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TWO-WEIGHT ESTIMATES FOR FOURIER OPERATORS AND BERNSTEIN INEQUALITY

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1. Introduction

Let $\mathbb{T}$ be the interval $[-\pi,\pi]$. A $2\pi-$periodic nonnegative integrable function $w:\mathbb{T}\to\mathbb{R}$ is called a weight function. We denote by $L_w^p(\mathbb{T})$, $1\le p<\infty$ the Banach function space of all measurable $2\pi-$periodic functions f, for which

$$\|f\|_{p,w}=\left(\int_{\mathbb{T}}|f(x)|^p w(x)\,dx\right)^{1/p}<\infty.$$

The boundedness problem of Cesaro and Abel-Poisson means of functions $f\in L_w^p(\mathbb{T})$ $(1<p<\infty)$ was studied in [5] and [2]. In the paper [5] it was been done the complete characterization of the weights w, for which Cesaro and Abel-Poisson means are bounded as operators from $L_w^p(\mathbb{T})$ to $L_w^p(\mathbb{T})$. Later on B. Muckenhoupt showed that the condition referred in [5] is equivalent to the condition $A_p(\mathbb{T})$, that is

$$\sup_I\frac{1}{|I|}\int_I w(x)\,dx\left(\frac{1}{|I|}\int_I w^{1-p'}(x)\,dx\right)^{p-1}<\infty, \tag{1}$$

where $p'=p/(p-1)$ and the supremum is taken over all intervals whose lengths are not greater than 2π (see [2]).

In two-weighted setting, B. Muckenhoupt has shown that (see [3]) the necessary and sufficient condition for the boundedness of the Abel-Poisson means as an operator from $L_w^p(\mathbb{T})$ to $L_v^p(\mathbb{T})$ is

$$\sup_I\frac{1}{|I|}\int_I v(x)\,dx\left(\frac{1}{|I|}\int_I w^{1-p'}(x)\,dx\right)^{p-1}<\infty. \tag{2}$$

The set of all pairs (v,w) of weights with the condition (2) is denoted by $\mathcal{A}_p(\mathbb{T})$. It was shown in [1] that the condition (2) is also necessary and sufficient for the boundedness of the Cesaro means σ_n^α from $L_w^p(\mathbb{T})$ to $L_v^p(\mathbb{T})$ where $\alpha>0$.

Let $w\in A_p(\mathbb{T})$ and $f\in L_w^p(\mathbb{T})$. It is well known that for the Steklov mean

$$f_h(x)=\frac{1}{h}\int_{x-h}^{x+h}f(t)\,dt,\quad h>0$$

the inequality

$$\|f_h\|_{p,w}\le c\ \|f\|_{p,w}$$

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holds, where the constant c is independent of h and f. Starting from this we define the modulus of continuity for the function $f \in L_w^p(\mathbb{T})$ as

$$\Omega(\delta, f)_{p,w} = \sup_{h \leq \delta} \|f - f_h\|_{p,w}, \quad \delta \geq 0.$$

In the present paper, we investigated the estimation problem of norms of the Cesaro and Abel-Poisson means from $L_w^p(\mathbb{T})$ to $L_v^q(\mathbb{T})$ where $1 < p \leq q < \infty$. These results were generalized to the two-dimensional case and applied to estimate the rate of convergence of Cesaro means and to obtain generalizations of the Bernstein inequality for trigonometric polynomials of one and two variables.

2. One-Dimensional Case

Definition 2.1. Let $1 < p \leq q < \infty$. A pair of weight functions (v, w) is said to be of class $\mathcal{A}_{p,q}(\mathbb{T})$ if the $A_{p,q}$-condition

$$\sup_I \left(\frac{1}{|I|} \int_I v(x)\, dx \right)^{1/q} \left(\frac{1}{|I|} \int_I w^{1-p'}(x)\, dx \right)^{1/p'} < \infty.$$

holds, where the supremum is taken over all intervals with $|I| \leq 2\pi$.

Let $\sigma_n^\alpha(\cdot, f)$ and $U_r(\cdot, f)$ denote the Cesaro and Abel-Poisson means of the function $f \in L_w^p(\mathbb{T})$, respectively. We obtained the following results.

