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BOUNDEDNESS CRITERION FOR THE CAUCHY SINGULAR INTEGRAL OPERATOR IN WEIGHTED GRAND LEBESGUE SPACES AND APPLICATION TO THE RIEMANN PROBLEM

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We present the necessary and sufficient condition for the Cauchy singular integral and Hardy-Littlewood maximal function defined on the Carleson curves to be bounded in weighted grand Lebesgue spaces. Then we solve the Riemann boundary value problem for analytic functions in the class of the Cauchy-type integrals with the density from the grand Lebesgue $L^{p)}(\Gamma)$ spaces. We consider the case when a coefficient in boundary condition is everywhere nonvanishing continuous function and the right side function belongs to the same $L^{p)}$ space. The solvability conditions are established and the explicit formulas for solutions are given.

Let $\Gamma = \{t \in \mathbf{C} : t = t(s), \ 0 \le s \le l > \infty\}$ be a simple rectifiable curve with a arc-length measure ν. In the sequel we use the notation:

$$D(t,r) := \Gamma \cap B(t,r), \ \ r > 0$$

where $B(t,r) = \{z \in \mathbf{C} : |z - t| < r\}$.

We recall that a rectifiable curve Γ is called Carleson curve (regular curve) if there exists a constant $c_0 > 0$ not depending on t and r such that

$$\nu D(t,r) \le c_0 r.$$

The weighted grand Lebesgue space $L_w^{p)}(\Gamma)$ $1 < p < \infty$ is a Banach function space defined by the norm

$$\|f\|_{L_w^{p)}(\Gamma)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{\nu\Gamma} \int_\Gamma |f(t)|^{p-\varepsilon} w(t) d\nu \right)^{\frac{1}{p-\varepsilon}} \tag{1}$$

where w is an almost everywhere positive integrable function on Γ (i.e. weight).

It is worth mentioning that the following continuous embeddings hold:

$$L_w^p(\Gamma) \subset L_w^{p)}(\Gamma) \subset L_w^{p-\varepsilon}(\Gamma).$$

The grand Lebesgue space $L^{p)}$ was introduced by T. Iwaniec and C. Sbordone [6].

Our goal is to give a complete characterization of that weight functions w which govern one-weighted norm inequalities in grand Lebesque spaces for the following two operators: the Cauchy singular integral

$$(S_\Gamma f)(t) = \frac{1}{\pi i} \int\limits_\Gamma \frac{f(\tau)}{\tau - t} d\tau$$

and the Hardy-Littlewood maximal function

$$(M_\Gamma f)(t) = \sup_{r>0} \frac{1}{r} \int\limits_{D(t,r)} |f(\tau)| d\nu$$

defined on the Carleson curves.

This problem in classical Lebesgue spaces for the Hardy-Littlewood function and Hilbert transform defined on the real line was solved in papers [12] and [5] respectively. Recently, for the same integral transforms defined on a finite interval, the solution of one-weighted problem in the grand Lebesgue spaces was done in [3] and [10]. G. David's well-known theorem states that for the boundedness of S_Γ in $L^p(\Gamma)$ it is necessary and sufficient that Γ should be the Carleson curve [2].

Theorem 1. *Let $1 < p < \infty$ and let Γ be the Carleson curve of finite length. Then the following conditions are equivalent:*

$$\text{(i)} S_\Gamma \ \text{ is bounded in } \ L_w^{p)}(\Gamma); \tag{2}$$

$$\text{(ii)} M_\Gamma \ \text{ is bounded in } \ L_w^{p)}(\Gamma); \tag{3}$$

$$\text{(iii)} \sup \frac{1}{r} \int\limits_{D(z,r)} w(\tau) d\nu \left(\frac{1}{r} \int\limits_{D(z,r)} w^{1-p'}(\tau) d\nu \right)^{p-1} < \infty \tag{4}$$

where the supremum is taken over all $z \in \Gamma$ and r, $0 < r < diam\Gamma$.

In the case of classical Lebesgue spaces for the equivalence of the boundedness of the operator S_Γ in $L_w^p(\Gamma)$ and (4) we refer to [1] and [7].

For the real line condition (4) coincides with the well-known B. Muckenhoupt's A_p condition.

In the sequel both of operators S_Γ and M_Γ we denote by T_Γ.

From Theorem 1 we deduce the following

Corollary. *Let $1 < p < \infty$. Operator T_Γ is bounded in $L^{p)}(\Gamma)$ if and only if condition (4) is satisfied i. e. Γ is the Carleson curve.*

Note that the following vector-valued analogy of Theorem 1 holds.

Let $f = (f_1, f_2, \ldots, f_n, \ldots)$ be a vector-valued function when f_k ($k = 1, 2, \ldots$) are measurable functions defined on Γ.

