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A REFINED INVERSE INEQUALITY OF APPROXIMATION IN WEIGHTED VARIABLE EXPONENT LEBESGUE SPACES

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Let $\mathbb{T} = [-\pi, \pi)$ and $w : \mathbb{T} \to \mathbb{R}^1$ be an almost everywhere positive, integrable function.

By $P(\mathbb{T})$ we denote the class of measurable, $2\pi-$ periodic functions on $\mathbb{T}$ such that

$$1 < p_- \le p(x) \le p_+ < \infty,$$

where

$$p_- = \operatorname*{essinf}_{x\in\mathbb{T}} p(x) \quad \text{and} \quad p_+ = \operatorname*{esssup}_{x\in\mathbb{T}} p(x).$$

By $L_w^{p(\cdot)}(\mathbb{T})$ we denote the weighted Banach function space of measurable 2π-periodic functions $f : \mathbb{T} \to \mathbb{R}^1$ such that

$$\|f\|_{p(\cdot),w} = \inf \left\{ \lambda > 0 : \int\limits_{\mathbb{T}} \left| \frac{f(x)w(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\} < \infty.$$

Definition 1. A variable exponent $p \in \widetilde{P}(\mathbb{T})$ if $p \in P(\mathbb{T})$ and

$$|p(x_1) - p(x_2)| \le \frac{c}{|\ln |x_1 - x_2||} \quad \text{for all } \ x_1, x_2 \in \mathbb{T}.$$

Definition 2. By the symbol $A_{p(\cdot)}$ we denote the class of weights satisfying the condition

$$\sup_{I\in J} \frac{1}{|I|^{p_I}} \|w^{p(\cdot)}\|_1 \left\| \frac{1}{w^{p(\cdot)}} \right\|_{p'(\cdot)/p(\cdot)} < \infty,$$

where

$$p_I = \left(\frac{1}{|I|} \int\limits_I \frac{dx}{p(x)} \right)^{-1}$$

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and J is the class of all intervals in $\mathbb{T}$.

Note that in [2] it was proved that when $p \in \widetilde{P}$ and $w \in A_{p(\cdot)}$, then the Hardy–Littlewood maximal function is bounded in $L_w^{p(\cdot)}$.

By $E_n(f)_{p(\cdot),w}$ we denote the best approximation of $f \in L_w^{p(\cdot)}$ by trigonometric polynomials of degree $\leq n$, i.e.

$$E_n(f)_{p(\cdot),w} = \inf \|f - T_k\|_{p(\cdot),w},$$

where the infimum is taken with respect to all trigonometric polynomials of degee $\leq n$.

The generalized modulus of a function $f \in L_w^{p(\cdot)}$ is defined as

$$\Omega_l(f,\delta) = \sup_{0<h_i<\delta} \left\| \Pi_{i=j}^{l}(I - A_{h_j})f \right\|_{p(\cdot),w}, \quad \delta > 0,$$

where I is the identity operator and

$$(A_{h_j}(f))(x) = \frac{1}{2h_j} \int\limits_{x-h_j}^{x+h_j} f(u)du.$$

Note that if $w \in A_{p(\cdot)}$, then $\Omega_l(f,\delta)$ is well–defined [2].

Theorem 1. *Let $p \in \widetilde{P}(I)$ and let $w^{-p_0} \in A_{(p(\cdot)/p_0)'}$ for some p_0, $1 < p_0 < p_-$. Then there exists a positive constant c such that*

$$\Omega_l\left(f, \frac{1}{n+1}\right) \leq \frac{c}{(n+1)^l} \left(\sum_{\nu=0}^{n} (\nu+1)^{\gamma l - 1} E_\nu^\gamma(f)_{p(\cdot),w} \right)^{1/\gamma}, \tag{1}$$

where $\gamma = \min(2, p_-)$.

Theorem 2. *Under the conditions of Theorem 1, if*

$$\sum_{\nu=1}^{\infty} \nu^{\gamma r - 1} E_\nu^\gamma(f)_{p(\cdot),w} < \infty,$$

for some $r \in \mathbb{N}$ with $\gamma = \min(2, p_-)$, then $f(x)$ has the absolutely continuous derivative $f^{(r-1)}(x)$ and $f^{(r)} \in L_w^{p(\cdot)}$ and there exists a positive constant c such that

$$\Omega_l\left(f^{(r)}, \frac{1}{n+1}\right)_{p(\cdot),w} \leq \frac{c}{(n+1)^r} \left(\sum_{\nu=0}^{n} (\nu+1)^{\gamma r - 1} E_\nu^\gamma(f)_{p(\cdot),w} \right)^{1/\gamma} + $$
$$+ c\left(\sum_{\nu=1}^{\infty} \nu^{\gamma r - 1} E_\nu^\gamma(f)_{p(\cdot),w} \right)^{1/\gamma}, \tag{2}$$

where $\gamma = \min(2, p_-)$.