Theorem 2.2. *Let $1 < p \leq q < \infty$. The inequality*

$$\|\sigma_n^\alpha(\cdot, f)\|_{q,v} \leq c\; n^{\frac{1}{p} - \frac{1}{q}} \|f\|_{p,w}, \quad \alpha > 0 \tag{3}$$

holds for arbitrary $f \in L_w^p(\mathbb{T})$, where the constant c does not depend on n and f, if and only if $(v, w) \in \mathcal{A}_{p,q}(\mathbb{T})$.

In the case $p = q$ the inequality (3) yields the convergence

$$\|\sigma_n^\alpha(\cdot, f) - f\|_{p,v} \to 0, \quad n \to \infty.$$

Moreover, if also $w \equiv v$ we can estimate the rate of convergence:

Theorem 2.3. *Let $1 < p < \infty$, $w \in A_p(\mathbb{T})$ and $f \in L_w^p(\mathbb{T})$. Then there exist a constant c, which does not depend on n and f, such that the estimate*

$$\|\sigma_n^\alpha(\cdot, f) - f\|_{p,w} \leq c\; n\; \Omega\left(\frac{1}{n}, f\right)_{p,w} \tag{4}$$

holds.

Remark. The similar estimate is true also in reflexive Orlicz spaces with weight.

Theorem 2.4. *Let $1 < p \leq q < \infty$. The necessary and sufficient condition for the validity of the inequality*

$$\|U_r(\cdot, f)\|_{q,v} \leq c\; (1 - r)^{\frac{1}{q} - \frac{1}{p}} \|f\|_{p,w} \tag{5}$$

for arbitrary $f \in L_w^p(\mathbb{T})$, where the constant c does not depend on r and f, is $(v, w) \in \mathcal{A}_{p,q}(\mathbb{T})$.

For $p = q$, Theorem 2.2 was obtained in [1] and Theorem 2.4 was proved by in [3].

Let us note that the analogue of the estimate (4) is true for $U_r(\cdot, f)$ where $p = q$ and $w \equiv v$.

3. Two-Dimensional Case

Let $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$ and $\mathbb{J}$ denote the set of all rectangles with the sides parallel to the coordinate axes.

Definition 3.1. The pair (v, w) is said to belongs to the class $\mathcal{A}_{p,q}\left(\mathbb{T}^2, \mathbb{J}\right)$ if the condition

$$\sup_{J\in\mathbb{J}}\left(\frac{1}{|J|}\int_J v\left(x,y\right)dxdy\right)^{1/q}\left(\frac{1}{|J|}\int_J w^{1-p'}\left(x,y\right)dxdy\right)^{1/p'} < \infty \tag{6}$$

holds.

Let $\sigma_{mn}^{(\alpha,\beta)}\left(\cdot,\cdot,f\right)$ and $U_{r\rho}\left(\cdot,\cdot,f\right)$ denote the Cesaro and Abel-Poisson means of the function $f \in L_w^p\left(\mathbb{T}^2\right)$, respectively. We have the two-dimensional analogues of Theorem 2.2 and Theorem 2.4.

Theorem 3.2. *Let $1 < p \le q < \infty$. The condition $(v, w) \in \mathcal{A}_{p,q}\left(\mathbb{T}^2, \mathbb{J}\right)$ is necessary and sufficient for validity of the inequality*

$$\left\|\sigma_{mn}^{(\alpha,\beta)}\left(\cdot,\cdot,f\right)\right\|_{q,v} \le c\ (mn)^{\frac{1}{p}-\frac{1}{q}}\ \|f\|_{p,w}, \quad \alpha > 0,\ \ \beta > 0 \tag{7}$$

for every $f \in L_w^p\left(\mathbb{T}^2\right)$, where the constant c is independent of m, n and f.

In the case $p = q$ Theorem 3.2 was proved in [1].

Theorem 3.3. *Let $1 < p \le q < \infty$. The inequality*

$$\left\|U_{r\rho}\left(\cdot,\cdot,f\right)\right\|_{q,v} \le c\,(1-r)^{\frac{1}{q}-\frac{1}{p}}\,(1-\rho)^{\frac{1}{q}-\frac{1}{p}}\ \|f\|_{p,w} \tag{8}$$

holds for arbitrary $f \in L_w^p\left(\mathbb{T}^2\right)$, where the constant c does not depend on r, ρ and f, if and only if $(v, w) \in \mathcal{A}_{p,q}\left(\mathbb{T}^2, \mathbb{J}\right)$.