Theorem 2. *Let $1 < p, \theta < \infty$ and let Γ be the Carleson curve of finite length. Then the inequality*

$$\left\| \left(\sum_{j=1}^{\infty} |T_\Gamma f_j(t)|^\theta \right)^{1/\theta} \right\|_{L_w^{p)}(\Gamma)} \leq c \left\| \left(\sum_{j=1}^{\infty} |f_j(t)|^\theta \right)^{1/\theta} \right\|_{L_w^{p)}(\Gamma)}$$

holds with a constant c independent of f if and only if condition (4) is fulfilled.

Definition. Let D be a simply connected bounded domain bounded by a rectifiable curve Γ. By $E^{p)}(D)$ $1 \leq p < \infty$ we denote a set of all analytic functions Φ in D for which there exists a sequence of closed curves $\Gamma_r \subset D$ converging to Γ such that

$$\sup_n \|\Phi\|_{L^{p)}(\Gamma_n)} < \infty.$$

Theorem 3. *Let $1 < p < \infty$ and Γ be a Carleson curve. For arbitrary $\Phi \in E^{p)}(D)$ the representation by the Cauchy type integral*

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{\varphi(\tau)}{\tau - z} d\tau$$

holds with $\varphi \in L^{p)}(\Gamma)$.

Note that the class $E^{p)}$ is an analogy of the Smirnov class of analytic functions.

Let Γ be an oriented rectifiable simple closed curve in the complex plane. We denote by D^+ and D^- the bounded and unbounded component of $\mathbf{C}\backslash\Gamma$, respectively.

One of the goal of this talk is to investigate the Riemann problem: find an analytic function Φ on the complex plane cut along Γ whose boundary values satisfy the conjugate condition

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma, \tag{5}$$

when G and g are given functions on Γ, and Φ^+ and Φ^- are boundary values of Φ on Γ from inside and outside Γ, respectively. This problem is also known as the problem of linear conjugation.

Problem (5) comes from Riemann [14]. Important results on which the posterior solution of problem (4) was based, were obtained by Yu. Sokhotski, D. Hilbert, I. Plemely and T. Carleman. A complete solutions of the

Riemann problem in the frame of Hölder continuous functions was given in the papers of Gakhov [4] and N. Muskhelishvili [13]. We refer also to the works [7–9], [11], [15] for investigation of the Riemann problem in classical L^p spaces.

Let

$$K^{p)}(\Gamma) = \left\{ \Phi(z) : \Phi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(\tau)d\tau}{\tau - z}, \ z \notin \Gamma \text{ with } \varphi \in L^{p)}(\Gamma) \right\}. \quad (6)$$

We proceed the solution of the Riemann problem in the following setting: let Γ be Carleson curve. Let G be a continuous function on Γ with the condition $G(t) \neq 0$, $t \in \Gamma$. Let $\mathit{æ} = \frac{1}{2\pi}[\arg G(t)]_\Gamma$. Find an analytic function $\Phi \in K^{p)}(\Gamma)$, $1 < p < \infty$, satisfying the condition (5), where $g \in L^{p)}(\Gamma)$.

Theorem. *The following statements hold:*

(i) *for* $\mathit{æ} \geq 0$, *problem* (5) *is unconditionally solvable in the class* $K^{p)}(\Gamma)$ *and all its solutions are given by*

$$\Phi(z) = \frac{X(z)}{2\pi i} \int_\Gamma \frac{g(\tau)}{X^+(\tau)(\tau - z)} d\tau + X(z)Q_{æ-1}(z) \quad (7)$$

with

$$X(z) = \begin{cases} \exp h(z), & z \in D^+ \\ (z - z_0)^{-æ} \exp h(z), & z \in D^-, \ z_0 \in D^+, \end{cases} \quad (8)$$

$$h(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\ln[(t - z_0)^{-æ}G(t)]}{t - z} \, dt, \quad (9)$$

where $Q_{æ-1}$ *is an arbitrary polynomial of degree* $æ - 1$ ($Q_{-1}(z) = 0$);

(ii) *for* $æ < 0$, *problem* (5) *is solvable in the class* $K^{p)}(\Gamma)$ *if and only if*

$$\int_\Gamma \frac{g(t)t^k}{X^+(t)} dt = 0, \quad k = 0, 1, \dots, |æ| - 1; \quad (10)$$

and under these conditions problem (5) *has the unique solution given by* (7) *with* $Q_{æ-1} = 0$.

In forthcoming talks we plan to discuss the Riemann problem in the case of oscillating coefficients G and approximation problems in grand Lebesgue spaces with weight.

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