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# BOUNDEDNESS CRITERIA FOR SINGULAR INTEGRALS IN WEIGHTED GRAND LEBESGUE SPACES

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*Boundedness criteria for the Calderón singular integral, Riesz transform and Cauchy singular integral in generalized weighted grand Lebesgue spaces $L^{p),\theta}_w$, $1 < p < \infty$, are studied. It is shown that an operator $K$ of this type is bounded in $L^{p),\theta}_w$ if and only if the weight $w$ satisfies the Muckenhoupt $A_p$ condition. Bibliography*: 15 *titles.*

## Introduction

The goal of this paper is to establish criteria for the boundedness of various singular integral operators in the generalized weighted grand Lebesgue space $L^{p),\theta}_w$, $1 < p < \infty$. In the unweighted case (denoted by $L^{p),\theta}$), such spaces go back to [1], where the existence and uniqueness of a solution to the following inhomogeneous $n$-harmonic equation were studied:

$$\operatorname{div} A(x, \nabla u) = \mu.$$

It should be emphasized that the grand Lebesgue spaces $L^{p)} := L^{p),1}$ first appeared in the paper [2], where the integrability problem of the Jacobian was treated under a minimal hypothesis. In particular, as was shown in [2], if $f = (f_1, \cdots, f_n) : \Omega \to \mathbb{R}^n$, where $\Omega$ is an open subset in $\mathbb{R}^n$, $n \geqslant 2$, then the Jacobian determinant of $f$ belongs to the class $L^1_{\mathrm{loc}}(\Omega)$ provided that $g \in L^{n)}$, where

$$g(x) := |Df(x)| = \{\sup |Df(x)y| : y \in S^{n-1}\}.$$

The one–weighted problem in grand Lebesgue spaces was first studied in [3], where necessary and sufficient conditions for the validity of the one–weight inequality for the Hardy–Littlewood maximal operator in $L^{p)}_w(0, 1)$ were established. The same problem for the Hilbert transform was investigated in [4]. In particular, it turned out that the Hardy–Littlewood maximal operator (the Hilbert transform) is bounded in $L^{p)}_w(0, 1)$ if and only if the weight $w$ belongs to the Muckenhoupt

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class $A_p$. We see below that the same assertion is valid in the case of the generalized grand Lebesgue spaces $L_w^{p),\theta}$.

## 1 Preliminaries

Let $\Omega$ be a bounded subset of $\mathbb{R}^n$, and let $w$ be a weight (i.e., an almost everywhere positive integrable function) on $\Omega$. The generalized weighted grand Lebesgue space $L_w^{p),\theta}(\Omega)$ $(1 < p < \infty)$ is a Banach space equipped with the norm

$$\|f\|_{L_w^{p),\theta}(\Omega)} = \sup_{0<\varepsilon<p-1} \left( \frac{\varepsilon^\theta}{|\Omega|} \int_\Omega |f(y)|^{p-\varepsilon} w(y) dy \right)^{1/(p-\varepsilon)}, \qquad 1 < p < \infty.$$

In the case $w \equiv$ const, the space $L_w^{p),\theta}(\Omega)$, denoted by $L^{p),\theta}(\Omega)$, is rearrangement invariant and is nonreflexive (cf., for example, [5, 6]).

In the case $\theta = 1$, the space $L^{p),\theta}(\Omega)$ is a grand Lebesgue space, denoted by $L^{p)}(\Omega)$.

In general, the weighted space $L_w^{p),\theta}(\Omega)$ is not rearrangement invariant (cf. also [3]).

It is easy to check the following continuous embeddings (cf. also [1, 3]):

$$L_w^p(\Omega) \subset L_w^{p),\theta_1}(\Omega) \subset L_w^{p),\theta_2}(\Omega) \subset L_w^{p-\varepsilon}(\Omega),$$

where $0 < \varepsilon < p - 1$ and $\theta_1 < \theta_2$.

Throughout the paper, constants (often different constants in the same series of inequalities) are, in general, denoted by $c$. The symbol $p'$ means the conjugate of $p$, $1 < p < \infty$, i.e., $p' := \dfrac{p}{p-1}$. We denote by $I$ a bounded interval in $\mathbb{R}$ and by $\overline{I}$ the closure of $I$.

## 2 Calderón Commutators and Riesz Transforms

The purpose of this section is to characterize the boundedness of the Calderón singular operator

$$\mathcal{C}_a f(x) = \int_I \frac{a(x) - a(t)}{(x-t)^2} f(t) dt, \quad x \in I,$$

where $a : \overline{I} \to \mathbb{R}$ and $a \in \text{Lip } 1$ on $\overline{I}$, and the Riesz transforms

$$(R_j f)(x) = \int_{I^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y) dy, \quad x = (x_1, \cdots, x_n), \quad x_j \in I, \quad 1 \leqslant j \leqslant n,$$

in the weighted spaces $L_w^{p),\theta}$, where $1 < p < \infty$ and $I$ is a bounded interval in $\mathbb{R}$.

We use the notation

$$\rho(E) := \int_E \rho(t) dt,$$

where $\rho$ is a weight on $\Omega$ and $E \subseteq \Omega$.

