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A NOTE ON EXTRAPOLATION AND MODULAR INEQUALITIES

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Abstract. In this note we present the Orlicz modular version of the well-known Littlewood-Paley's theorem. The result is based on a certain extrapolation theorem established in the given paper.

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1. Some Definitions and Auxiliary Statements

By the symbol Φ we denote a set of all functions $\varphi : R^1 \to R^1$ which are nonnegative, even, increasing on $[0,\infty)$ and such that $\varphi(0+) = 0$, $\lim\limits_{t\to\infty} \varphi(t) = \infty$.

Definition 1. A function $\varphi \in \Phi$ is said to be the Young function if φ is convex and

$$\lim_{t\to 0+} \frac{\varphi(t)}{t} = \lim_{t\to\infty} \frac{t}{\varphi(t)} = 0.$$

Definition 2. A nonnegative function $\varphi : [0,\infty) \to [0,\infty)$ is quasiconvex if there exists a Young function ω and a constant $c \geq 1$ such that

$$\omega(t) \leq \varphi(t) \leq c\,\omega(c\,t), \quad t \geq 0.$$

A quasiconvex function can be associated with its complementary function, that is the function $\widetilde{\varphi}$ defined by

$$\widetilde{\varphi}(t) = \sup_{s\geq 0} \big(s\,t - \varphi(s)\big).$$

The subadditivity of a supremum implies that $\widetilde{\varphi}$ is always a Young function. Moreover, $\widetilde{\widetilde{\varphi}} \leq \varphi$. The equality holds if φ itself is a Young function.

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Definition 3. A function $\varphi \in \Phi$ satisfies the Δ_2 condition ($\varphi \in \Delta_2$) if there exists $c > 0$ such that

$$\varphi(2t) \le c\,\varphi(t), \quad t > 0.$$

In the sequel, we will need the following propositions.

Proposition 1. *Let $h \in \Phi$. Then the following two conditions are equivalent:*

(i) *h^α is quasiconvex for some $\alpha \in (0,1]$;*

(ii) *$\widetilde{h} \in \Delta_2$ and h is quasiconvex.*

(*See* [1], [2], *Lemma* 6.1.6.)

Proposition 2. *Let $\varphi \in \Phi$. Then the following statements are equivalent:*

(i) *φ is quasiconvex on $[0,\infty)$;*

(ii) *the inequality*

$$\varphi\big(tx_1 + (1-t)x_2\big) \le c_1\big(t_1\varphi(c_1x_1) + (1-t)\,\varphi(c_1x_2)\big)$$

holds for all $x_1, x_2 \in [0,\infty)$ and all $t \in (0,1)$ with a constant c_1 independent of x_1, x_2 and t.

(*See* [1], *Lemma* 1.1.1.)

Proposition 3. *Let $\varphi \in \Phi$. The following conditions are equal:*

(i) *φ is quasiconvex;*

(ii) *there is a positive constant ε such that*

$$\widetilde{\varphi}\bigg(\varepsilon\,\frac{\varphi(t)}{t}\bigg) \le \varphi(t), \quad t > 0.$$

When φ is convex, the inequality holds with $\varepsilon = 1$.

(*See* [2], *Lemma* 1.1.1.)

Let (X, d, μ) be a quasimetric measure space satisfying the following so-called doubling condition: There exists a positive constant $c > 0$ such that

$$\mu\, B(x, 2r) \le c\,\mu\, B(x,r)$$

for an arbitrary ball with center at x, of radius r. Let

$$M\,f(x) = \sup_{r>0} \frac{1}{\mu\, B(x,r)} \int\limits_{B(x,r)} |f(y)|\, d\mu$$

be the Hardy-Littlewood maximal function defined for an arbitrary locally μ-integrable function.

Theorem A ([1], Theorem 1.2.1)**.** *Let $\varphi \in \Phi$. Then the following statements are equivalent:*

(i) *there exists a positive constant c_1 such that the inequality*

$$\int\limits_X \varphi\big(M\,f(x)\big)\, d\mu \le c_1 \int\limits_X \varphi\big(c_1\, f(x)\big)\, d\mu$$

holds;

(ii) *the function* φ^α *is quasiconvex for some* $\alpha \in (0,1)$.

(*See also* [2], *Theorem* 6.4.4 *for* $w \equiv 1$.)

Definition 4. A nonnegative locally integrable function w is said to be of the class A_1 if

$$M\,w(x) \le c\,w(x)$$

for almost all $x \in X$ in a μ-measure sense.

2. Main Results

By $\mathcal{F}$ we denote a family of ordered pairs (f,g) of μ-measurable nonnegative functions defined on the measure space (X,d,μ).

Theorem 1. *Let* $\varphi(t^{\frac{1}{p_0}})$ *be a Young function satisfying the* Δ_2 *condition for some* $p_0 > 1$.

