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Mathematics

Boundedness Criterion for the Cauchy Singular Integral Operator and Maximal Functions in Weighted Grand Lebesgue Spaces

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ABSTRACT. We present the necessary and sufficient condition for the Cauchy singular integral and Hardy-Littlewood maximal function defined on the Carleson curves to be bounded in weighted Grand Lebesgue Spaces. *© 2009 Bull. Georg. Natl. Acad. Sci.*

***Key words**: Grand Lebesgue space, Carleson curve, Cauchy singular integral, maximal function, weights.*

1. Introduction

Let $\Gamma = \{t \in \mathbb{C}: \quad t = t(s), \quad 0 \le s \le l < \infty\}$ be a simple rectifiable curve with a arc-length measure ν. In the sequel we use the notation:

$$D(t,r) := \Gamma \cap B(t,r), \quad r > 0,$$

where $B(t,r) = \{z \in \mathbb{C}: \quad |z-t| < r\}$, $t \in \Gamma$.

We recall that a rectifiable curve Γ is called Carleson curve (regular curve) if there exists a constant $c_0 > 0$ not depending on t and r such that

$$\nu D(t,r) \le c_0 r .$$

The weighted grand Lebesgue space $L_w^{p)}(\Gamma)$ $(1 < p < \infty)$ is a Banach function space defined by the norm

$$\|f\|_{L_w^{p)}(\Gamma)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{\nu\Gamma} \int_\Gamma |f(t)|^{p-\varepsilon} w(t)\, d\nu \right)^{\frac{1}{p-\varepsilon}}, \tag{1}$$

where w is an almost everywhere positive integrable function on Γ (i. e. weight).

It is worth mentioning that the following continuous embeddings hold:

$$L_w^p(\Gamma) \subset L_w^{p)}(\Gamma) \subset L_w^{p-\varepsilon}(\Gamma).$$

The grand Lebesgue space $L^{p)}$ was introduced by T. Iwaniec and C. Sbordone [5].

Our goal is to give a complete characterization of weight functions w governing the $L_w^{p)}(\Gamma)$ - boundedness of the following two operators: the Cauchy singular integral

$$\left(S_\Gamma f\right)(t)=\frac{1}{\pi i}\int_\Gamma \frac{f(\tau)\,d\tau}{\tau-t}$$

and the Hardy-Littlewood maximal function

$$\left(M_\Gamma f\right)(t)=\sup_{r>0}\frac{1}{r}\int_{D(t,r)}\left|f(\tau)\right|d\nu$$

defined on the Carleson curves.

This problem in classical Lebesgue spaces for the Hardy-Littlewood functions and Hilbert transforms defined on the real line was solved in papers [8] and [4] respectively. Recently, for the same integral transforms defined on a finite interval, the solution of one-weighted problem in the grand Lebesgue spaces was done in [3] and [7]. G. David's well-known theorem states that for the boundedness of S_Γ in $L^p(\Gamma)$ it is necessary and sufficient that Γ be the Carleson curve.

2. Main Results.

Theorem 1. *Let* $1<p<\infty$ *and let* Γ *be the Carleson curve of finite length.*
Then the following conditions are equivalent:

i) S_Γ *is bounded in* $L_w^{p)}(\Gamma)$; (2)

ii) M_Γ *is bounded in* $L_w^{p)}(\Gamma)$; (3)

iii) $$\sup\frac{1}{r}\int_{D(z,r)}w(\tau)\,d\nu\left(\frac{1}{r}\int_{D(z,r)}w^{1-p'}(\tau)\,d\nu\right)^{p-1}<\infty, \tag{4}$$

where the supremum is taken over all $z\in\Gamma$ *and r,* $0<r<diam\Gamma$.

For the equivalence of the boundedness of the operator S_Γ in $L_w^p(\Gamma)$ and condition (4) in the case of classical Lebesgue spaces we refer to [1] and [6].

For the real line condition (4) coincides with the well-known B. Muckenhoupt's A_p condition.

In the sequel both of the operators S_Γ and M_Γ we denote by T_Γ.

From Theorem 1 we deduce the following

Corollary. *Let* $1<p<\infty$. *Operator* T_Γ *is bounded in* $L^{p)}(\Gamma)$ *if and only if condition* (4) *is satisfied, i. e.* Γ *is the Carleson curve.*

Note that the following vector-valued analogy of Theorem 1 holds.

Let $f=\left(f_1,f_2,\dots,f_n,\dots\right)$ be a vector-valued function when f_k $\left(k=1,2,\dots\right)$ are measurable functions defined on Γ.

Theorem 2. *Let* $1<p,\ \theta<\infty$ *and let* Γ *be the Carleson curve of finite length. Then the inequality*

$$\left\|\left(\sum_{j=1}^{\infty}\left|T_\Gamma f_j(t)\right|^\theta\right)^{1/\theta}\right\|_{L_w^{p)}(\Gamma)}\le c\left\|\left(\sum_{j=1}^{\infty}\left|f_j(t)\right|^\theta\right)^{1/\theta}\right\|_{L_w^{p)}(\Gamma)}$$

holds with a constant c independent of f if and only if condition (4) *is fulfilled.*

Definition. Let G be a simply connected bounded domain bounded by a rectifiable curve Γ. By $E^{p)}(G)$ $\left(1\le p<\infty\right)$ we denote a set of all analytic functions Φ in G for which there exists a sequence of closed curves $\Gamma_n\subset G$ converging to Γ such that

$$\sup_n\|\Phi\|_{L^{p)}(\Gamma_n)}<\infty.$$

Theorem 3. *Let* $1 < p < \infty$ *and* Γ *be a Carleson curve. For arbitrary* $\Phi \in E^{p)}(G)$ *the representation by the Cauchy type integral*

$$\Phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\tau)}{\tau - z} d\tau$$

holds with $\varphi \in L^{p)}(\Gamma)$.

Note that the class $E^{p)}$ is an analogy of the Smirnov class of analytic functions.

The proofs of these results will appear in "Journal of Function Spaces and Applications".

In the forthcoming papers the above-presented results will be applied to nonlinear harmonic analysis, approximation theory, boundary value problems for analytic and harmonic functions and the theory of singular integral equations.

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