**Definition 2.1.** Let $1 < p < \infty$. We say that a weight $w$ belongs to the *Muckenhoupt class* $A_p(I)$ $\big(w \in A_p(I)\big)$ if

$$A_p(w, I) := \sup_J \left(\frac{1}{|J|} w(J)\right)\left(\frac{1}{|J|} w^{1-p'}(J)\right)^{p-1} < \infty, \tag{2.1}$$

where the supremum is taken over all subintervals $J$ of $I$.

Let $1 < p < \infty$. In his celebrated paper [7], Muckenhoupt introduced the classes of weights $A_p$ and showed that the Hardy–Littlewood maximal operator is bounded in the weighted classical Lebesgue spaces $L^p_w$ if and only if $w \in A_p$. Later, it was proved [8] that the Hilbert transform in $L^p_w$ is bounded if and only if $w$ satisfies the Muckenhoupt condition $A_p$.

**Lemma 2.1.** *Let $1 < p < \infty$. Suppose that $w \in A_p(I)$. Then there are positive constants $\sigma$ and $L$ such that $w \in A_{p-\sigma}(I)$ and*

$$\|\mathcal{C}_a\|_{L^{p-\varepsilon}_w \to L^{p-\varepsilon}_w} \leqslant L$$

*for all $0 < \varepsilon < \sigma$.*

**Proof.** Since the class $A_p$ is open, for given $w \in A_p(I)$ there exists a small constant $\sigma$ such that $w \in A_{p-\sigma}(I)$. Using the boundedness of $\mathcal{C}_a$ in $L^s_w$ with $w \in A_s$, $1 < s < \infty$ (cf. [9]), and the Riesz–Thorin interpolation theorem, we get the boundedness inequality in $L^{p-\varepsilon}_w(I)$ for every $0 < \varepsilon < \sigma$ with a constant independent of $\varepsilon$. Indeed, there exist positive constants $M_i$ ($M_i > 1$), $i = 1, 2$, such that for all simple functions $\psi$

$$\|\mathcal{C}_a\psi\|_{L^{p-\sigma}_w(I)} \leqslant M_1\|\psi\|_{L^{p-\sigma}_w(I)}$$

and

$$\|\mathcal{C}_a\psi\|_{L^{p}_w(I)} \leqslant M_2\|\psi\|_{L^{p}_w(I)},$$

For arbitrary $0 < \varepsilon < \sigma$ there exists $0 < t_\varepsilon < 1$ such that

$$\frac{1}{p-\varepsilon} = \frac{1-t_\varepsilon}{p-\sigma} + \frac{t_\varepsilon}{p}.$$

By the Riesz–Thorin interpolation theorem (cf., for example, [10, p. 16]), the inequality

$$\|\mathcal{C}_a\psi\|_{L^{p-\varepsilon}_w(I)} \leqslant M_1^{1-t_\varepsilon} M_2^{t_\varepsilon}\|\psi\|_{L^{p-\varepsilon}_w(I)}$$

is fulfilled and $\mathcal{C}_a$ is uniquely extendable to $L^{p-\varepsilon}_w$ preserving the last inequality. Hence

$$\|\mathcal{C}_a f\|_{L^{p-\varepsilon}_w(I)} \leqslant L\|f\|_{L^{p-\varepsilon}_w(I)} \quad (0 < \varepsilon < \sigma),$$

where $L = M_1 M_2$. □

**Lemma 2.2.** *Suppose that $1 < p < \infty$, $\theta > 0$, and $I$ is a bounded interval. Then there is a positive constant $c$ such that for all $f \in L^p_w(I)$ and intervals $J \subset I$*

$$\|f\chi_J\|_{L^{p),\theta}_w(I)} \leqslant cw(J)^{-1/p}\|f\chi_J\|_{L^p_w(I)}\|\chi_J\|_{L^{p),\theta}_w(I)}.$$

**Proof.** Let $f \geqslant 0$. By the Hölder inequality,

$$\|f\chi_J\|_{L^{p),\theta}_w(I)} = \sup_{0<\varepsilon\leqslant p-1}\left(\varepsilon^\theta\int_J f^{p-\varepsilon}w\right)^{\frac{1}{p-\varepsilon}} = \sup_{0<\varepsilon\leqslant p-1}\left(\varepsilon^\theta\int_J f^{p-\varepsilon}w^{\frac{p-\varepsilon}{p}}w^{\frac{\varepsilon}{p}}\right)^{\frac{1}{p-\varepsilon}}$$

$$\leqslant \sup_{0<\varepsilon\leqslant p-1}\varepsilon^{\frac{\theta}{p-\varepsilon}}\left(\int_J\left(f^{p-\varepsilon}w^{\frac{p-\varepsilon}{p}}\right)^{\frac{p}{p-\varepsilon}}\right)^{\frac{1}{p}}\cdot\left(\int_J\left(w^{\frac{\varepsilon}{p}}\right)^{\frac{p}{\varepsilon}}\right)^{\frac{\varepsilon}{p(p-\varepsilon)}}$$

$$= \sup_{0<\varepsilon\leqslant p-1}\varepsilon^{\frac{\theta}{p-\varepsilon}}\left(\int_J f^pw\right)^{\frac{1}{p}}\left(\int_J w\right)^{\frac{\varepsilon}{p(p-\varepsilon)}}$$

$$= \sup_{0<\varepsilon\leqslant p-1}\varepsilon^{\frac{\theta}{p-\varepsilon}}\|f\chi_J\|_{L^p_w(I)}\cdot\big(w(J)\big)^{-\frac{1}{p}}(w(J))^{\frac{1}{p-\varepsilon}}$$

$$= \big(w(J)\big)^{-\frac{1}{p}}\|f\chi_J\|_{L^p_w(I)}\sup_{0<\varepsilon\leqslant p-1}\left(\varepsilon^\theta\int_I(\chi_J wx)^{p-\varepsilon}\right)^{\frac{1}{p-\varepsilon}}$$

$$= \big(w(J)\big)^{-\frac{1}{p}}\|f\chi_J\|_{L^p_w(I)}\|\chi_J\|_{L^{p),\theta}_w(I)}.$$

The lemma is proved. □

The following lemma will also be useful for us.