Let there exist a constant $c > 0$ *such that for arbitrary pairs* $(f,g) \in \mathcal{F}$ *and arbitrary weight function* $w \in A_1$ *the inequality*

$$\int\limits_X f^{p_0}(x)\,w(x)\,d\mu \le C \int\limits_X g^{p_0}(x)\,w(x)\,d\mu \tag{1}$$

holds when the left–hand side is finite.

Then there exists a constant C_1 *such that*

$$\int\limits_X \varphi(f)(x)\,d\mu \le C \int\limits_X \varphi(g)(x)\,d\mu \tag{2}$$

for any $(f,g) \in \mathcal{F}$ *such that the left-hand side is finite.*

Let $\mathbb{T}$ *be the interval* $[-\pi,\pi]$ *and* $F \in L^1(\mathbb{T})$.

$$F(x) \sim \frac{1}{2}\,a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

be the Fourier series.

We introduce the notations:

$$A_k(x) := (a_k \cos kx + b_k \sin kx), \quad \delta_0 := \frac{1}{2}\,a_0$$

and

$$\delta_k := \sum_{j=2^{k-1}}^{2^k-1} A_j(x).$$

Theorem 2. *Let* $\varphi\big(t^{\frac{1}{p_0}}\big)$ *be a Young function for some* $p_0>1$ *satisfying the* Δ_2 *condition. Then there exist two positive constants* c_1 *and* c_2 *such that*

$$c_1\int\limits_{\mathbb{T}}\varphi(F)(x)\,dx\le\int\limits_{\mathbb{T}}\varphi\bigg(\Big(\sum_{k=0}^{\infty}\delta_k^2\Big)^{1/2}\bigg)dx\le c_2\int\limits_{\mathbb{T}}\varphi(F)(x)\,dx \tag{3}$$

for arbitrary $F\in L^1(\mathbb{T})\cap L^{\varphi}(\mathbb{T})$.

3. Proofs

Proof of Theorem 1. We will essentially use the idea of proving Theorem 3.1 from [3] which in its turn is based on the well-known extrapolation method of J. L. Rubio de Francia [4]. In fact, we present a modification of the above-mentioned proof.

Let

$$\psi:=\varphi\big(u^{\frac{1}{p_0}}\big).$$

Under our notation

$$\widetilde{\psi}(t)=\sup_{s>0}\big(ts-\psi(s)\big)$$

is the complementary function to ψ. We supposed that $\psi\in\Delta_2$. According to Propositions 1 and 2, we find that $\widetilde{\psi}^{\,\alpha}$ is quasiconvex for some α, $0<\alpha<1$ and

$$\widetilde{\psi}(\theta\,t)=\big[\widetilde{\psi}^{\,\alpha}(\theta\,t+(1-\theta)\cdot\theta)\big]^{1/\alpha}\le a_1^{1/\alpha}\,\theta^{1/\alpha}\,\widetilde{\psi}(a_1t) \tag{4}$$

for $0<\theta<1$ and some $a_1\ge1$.

On the other hand, by Theorem A we have

$$\int\limits_X\widetilde{\psi}\big(M\,f(x)\big)\,d\mu\le a_2\int\limits_X\widetilde{\psi}(a_2\,f)(x)\,d\mu,$$

since $\widetilde{\psi}^{\,\alpha}$ is quasiconvex.

Let $a_0=\max\big\{a_1,a_1^{1/\alpha},a_2\big\}$. It is clear that $a_0\ge1$. Therefore we have two estimates:

$$\int\limits_X\widetilde{\psi}\bigg(\frac{M\,f(x)}{a_0}\bigg)\,d\mu\le a_0\int\limits_X\widetilde{\psi}(f)(x)\,d\mu \tag{5}$$

and

$$\widetilde{\psi}(\theta\,t)\le a_0\,\theta^{1/\alpha}\,\widetilde{\psi}(a_0\,t). \tag{6}$$

Let θ, $0<\theta<1$ to be chosen later on.

Let

$$0\le h(x)=\frac{\theta\,\psi(f^{p_0})}{a_0\,f^{p_0}}.$$

Define now the function

$$R\,h(x) := \frac{2a_0-1}{2a_0}\sum_{k=0}^{\infty}\frac{1}{(2a_0)^k}\,\frac{M^k\,h(x)}{a_0}\,,$$

where M^k is the k-th iteration of the Hardy-Littlewood function M.

Arguing similarly to the arguments given in [3], we can easily see that $R(h)$ satisfies the following conditions:

$$\text{(i)}\qquad h(x) \le \frac{2a_0}{2a_0-1}\,R\,h(x); \tag{7}$$

$$\text{(ii)}\qquad \int_X \widetilde{\psi}(R\,h)(x) \le \frac{2a_0-1}{2a_0}\int_X \psi(h)(x); \tag{8}$$

$$\text{(iii)}\qquad M(Rh)(x) \le 2a_0^2\,R\,h(x). \tag{9}$$

The last property means that $R(h) \in A_1$ with a constant, independent of f.

