0$, then $f\in \mathrm{Lip}_{\Gamma}\,\alpha(p(\cdot),\rho)$.

Now combining this corollary with Theorem 6 we obtain the following constructive description of the classes $\mathrm{Lip}\,_{\Gamma}\alpha(p(\cdot),\rho)$.

Theorem 10. *Let G be a simply connected domain with the boundary, satisfying condition* (1). *Let also $E_{p(\cdot)}(G,\rho)$ be the spaces with the pair $(p,\rho) \in \Re$. If $\alpha > 0$, then the following statements are equivalent:*

(i) $f \in \operatorname{Lip}_{\Gamma}\alpha\left(p\left(\cdot\right),\rho\right)$,

(ii) $E_m(f)_{E_{p(\cdot)}(G,\rho)} = O\left(m^{-\alpha}\right), \quad m = 1,2,\dots$.

The statement of Theorem 11 for the nonweighted Smirnov spaces $E_p(G)$, and constant $p > 1$, in terms of the usual modulus of continuity in the spaces $L^p(\mathbf{T})$ of $f \circ \psi$, was obtained by Alper in [1] .

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