**Lemma 2.3.** *Suppose that $1 < p < \infty$, $\theta > 0$, and $I$ is a bounded interval. If an operator $\mathcal{C}_a$ is bounded in $L^{p),\theta}_w(I)$ and there exists a constant $m$ such that*

$$0 < m \leqslant |a'(t)|,$$

*then*

$$w^{1-p'}(I) < \infty. \tag{2.2}$$

**Proof.** It suffices to show that (2.2) holds for all intervals $J \subset I$ such that $|J| < 1/2$. Assume the contrary. Let

$$w^{1-p'}(J) = \infty \tag{2.3}$$

for some $J := (a, b)$. Without loss of generality, we can assume that $I = [0, 1]$ and $b < 1$. Note that (2.3) implies the existence of a function $g \in L^p_w(J)$, $g \geqslant 0$, such that

$$\int_J g(x)dx = \infty.$$

We set

$$f_J := g\chi_J.$$

Then $f_J \in L^p_w(I)$. On the other hand,

$$|\mathcal{C}_a f_J(x)| = \Big| \int\limits_J \frac{a(x)-a(t)}{(x-t)^2} g(t)dt \Big| = \Big| \frac{a(x)-a(\xi)}{x-\xi} \Big| \int\limits_J \frac{g(t)dt}{x-t} \geqslant m \int\limits_J g(t)dt = \infty,$$

where $\xi \in J$ and $x$ is an arbitrary point in $(b,1)$.

Here, we used the continuity of the function

$$\varphi_x(t) = \frac{a(x)-a(t)}{x-t}$$

on $[a,b]$ for arbitrary $x \in (b,1)$. By Lemma 2.2, $f_J \in L^{p),\theta}_w(I)$. But

$$\|\mathcal{C}_a f_J\|_{L^{p),\theta}_w(I)} = \infty,$$

which contradicts the boundedness of $\mathcal{C}_a$ in $L^{p),\theta}_w(I)$. □

We formulate and prove the main result of this section.

**Theorem 2.1.** *Suppose that $I$ is a finite interval, $1 < p < \infty$, and $\theta > 0$. Then $\mathcal{C}_a$ is bounded in $L^{p),\theta}_w(I)$ if $w \in A_p(I)$.*

*Conversely, if $\mathcal{C}_a$ is bounded in $L^{p),\theta}_w$ under the assumption that there exists a constant $m$ such that $0 < m \leqslant |a'(x)|$ for all $x \in I$, then $w \in A_p(I)$.*

**Proof.** For the sake of simplicity, we assume that $I := [0,1]$.

*Sufficiency.* Note that $\mathcal{C}_a f(x)$ exists almost everywhere on $[0,1]$ for $f \in L^{p),\theta}_w(I)$. Indeed, since $w \in A_p(I)$, there is a small positive number $\varepsilon$ such that $w \in A_{p-\varepsilon}(I)$. Further, by the definition of the space $L^{p),\theta}_w(I)$, we have $f \in L^{p-\varepsilon}_w(I)$. Consequently, the existence of $\mathcal{C}_a f(x)$ follows immediately.

By Lemma 2.1, we conclude that there is a constant $\sigma \in (0, p-1)$ such that

$$\|\mathcal{C}_a f\|_{L^{p-\varepsilon}_w(I)} \leqslant c\|f\|_{L^{p-\varepsilon}_w(I)}, \quad \varepsilon \in (0,\sigma],$$

where $c$ is a constant independent of $f$ and $\varepsilon$.

We fix $\varepsilon \in (\sigma, p-1)$. Then

$$\frac{p-\sigma}{p-\varepsilon} > 1.$$

Using the Hölder inequality with the exponent $(p-\sigma)/(p-\varepsilon)$ and observing that

$$\Big(\frac{p-\sigma}{p-\varepsilon}\Big)' = \frac{p-\sigma}{\varepsilon-\sigma},$$

we find

$$\|\mathcal{C}_a f\|_{L^{p-\varepsilon}_w(I)} \leqslant \Big( \int\limits_I |\mathcal{C}_a f(x)|^{p-\sigma} w(x)dx \Big)^{1/(p-\sigma)} w(I)^{(\varepsilon-\sigma)/[(p-\sigma)(p-\varepsilon)]}. \tag{2.4}$$

Further, since $\sigma < p-1$ and $\varepsilon \in (\sigma, p-1)$, we have

$$0 < \frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)} < \frac{p-1-\sigma}{p-\sigma}, \qquad\qquad (p-1)\sigma^{-1/(p-\sigma)} > 1.$$