By virtue of (7), we have

$$\int_X \varphi(f)(x)\,d\mu = \int_X \psi(f^{p_0})(x)\,d\mu = \frac{a_0}{\theta}\int_X f^{p_0}(x)\,h(x)\,dx \le$$

$$\le \frac{2a_0^2}{(2a_0-1)\theta}\int_X f^{p_0}(x)\,R\,h(x)\,d\mu. \tag{10}$$

Let us now prove that

$$\int_X f^{p_0}(x)\,R\,h(x)\,d\mu < \infty. \tag{11}$$

Using the Young inequality, we obtain

$$\int_X f^{p_0}(x)\,R\,h(x)\,d\mu \le \int_X \psi(f(x))\,d\mu + \int_X \widetilde{\psi}(R\,h(x))\,dx.$$

But according to our assumption, the first term on the right-hand side is finite. Taking into account (8) and (6), for the second summand we have

$$\int_X \overline{\psi}(R\,h(x))\,d\mu \le \frac{2a_0-1}{a_0}\int_X \overline{\psi}(h(x))d\mu = \frac{2a_0-1}{a_0}\int_X \overline{\psi}\left(\frac{\theta\,\psi(f^{p_0})(x)}{a_0 f^{p_0}(x)}\right)d\mu \le$$

$$\le \frac{2a_0-1}{a_0}\,a_0\,\theta^{1/\alpha}\int_X \overline{\psi}\left(\frac{\psi(f^{p_0})(x)}{a_0\,f^{p_0}(x)}\right)d\mu.$$

Applying Proposition 3, the latter estimate implies

$$\int_X \overline{\psi}(R\,h(x))\,d\mu \le (2a_0-1)\,\theta^{1/\alpha}\int_X \psi(f^{p_0})(x)\,d\mu.$$

Thus

$$\int_X \overline{\psi}(R\,h(x))\,d\mu \le (2a_0-1)\,\theta^{1/\alpha} \int_X \varphi(f)(x)\,d\mu \tag{12}$$

and hence the proof of inequality (11) is complete.

Taking into account the assumption of the theorem and the condition $R\,h \in A_1$, we obtain

$$\int_X \varphi(f^{p_0})(x)\,d\mu \le \frac{2a_0^2}{(2a_0-1)\,\theta}\,C \int_X g^{p_0}(x)\,R\,h(x)\,d\mu.$$

Using the Young inequality on the right-hand side of the latter inequality, we can conclude that

$$\int_X \psi(f^{p_0})\,d\mu \le \frac{2a_0^2}{(2a_0-1)\,\theta}\,C\bigg(\int_X \psi(g^{p_0})(x)\,d\mu + \int_X \widetilde{\psi}(R\,h)(x)\,d\mu\bigg).$$

Then by virtue of (12), we have

$$\int_X \psi(f^{p_0})(x)\,d\mu \le \frac{2a_0^2}{(2a_0-1)\,\theta}\,C \int_X \psi(g^{p_0})(x)\,d\mu + \\ + 2a_0^2(C+1)\,\theta^{\frac{1-\alpha}{\alpha}} \int_X \varphi(f)(x)\,d\mu.$$

Choose now $\theta = \big(4a_0^2(C+1)\big)^{-\frac{\alpha}{1+\alpha}}$. It is clear that $0<\theta<1$.

Therefore

$$\int_X \varphi(f)(x)\,d\mu \le \frac{2a_0^2}{(2a_0-1)\,\theta}\,C \int_X \varphi(g)(x)\,d\mu + \frac{1}{2}\int_X \varphi(f)(x)\,d\mu.$$

The last inequality provides us with a desired result. □

Proof of Theorem 2. We need the following result due to D. S. Kurtz (see [5], [6]).

Let $w \in A_p$, $1<p<\infty$, then

$$c_1\|f\|_{L^p_w} \le \bigg\|\Big(\sum_{k=0}^{\infty}\delta_k^2\Big)^{1/2}\bigg\|_{L^p_w} \le c_2\|f\|_{L^p_w}.$$

But arbitrarily $w \in A_1$ belongs to the A_p class, too.

Now we derive the right part of a chain of desired inequalities taken in Theorem 1,

$$f := \bigg(\sum_{k=0}^{\infty}\delta_k^2\bigg)^{1/2} \quad \text{and} \quad g := |F|.$$

As for the left part, we have to change the role of the functions. □

Remark. It is evident that in a similar way we can derive modular inequalities for various operators of harmonic analysis.

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