4. Generalizations of Bernstein Inequality

By aim of the inequalities (3) and (7) we prove generalizations of two-weighted Bernstein inequalities obtained in [1] for trigonometric polynomials of one and two variables.

By using (3) we obtain the following inequality.

Theorem 4.1. *Let $1 < p \le q < \infty$ and $(v, w) \in \mathcal{A}_{p,q}(\mathbb{T})$. Then for every trigonometric polynomial T_n of degree not greater than n, the inequality*

$$\left\|T_n'\right\|_{q,v} \le c\ n^{1+\frac{1}{p}-\frac{1}{q}}\ \|T_n\|_{p,w} \tag{9}$$

holds, where the constant c is independent of n.

The following inequality follows from (7).

Theorem 4.2. *Let $1 < p \le q < \infty$, $(v, w) \in \mathcal{A}_{p,q}\left(\mathbb{T}^2, \mathbb{J}\right)$ and let $T_{mn}(x, y)$ be a trigonometric polynomial of degree $\le m$ with respect to x and of degree $\le n$ with respect to y. Then there exists a constant c which does not depend on m and n such that the inequality*

$$\left\|\frac{\partial^2 T_{mn}}{\partial x \partial y}\right\|_{q,v} \le c\,(mn)^{1+\frac{1}{p}-\frac{1}{q}}\ \|T_{mn}\|_{p,w} \tag{10}$$

holds.

Let $E_n(f)_{p,w}$ $(n = 0, 1, 2, \dots)$ denote the order of best approximation of the function $f \in L_w^p(\mathbb{T})$ by trigonometric polynomials of degree not exceeding n. As a result of Theorem 4.1 we can state the following theorem.

Theorem 4.3. *Let $1 < p \leq q < \infty$, $(v,w) \in \mathcal{A}_{p,q}(\mathbb{T})$ and $f \in L_w^p(\mathbb{T})$. If the series*

$$\sum_{k=1}^{\infty} k^{\frac{1}{p}-\frac{1}{q}} E_k(f)_{p,w}$$

is convergent, then f is absolutely continuous, $f' \in L_v^q(\mathbb{T})$ and the estimate

$$E_n(f')_{q,v} \leq c\left(n^{1+\frac{1}{p}-\frac{1}{q}} E_n(f)_{p,w} + \sum_{k=n+1}^{\infty} k^{\frac{1}{p}-\frac{1}{q}} E_k(f)_{p,w}\right), \quad n = 1, 2, \dots$$

holds.

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References

1. A. Guven, V. Kokilashvili, On the mean summability by Cesaro method of Fourier trigonometric series in two-weighted setting. *J. Inequal. Appl. Article ID* 41837, 2006, 1–15.
2. B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.* **165**(1972), 207–226.
3. B. Muckenhoupt, Two weight function norm inequalities for the Poisson integral. *Trans. Amer. Math. Soc.* **210**(1975), 225–231.
4. A. D. Nakhman, B. P. Osilenker, Estimates of weighted norms of some operators generated by multiple trigonometric Fourier series. (Russian) *Izv. Vyssh. Uchebn. Zaved. Mat.* **239**(1982), No. 4, 39–50.
5. M. Rosenblum, Summability of Fourier series in $L^p(d\mu)$. *Trans. Amer. Math. Soc.* **165**(1962), No. 1, 32–42.
6. A. F. Timan, Theory of approximation of functions of a real variable. *International Series of Monographs in Pure and Applied Mathematics, Vol.* 34. *A Pergamon Press Book. The Macmillan Co., New York* 1963.
7. Ts. Tsanava, On the Fourier operators in weighted Lebesgue spaces. *Proceedings of A. Razmadze Mathematical Institute* **138**(2005), 107–109.
8. A. Zygmund, Trigonometric series. 2nd ed. Vols. I, II. *Cambridge University Press, New York* 1959.

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