Using (2.4) and the Hölder inequality, we find

$$\begin{aligned}
\|\mathcal{C}_a f\|_{L_w^{p),\theta}(I)} &= \max\Big\{ \sup_{0<\varepsilon\leqslant\sigma} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|\mathcal{C}_a f\|_{L_w^{p-\varepsilon}(I)}, \sup_{\sigma<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|\mathcal{C}_a f\|_{L_w^{p-\varepsilon}(I)} \Big\} \\
&\leqslant \max\Big\{ \sup_{0<\varepsilon\leqslant\sigma} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|\mathcal{C}_a f\|_{L_w^{p-\varepsilon}(I)}, \sup_{\sigma<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|\mathcal{C}_a f\|_{L_w^{p-\sigma}} w(I)^{\frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)}} \Big\} \\
&= \max\Big\{ \sup_{0<\varepsilon\leqslant\sigma} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|\mathcal{C}_a f\|_{L_w^{p-\varepsilon}(I)}, \sup_{\sigma<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \sigma^{-\frac{\theta}{p-\sigma}} \sigma^{\frac{\theta}{p-\sigma}} \|\mathcal{C}_a f\|_{L_w^{p-\sigma}} w(I)^{\frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)}} \Big\} \\
&\leqslant \max\Big\{ \sup_{0<\varepsilon\leqslant\sigma} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|\mathcal{C}_a f\|_{L_w^{p-\varepsilon}(I)}, \\
&\qquad \Big( \sup_{\sigma<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \sigma^{-\frac{\theta}{p-\sigma}} w(I)^{\frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)}} \Big) \Big( \sup_{0<\varepsilon\leqslant\sigma} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|\mathcal{C}_a f\|_{L_w^{p-\varepsilon}(I)} \Big) \Big\}
\end{aligned}$$

Using the notation

$$\begin{aligned}
S &:= \sup_{0<\varepsilon\leqslant\sigma} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|\mathcal{C}_a f\|_{L_w^{p-\varepsilon}(I)}, \\
T &:= \sup_{\sigma<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \sigma^{-\frac{\theta}{p-\sigma}} w(I)^{\frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)}},
\end{aligned}$$

we find

$$\begin{aligned}
\|\mathcal{C}_a f\|_{L_w^{p),\theta}(I)} &\leqslant \max\{1, T\} \cdot S \\
&\leqslant \max\Big\{ 1, \sup_{\sigma<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \sigma^{-\frac{\theta}{p-\sigma}} (1+w(I))^{\frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)}} \Big) \cdot S.
\end{aligned}$$

We set

$$\begin{aligned}
h(\varepsilon) &:= \varepsilon^{\frac{\theta}{p-\varepsilon}}, \\
g(\varepsilon) &:= (1+w(I))^{\frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)}}.
\end{aligned}$$

We see that $h$ and $g$ are increasing in $(\sigma, p-1)$.

Therefore, using the boundedness of the Calderón singular integral operator in the classical $L_w^p$ spaces [9] and Lemma 2.1, we conclude that

$$\begin{aligned}
\|\mathcal{C}_a f\|_{L_w^{p),\theta}(I)} &\leqslant c \max\Big\{ 1, (p-1)^\theta \sigma^{-\frac{\theta}{p-\sigma}} (1+w(I))^{\frac{p-1-\sigma}{p-\sigma}} \Big\} \sup_{0<\varepsilon\leqslant\sigma} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{L_w^{p-\varepsilon}(I)} \\
&\leqslant c(p-1)^\theta \sigma^{-\frac{\theta}{p-\sigma}} (1+w(I))^{\frac{p-1-\sigma}{p-\sigma}} \|f\|_{L_w^{p),\theta}(I)}.
\end{aligned}$$

*Necessity.* Since $w$ is integrable on $I$, se have

$$\|1\|_{L^{p)}_w(I)} < \infty.$$

It suffices to prove that

$$\sup_{\substack{J\subset I\\ |J|\leqslant 1/4}} \left(\frac{1}{|J|}\int_J w(x)dx\right)\left(\frac{1}{|J|}\int_J w^{1-p'}(t)dt\right)^{p-1} < \infty.$$

Let $J := (a,b)$ be a subinterval of $I$ such that $b-a \leqslant 1/4$. We set

$$J' := (b, 2b-a)$$

if $(b, 2b-a)$; otherwise,

$$J' := (2a-b, a).$$

First we claim that there is a positive constant $c$ independent of $J$ such that

$$\|\chi_J\|_{L^{p),\theta}_w(I)} \leqslant c\|\chi_{J'}\|_{L^{p),\theta}_w(I)}. \tag{2.5}$$

Indeed, without loss of generality we can assume that $J' = (b, 2b-a)$. Then $t - x \leqslant 2|J|$ whenever $t \in J'$ and $x \in J$. Let $f := \chi_{J'}$. Then

$$|\mathcal{C}_a f(x)| = \left|\int_{J'} \frac{a(x)-a(t)}{(x-t)^2}dt\right| = \left|\frac{a(x)-a(\xi)}{x-\xi}\right|\int_{J'}\frac{dt}{t-x} \geqslant \frac{1}{2}m$$

for arbitrary $x \in J$ and some $\xi \in [b, 2b-a]$. Thus,

$$\|\mathcal{C}_a f\|_{L^{p),\theta}_w(I)} \geqslant \|\mathcal{C}_a f\|_{L^{p),\theta}_w(J)} \geqslant m\left\|\int_{J'}\frac{dt}{t-x}\right\|_{L^{p),\theta}_w(J)} \geqslant \frac{1}{2}m\|\chi_J\|_{L^{p),\theta}_w(I)}.$$

On the other hand,

$$\|f\|_{L^{p),\theta}_w(I)} = \|\chi_{J'}\|_{L^{p),\theta}_w(I)}.$$

Consequently, the boundedness of $\mathcal{C}_a$ in $L^{p),\theta}_w(I)$ implies (2.5).