The proofs are based on our results on the weighted extrapolation theorem [4] and its corollaries, in particular on an analogue of the Littlewood–Paley theorem in weighted variable Lebesgue spaces

Theorem A [4]**.** *Let f be a 2π–periodic function and*

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} A_k(x), \quad A_k(x) = a_k \cos kx + b_k \sin kx.$$

Let $w^{-p_0} \in A_{(p(\cdot)/p_0)'}$ for some p_0, $1 < p_0 < p_-$. Then there exist constants c_1 and c_2 such that

$$c_1 \|f\|_{p(\cdot),w} \le \left\| \left(\sum_{\nu=0}^{\infty} \left| \sum_{k=2^{\nu-1}}^{2^\nu - 1} A_k(x) \right|^2 \right)^{1/2} \right\|_{p(\cdot),w} \le c_2 \|f\|_{p(\cdot),w}.$$

In the previous theorem we assume that $A_{2^{-1}}(x) = 0$.

The following two simple lemmas are needed in the proof.

Lemma 1. *Let $1 < p_- \le 2$. Then for an arbitrary system of functions $\{\varphi_j(x)\}_{j=1}^m$, $\varphi_j \in L_w^{p(\cdot)}$ we have*

$$\left\| \left(\sum_{j=1}^{m} \varphi_j^2 \right)^{1/2} \right\|_{p(\cdot),w} \le c \left(\sum_{j=1}^{m} \|\varphi_j\|_{p(\cdot),w}^{p_-} \right)^{1/p_-}$$

with a constant c independent of φ_j and m.

Lemma 2. *Let $p_- > 2$. Then for an arbitrary system of functions $\{\varphi_j(x)\}_{j=1}^m$, $\varphi_j \in L_w^{p(\cdot)}$ we have*

$$\left\| \left(\sum_{j=1}^{m} \varphi_j^2 \right)^{1/2} \right\|_{p(\cdot),w} \le c \left(\sum_{j=1}^{m} \|\varphi_j\|_{p(\cdot),w}^{2} \right)^{1/2}.$$

Estimates (1) and (2) are sharper versions of the estimates obtained in [3]; they completely recover results known for the constant exponent case (see [1], [5]).

The weight function

$$w(x) = \Pi_{k=1}^{n} |x - x_k|^{\beta_k}, \;\; x_k \in \mathbb{T}, \; x_j \ne x_k \;\text{ if } j \ne k,$$

where

$$-\frac{1}{p(x_k)} < \beta_k < \frac{1}{p'(x_k)}$$

satisfies the conditions of Theorems 1 and 2. For more general weights we refer to [4].

Sketch of the proof of Theorem 1. Let $2^m \le n+1 < 2^{m+1}$. By $S_n(f,x)$ we denote the partial sums of $f \in L_w^{p(\cdot)}$. We have

$$\Omega_l\left(f, \frac{1}{n+1}\right) \le \Omega_l\left(f - S_{2^m}, \frac{1}{n+1}\right)_{p(\cdot),w} + \Omega_l\left(S_{2^m}, \frac{1}{n+1}\right)_{p(\cdot,w} \le$$

$$\le cE_{2^m}(f)_{p(\cdot),w} + \Omega_l\left(S_{2^m}, \frac{1}{n+1}\right)_{p(\cdot),w}.$$

Thus

$$\Omega_l\left(S_{2^m}, \frac{1}{n+1}\right) \le \frac{c}{(n+1)^l}\Big\{\|S_1^{(l)} - S_0^{(l)}\|_{p(\cdot),w} +$$

$$+ \Big\|\sum_{i=0}^{m-1}(S_{2^{i+1}}^{(l)} - S_{2^i}^{(l)})\Big\|_{p(\cdot),w}\Big\}. \tag{3}$$

For the first term on the right side we have

$$\|S_1^{(l)} - S_0^{(l)}\|_{p(\cdot),w} \le c(|a_1| + |b_1|) \le cE_0(f)_{p(\cdot),w}. \tag{4}$$

Let now

$$B_{k,\mu} := a_k \cos\left(k + \mu\frac{\pi}{2}\right)x + b_k \sin\left(k + \mu\frac{\pi}{2}\right)x.$$

Applying Theorem A, we get

$$\Big\|\sum_{i=0}^{m-1}\{S_{2^{i=1}}^{(l)} - S_{2^i}^{(l)}\}\Big\|_{p(\cdot),w} \le \Big\|\Big(\sum_{i=0}^{m-1}\Big|\sum_{k=2^i+1}^{2^{i=1}} k^l B_{k,l}(x)\Big|^2\Big)^{1/2}\Big\|_{p(\cdot),w}.$$

Now by using Lemmas 1 and 2 we conclude that

$$\Big\|\sum_{i=0}^{m-1}\{S_{2^{i=1}}^{(l)} - S_{2^i}^{(l)}\}\Big\|_{p(\cdot),w} \le \Big\|\Big(\sum_{i=0}^{m-1}\Big|\sum_{k=2^i+1}^{2^{i=1}} k^l B_{k,l}(x)\Big|^\gamma\Big)^{1/\gamma}\Big\|_{p(\cdot),w}, \tag{5}$$

where $\gamma = \min(2, p_-)$.

By means of the Abel transformation and the estimate

$$\|f(x) - S_k(f,\cdot)\|_{p(\cdot),w} \le cE_n(f)_{p(\cdot),w}$$

we derive the inequality

$$\Big\|\sum_{k=2^i+1}^{2^{i+1}} k^l B_{k,l}(x)\Big\|_{p(\cdot),w} \le c2^{il}E_{2^j}(f)_{p(\cdot,w}. \tag{6}$$

Finally from (3) – (6) we obtain the desired estimate (1).

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