Consider the test function $f = w^{1-p'}\chi_J$. For $x \in J'$ we have

$$|\mathcal{C}_a f(x)| = \left|\int_J \frac{a(x)-a(t)}{(x-t)^2}w^{1-p'}(t)dt\right| \geqslant \frac{m}{2|J|}\int_J w^{1-p'}(t)dt$$

which implies the estimate

$$\|\mathcal{C}_a f\|_{L^{p),\theta}_w(I)} \geqslant c\left(\frac{1}{|J|}\int_J w^{1-p'}(t)dt\right)\|\chi_{J'}\|_{L^{p),\theta}_w(0,1)}.$$

Taking into account the last estimate, the boundedness of $\mathcal{C}_a$ in $L^{p)}_w(I)$, Lemma 2.2, and the inequality (2.5), we get

$$\frac{1}{|J|} w^{1-p'}(J) \|\chi_{J'}\|_{L^{p),\theta}_w(I)} \leqslant c \|\mathcal{C}_a f\|_{L^{p),\theta}_w(I)} \leqslant c \|f\|_{L^{p),\theta}_w(J)}$$

$$\leqslant c \big(w(J)\big)^{-1/p} \Big( \int\limits_J w^{1-p'}(t)dt \Big)^{1/p} \|\chi_{J'}\|_{L^{p),\theta}_w(I)}.$$

By Lemma 2.3 ,

$$|J|^{-1} \big(w(J)\big)^{1/p} \big(w^{1-p'}(J)\big)^{1/p'} \leqslant c,$$

where the positive constant $c$ is independent of $J$. □

**Theorem 2.2.** *Let $I$ be a bounded interval, and let $1 < p < \infty$. Then the Riesz transforms are bounded in $L^{p),\theta}_w(I^n)$ for all $i = 1, \dots, n$, if and only if $w \in A_p(I^n)$, i.e.,*

$$\sup_Q \Big( \frac{1}{|Q \cap I^n|} w(Q \cap I^n) \Big) \Big( \frac{1}{|Q \cap I^n|} w^{1-p'}(Q \cap I^n) dt \Big)^{p-1} < \infty,$$

*where the supremum is taken over all $n$-dimensional cubes $Q$ with centers at $I^n$.*

The proof is similar to that of Theorem 2.1.

Finally, we note that the one-weight theorem for the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{J \ni x} \frac{1}{|J|} \int\limits_J |f(y)| dy, \quad x \in I,$$

remains true for the generalized grand Lebesgue spaces $L^{p),\theta}_w$ under the condition that $w \in A_p$ (cf. [3] in the case $\theta = 1$). Namely, the following assertion holds.

**Theorem 2.3.** *Suppose that $I$ is a bounded interval, $1 < p < \infty$, and $\theta > 0$. Then the Hardy–Littlewood maximal operator $M$ is bounded in $L^{p),\theta}_w(I)$ if and only if $w \in A_p$.*

**Proof.** We follow [3].

*Necessity.* Consider an interval $J \subset I$. By the definition of the maximal operator,

$$\frac{1}{|J|} \int\limits_J |f(t)| dt \leqslant M(f \cdot \chi_J)(x), \quad x \in J.$$

By the boundedness of $M$ in $L^{p),\theta}_w(I)$,

$$\Big( \frac{1}{|J|} \int\limits_J |f(y)| dy \Big) \|\chi_J\|_{L^{p),\theta}_w(I)} = \Big\| \Big( \frac{1}{|J|} \int\limits_J |f(y)| dy \Big) \cdot \chi_J \Big\|_{L^{p),\theta}_w(I)}$$

$$\leqslant \|M(f \cdot \chi_J)\|_{L^{p),\theta}_w(I)} \leqslant c \|f \cdot \chi_J\|_{L^{p),\theta}_w(I)}$$

$$= c \sup_{0 < \varepsilon \leqslant p-1} \Big( \varepsilon^\theta \int\limits_J |f(y)|^{p-\varepsilon} w(y) dy \Big)^{\frac{1}{p-\varepsilon}}$$

$$= c \sup_{0<\varepsilon\leqslant p-1} \left( \varepsilon^{\theta} \int\limits_J |f(y)|^{p-\varepsilon} w^{\frac{p-\varepsilon}{p}}(y) w^{\frac{\varepsilon}{p}}(y) dy \right)^{\frac{1}{p-\varepsilon}}$$

$$\leqslant c \sup_{0<\varepsilon\leqslant p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left( \int\limits_J \left( |f(y)|^{p-\varepsilon} w^{\frac{p-\varepsilon}{p}}(y) \right)^{\frac{p}{p-\varepsilon}} dy \right)^{\frac{1}{p}} \left( \int\limits_J \left( w^{\frac{\varepsilon}{p}}(y) \right)^{\frac{p}{\varepsilon}} dy \right)^{\frac{\varepsilon}{p(p-\varepsilon)}}$$

$$= c \left( \int\limits_J |f(y)|^p w(y) dy \right)^{1/p} \sup_{0<\varepsilon\leqslant p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} w(J)^{\frac{\varepsilon}{p(p-\varepsilon)}}$$

$$= c w^{-1/p}(J) \left( \int\limits_J |f(y)|^p w(y) dy \right)^{\frac{1}{p}} \sup_{0<\varepsilon\leqslant p-1} \left( \varepsilon^{\frac{\theta}{p-\varepsilon}} w(J)^{\frac{1}{p-\varepsilon}} \right)$$

$$= c w^{-1/p}(J) \left( \int\limits_J |f(y)|^p w(y) dy \right)^{\frac{1}{p}} \|\chi_J\|_{L^{p),\theta}_w(I)}.$$

Consequently,

$$\frac{1}{|J|} \int\limits_J |f(y)| dy \leqslant c w^{-1/p}(J) \left( \int\limits_J |f(y)|^p w(y) \right)^{1/p}. \tag{2.6}$$

Substituting

$$f = w^{-\frac{1}{p-1}} \chi_J$$

into (2.6), we have $w \in A_p(I)$.

*Sufficiency.* The proof is similar to that of Theorem 2.1. In this case, we need to use Muckenhoupt's [7] result on the one–weight inequality for $M$ (cf. also [3] in the case $\theta = 1$). □

## 3 Cauchy Singular Integral Operator and Maximal Functions on Curves

In this section, we present a necessary and sufficient condition for the Cauchy singular integral and Hardy-Littlewood maximal function defined on Carleson curves to be bounded in weighted generalized grand Lebesgue spaces.

Let

$$\Gamma = \{t \in \mathbf{C} : \ t = t(s), \ \ 0 \leqslant s \leqslant l < \infty\}$$

be a simple rectifiable curve of finite length with an arc-length measure $\nu$. We set

$$D(t,r) := \Gamma \cap B(t,r), \quad r > 0,$$

where

$$B(t,r) = \{z \in \mathbf{C} : \ |z - t| < r\}.$$

We recall that a rectifiable curve $\Gamma$ is called a *Carleson curve* (a *regular curve*) if there exists a constant $c_0 > 0$ such that

$$\nu D(t,r) \leqslant c_0 r$$

for arbitrary $t \in \Gamma$ and $r > 0$.

The weighted grand Lebesgue space $L_w^{p),\theta}(\Gamma)$, $1 < p < \infty$, $\theta > 0$, is a Banach function space equipped with the norm

$$\|f\|_{L_w^{p),\theta}(\Gamma)} = \sup_{0<\varepsilon<p-1} \left( \frac{\varepsilon^\theta}{\nu\Gamma} \int_\Gamma |f(t)|^{p-\varepsilon} w(t) d\nu \right)^{\frac{1}{p-\varepsilon}},$$

where $w$ is an almost everywhere positive integrable function on $\Gamma$ (i.e., $w$ is a weight).

Our goal is to characterize weight functions $w$ governing the one-weighted norm inequality for the following two operators: the Cauchy singular integral

$$(S_\Gamma f)(t) = \frac{1}{\pi i} \int_\Gamma \frac{f(\tau)}{\tau - t} d\tau$$

and the Hardy-Littlewood maximal function

$$(M_\Gamma f)(t) = \sup_{r>0} \frac{1}{r} \int_{D(t,r)} |f(\tau)| d\nu$$

defined on Carleson curves.

A well-known theorem due to David [11] asserts that $S_\Gamma$ is bounded in $L^p(\Gamma)$ if and only if $\Gamma$ is a Carleson curve.

The main result of this section is contained in the following assertion.

**Theorem 3.1.** *Let $\Gamma$ be a Carleson curve, and let $w \in A_p(\Gamma)$, $1 < p < \infty$, i.e.,*

$$\sup \frac{1}{r} \int_{D(z,r)} w(\tau) d\nu \left( \frac{1}{r} \int_{D(z,r)} w^{1-p'}(\tau) d\nu \right)^{p-1} < \infty, \tag{3.1}$$

*where the supremum is taken over all $z \in \Gamma$ and $r$ such that $0 < r < \operatorname{diam}\Gamma$. Let $\theta > 0$. Then*

(i) $\qquad S_\Gamma$ *is bounded in* $L_w^{p),\theta}(\Gamma)$, $\qquad$ (3.2)

(ii) $\qquad M_\Gamma$ *is bounded in* $L_w^{p),\theta}(\Gamma)$. $\qquad$ (3.3.)

*If for a rectifiable curve $\Gamma$ and a weight $w$ the operator $S_\Gamma$ (respectively, $M_\Gamma$) is bounded in $L_w^{p),\theta}(\Gamma)$, $p > 1$, $\theta > 0$, then the condition* (3.1) *is fulfilled.*

We refer to [12, 13] for the equivalence of the boundedness of $S_\Gamma$ in $L_w^p(\Gamma)$ and the condition (3.1) in the case of the classical Lebesgue spaces.

In the case of the real line, the condition (3.1) coincides with the well-known Muckenhoupt $A_p(I)$ condition.

From Theorem 3.1 we deduce the following assertion.

**Corollary 3.1.** *Let $1 < p < \infty$. An operator $S_\Gamma$ (respectively, $M_\Gamma$) is bounded in $L^{p),\theta}(\Gamma)$ if and only if $\Gamma$ is a Carleson curve.*

**Proof of Theorem 3.1.** *Sufficiency* can be derived in the same way as in the case of the Hilbert transform. We focus only on the *necessity* for the Cauchy singular integral.

Our goal is to prove that the boundedness of $S_\Gamma$ in $L^{p),\theta}_w(\Gamma)$ implies

$$\sup_{\substack{t\ni\Gamma\\ 0<r<\frac{d}{3}}} \frac{1}{r}\int\limits_{D(t,r)} w(\tau)d\nu\left(\frac{1}{r}\int\limits_{D(t,r)} w^{1-p'}d\nu\right)^{p-1} < \infty,$$

where

$$d = \min_{t\in\Gamma}\max_{\tau\in\Gamma}|t-\tau|.$$

We fix $t \in \Gamma$. Let $x \in \Gamma$ be such that $|x-t| = 3r$. For a nonnegative function $f \in L^{p),\theta}_w(\Gamma)$ we introduce a function $g \in L^{p),\theta}_w(\Gamma)$ by the formula

$$g(t) = f(\tau)\frac{d\tau}{|d\tau|}e^{i\arg(t-x)}\cdot\chi_{D(t,r)}$$

(cf., for example, [12, p. 128]).

It is easy to see that for almost all $z \in D(x,r)$

$$(S_\Gamma g)(z) = \frac{1}{\pi i}\int\limits_{D(t,r)}\frac{g(\tau)}{\tau-z}d\tau = \frac{1}{\pi i}\int\limits_{D(t,r)}\frac{f(\tau)}{|\tau-z|}e^{i\alpha(\tau,z)}d\nu,$$

where $\alpha(\tau,z) := \arg(t-x) - \arg(\tau-z)$. Since

$$|\tau-z| \leqslant |t-x| + 2r = 5r$$

and

$$|\sin\alpha(\tau,z)| \leqslant \frac{r}{\left(\frac{3}{2}\right)r} = \frac{2}{3}, \quad \cos\alpha(\tau,z) \geqslant \left(1-\left(\frac{2}{3}\right)^2\right)^{\frac{1}{2}} = \frac{\sqrt{5}}{3}$$

for $\tau \in D(t,r)$ and $z \in D(x,r)$, for such $\tau$ and $z$ we have

$$|S_\Gamma g(z)| \geqslant \frac{1}{\pi}\int\limits_{D(t,r)}\frac{f(\tau)}{|\tau-z|}\cos\alpha(\tau,z)d\nu \geqslant \frac{\sqrt{5}}{3\pi}\int\limits_{D(t,r)}\frac{f(\tau)}{|\tau-z|}d\nu \geqslant \frac{c}{r}\int\limits_{D(t,r)} f(\tau)d\nu \tag{3.4}$$

for almost all $z \in \Gamma(x,r)$.

From (3.4) it follows that

$$\|S_\Gamma g(z)\|_{L^{p),\theta}_w(D(x,r))} = \sup_{0<\delta\leqslant p-1}\left(\frac{\delta^\theta}{\nu D(x,r)}\int\limits_{D(x,r)}|Sg(z)|^{p-\delta}w(z)d\nu\right)^{\frac{1}{p-\delta}}$$

$$\geqslant \left(\frac{c}{r}\int\limits_{D(t,r)} f(\tau)d\nu\right)\sup_{0<\delta\leqslant p-1}\left(\frac{\delta^\theta}{\nu D(x,r)}\int\limits_{D(x,r)} w(z)d\nu\right)^{\frac{1}{p-\delta}}.$$

Thus, we have

$$\|S_\Gamma g(z)\|_{L_w^{p),\theta}(D(x,r))} \geqslant \int\limits_{D(t,r)} f(\tau)d\nu \|\chi_{D(x,r)}\|_{L_w^{p),\theta}(\Gamma)}. \tag{3.5}$$

From the boundedness inequality and (3.5) it follows that

$$\|g(z)\|_{L_w^{p),\theta}(D(x,r))} \geqslant \frac{c}{r} \int\limits_{D(t,r)} f(\tau)d\nu \|\chi_{D(x,r)}\|_{L_w^{p),\theta}(\Gamma)}. \tag{3.6}$$

But

$$\|g(z)\|_{L_w^{p),\theta}(D(x,r))} \leqslant \|f\|_{L_w^{p),\theta}(D(t,r))}.$$

Now, (3.6) yields

$$\frac{1}{r} \int\limits_{D(t,r)} f(\tau)d\nu \|\chi_{D(x,r)}\|_{L_w^{p),\theta}(\Gamma)} \leqslant c\|f\|_{L_w^{p),\theta}(D(t,r))}$$

for arbitrary $f \geqslant 0$.

For $f \equiv 1$ the last inequality implies

$$\frac{1}{r}\nu D(t,r)\|\chi_{D(x,r)}\|_{L_w^{p),\theta}(\Gamma)} \leqslant c\|\chi_{\Gamma(t,r)}\|_{L_w^{p),\theta}(\Gamma)}. \tag{3.7}$$

Since $\Gamma$ is rectifiable, we have

$$\|\chi_{D(x,r)}\|_{L_w^{p),\theta}(\Gamma)} \leqslant c\|\chi_{D(t,r)}\|_{L_w^{p),\theta}(\Gamma)}.$$

Exchanging the roles of $D(x,r)$ and $D(t,r)$, we obtain the inequality

$$\|\chi_{D(t,r)}\|_{L_w^{p),\theta}(\Gamma)} \leqslant c\|\chi_{D(x,r)}\|_{L_w^{p),\theta}(\Gamma)} \tag{3.8}$$

for arbitrary $t$, $z$, and $x$ such that $|t-x| = 3r$, $z \in \Gamma(x,r)$, and $0 < r \leqslant \frac{d}{3}$.

Now, we set

$$g(\tau) = w^{1-p'}(\tau)\chi_{D(t,r)}$$

in the definition of $g$. According to (3.4), we have

$$|S_\Gamma g(z)| \geqslant \frac{c}{r} \int\limits_{D(t,r)} w^{1-p'}(\tau)d\nu$$

for almost all $z \in D(x,r)$.

The boundedness of $S_\Gamma$ and the last inequality imply

$$\begin{aligned} \|g(z)\|_{L_w^{p),\theta}(\Gamma)} &\geqslant c\|S_\Gamma g\|_{L_w^{p),\theta}(D(x,r)} \\ &\geqslant \frac{c}{r} \int\limits_{D(t,r)} w^{1-p'}(\tau)d\nu \cdot \sup_{0<\delta\leqslant p-1} \left( \frac{\delta^\theta}{\nu D(x,r)} \int\limits_{D(x,r)} w(z)d\nu \right)^{\frac{1}{p-\delta}} \\ &= \frac{c}{r} \int\limits_{D(t,r)} w^{1-p'}(\tau)d\nu \|\chi_{D(x,r)}\|_{L_w^{p),\theta}(D(x,r)}. \end{aligned} \tag{3.9}$$

On the other hand,

$$\|g\|_{L_w^{p),\theta}(\Gamma)} \leqslant \|w^{1-p'}\chi_{D(t,r)}\|_{L_w^{p),\theta}(\Gamma)}.$$

From the last inequality and (3.9) we deduce that

$$\frac{1}{r}\int\limits_{D(t,r)} w^{1-p'}(\tau)d\nu\ \|\chi_{D(x,r)}\|_{L_w^{p),\theta}(D(x,r))} \leqslant c\|w^{1-p'}\chi_{D(t,r)}\|_{L_w^{p),\theta}(\Gamma)}. \tag{3.10}$$

Now we need the inequality

$$\|f\chi_{D(t,r)}\|_{L_w^{p),\theta}(\Gamma)} \leqslant c\,(wD(t,r))^{-\frac{1}{p}}\,\|f\chi_{D(t,r)}\|_{L_w^p(\Gamma)}\cdot\|\chi_{D(t,r)}\|_{L_w^{p),\theta}(\Gamma)} \tag{3.11}$$

for arbitrary $f$. It is proved in the same way as in the case of a finite interval (cf. Lemma 2.2).

Using (3.11), from (3.10) we find

$$\frac{1}{r}\int\limits_{D(t,r)} w^{1-p'}(\tau)d\nu\|\chi_{D(x,r)}\|_{L_w^{p)}} \leqslant c(wD(t,r))^{-\frac{1}{p}}\Bigg(\int\limits_{D(t,r)} w^{1-p'}(\tau)d\nu\Bigg)^{\frac{1}{p}}\|\chi_{D(t,r)}\|_{L_w^{p),\theta}(\Gamma)}.$$

Taking (3.8) into account, from this inequality we find

$$\frac{1}{r}\int\limits_{D(t,r)} w^{1-p'}(\tau)d\nu\|\chi_{\Gamma(x,r)}\|_{L_w^{p),\theta}(\Gamma)} \leqslant c(wD(t,r))^{-\frac{1}{p}}\Bigg(\int\limits_{D(t,r)} w^{1-p'}(\tau)d\nu\Bigg)^{\frac{1}{p}}\|\chi_{D(x,r)}\|_{L_w^{p),\theta}(\Gamma)}.$$

Hence

$$\frac{1}{r}(wD(t,r))^{\frac{1}{p}}\Bigg(\int\limits_{D(t,r)} w^{1-p'}(\tau)d\nu\Bigg)^{\frac{1}{p'}} \leqslant c.$$

Thus, $w \in A_p(\Gamma)$. □

Arguing in a similar way as in the proof of Theorems 2.1 and 3.1, we obtain the following assertion.

**Theorem 3.2.** *Suppose that $\Gamma$ is a Carleson curve of finite length, $1 < p < \infty$, and $\theta > 0$. Let $a : \Gamma \to \mathbf{C}$ and $a \in \mathrm{Lip}\ 1$ on $\Gamma$. Then the operator*

$$\mathcal{C}_{a,\Gamma}f(t) = \int\limits_{\Gamma} \frac{a(t)-a(\tau)}{(t-\tau)^2}f(\tau)d\tau$$

*is bounded in $L_w^{p),\theta}(\Gamma)$ for $w \in A_p(\Gamma)$. Conversely, if there exists a constant $m$ such that $0 < m \leqslant |a'(t)|$ for arbitrary $t \in \Gamma$ and $\mathcal{C}_{a,\Gamma}$ is bounded in $L_w^{p),\theta}(\Gamma)$, then $w \in A_p(\Gamma)$.*

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