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Mathematics

On the Approximation of Periodic Functions within the Frame of Grand Lebesgue Spaces

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ABSTRACT. The paper announces the boundedness criteria of majorants for β-order ($\beta > 0$) Cesáro and Abel-Poisson means in weighted grand Lebesgue spaces $L^{p),\theta}$ ($1 < p < \infty$, $\theta > 0$). It is claimed that in these spaces the mean continuity property fails to hold. The fractional order moduli of smoothness are introduced in a subspace of $L^{p),\theta}$ where the set of smooth functions is dense. Using these characteristics, direct and inverse inequalities of the constructive theory of functions are obtained. The rate of approximation by β-order ($\beta > 0$) Cesáro means is estimated for a function from the above-mentioned subspace. In weighted grand Lebesgue spaces, an analog of the well-known Bernstein inequality is established for the derivatives of trigonometric polynomials.

Key words: *Grand Lebesgue space, majorants of linear summability means, trigonometric polynomials, direct and inverse inequalities, mean continuity, Bernstein type inequality, logarithmic-fractional order derivative.*

The present paper announces the authors' recent results on the approximation of functions by Fourier operators in some new function spaces.

Let $\mathbb{T} = (-\pi, \pi)$ and $1 < p < \infty$, $\theta > 0$. The weighted grand Lebesgue space of 2π-periodic functions is defined by the norm

$$\|f\|_{L_w^{p),\theta}} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon^\theta}{2\pi} \int_T |f(x)|^{p-\varepsilon} w(x)\,dx \right)^{\frac{1}{p-\varepsilon}}.$$

Here w is a weight function, i.e. an a. e. positive function which is integrable on T. When $w \equiv 1$, we set $L_w^{p),\theta} = L^{p),\theta}$.

In the weighted case, grand Lebesgue spaces on the bounded sets of a Euclidean space were introduced by T. Iwaniec and C. Sbordone [1] for $\theta = 1$, and by L. Greco, T. Iwaniec and C. Sbordone [2] for $\theta > 1$. It is a well-established fact that these spaces are non-reflexive non-separable ones.

For the boundedness problems in $L_w^{p),\theta}$ for various integral operators we refer the reader to [3-6].

It is well known that the following continuous embeddings hold:

$L_w^p \to L_w^{p),\theta} \to L_w^{p-\varepsilon}$, $\quad 0<\varepsilon<p-1$.

Let $f \in L^1(\mathbb{T})$ and

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k(f)\cos kx + b_k(f)\sin kx\right) \tag{1}$$

be its Fourier series.

We denote by $\sigma_n^\beta(f,x)$ $(\beta>0)$ and $u_r(f,x)$ the Cesáro and Abel-Poisson summability means, respectively.

A weight function w is said to be of the Muckenhoupt class A_p $(1<p<\infty)$ if

$$\sup\left(\frac{1}{|I|}\int_I w(x)\,dx\right)\left(\frac{1}{|I|}\int_I w^{1-p'}(x)\,dx\right)^{p-1} < \infty,$$

where the supremum is taken over all intervals whose length is less than 2π.

Theorem 1. *Let* $1<p<\infty$ *and* $\theta>1$. *The following statements are equivalent:*

i) *There exists a positive constant* $c_1>0$ *such that*

$$\left\| \sup_n \left|\sigma_n^\beta(f,x)\right| \right\|_{L_w^{p),\theta}} \le c_1 \|f\|_{L_w^{p),\theta}}$$

for an arbitrary function $f \in L_w^{p),\theta}$,

ii) *There exists a positive constant* $c_2>0$ *such that* $\left\|\sup\left|u_r(f,x)\right|\right\|_{L_w^{p),\theta}} \le c_2\|f\|_{L_w^{p),\theta}}$;

for an arbitrary function $f \in L_w^{p),\theta}$,

iii) $w \in A_p(\mathbb{T})$.

Note that an analogous statement is valid for more general means of the linear method of summability.

As has been mentioned above, the space $L^{p),\theta}$ is a non-separable one. The closure of L^p by the norm of $L^{p),\theta}$ does not coincide with the latter space.

We denote by $\dot{L}^{p),\theta}$ the closure of L^p with respect to the norm of $L^{p),\theta}$. As is known [7], $\dot{L}_w^{p),\theta}$ is a subspace of the space $L^{p),\theta}$ of functions satisfying

$$\lim_{\varepsilon\to 0} \varepsilon^\theta \int_T |f(x)|^{p-\varepsilon} w(x)\,dx = 0.$$

For functions from $\dot{L}^{p),\theta}$ we obtain in a standard manner

Corollary. *Let* $1<p<\infty$ *and* $\theta>1$. *Let* $w \in A_p(\mathbb{T})$. *Then for* $f \in \dot{L}_w^{p),\theta}$ *we have*

$$\lim_{n\to\infty} \left\|\sigma_n^\beta(f,\cdot) - f\right\|_{L_w^{p),\theta}} = 0$$

and

$$\lim_{n\to\infty}\left\|u_r(f,\cdot)-f\right\|_{L_w^{p),\theta}}=0.$$

Theorem 3. *Let* $1<p<\infty$ *and* $\theta>1$. *The mean continuity property fails to hold in the space* $L^{p),\theta}$, *i.e. there exists a function* $f\in L^{p),\theta}$ *such that*

$$\lim_{h\to 0}\left\|f(\cdot+h)-f(\cdot)\right\|_{L^{p),\theta}}\neq 0.$$

Theorem 3 shows that in the space $L^{p),\theta}$ it is impossible to define the moduli of smoothness by the translation operator but in $\dot{L}^{p),\theta}$ the moduli of smoothness can be introduced in the traditional way.

Here we employ the moduli of smoothness of fractional order. Assume that $r\geq 0$. Let Δ_r^h denote

$$\Delta_r^h f(x)=\sum_{k=0}^{n}(-1)^k\binom{r}{k}f\left(x+(r-k)h\right)$$

which is the *r*-th order difference of the function *f*.

Here

$$\binom{r}{k}=\frac{r(r-1)\cdots(r-k+1)}{k!}$$

for $k>1$, and

$$\binom{r}{k}=0$$

for $k=1$.

For $f\in\dot{L}^{p),\theta}(\mathbb{T})$ we set

$$\omega_r(f,\delta)_{L^{p),\theta}}=\sup_{|h|\leq\delta}\left\|\Delta_h^r f(\cdot)\right\|_{L^{p),\theta}}$$

and

$$E_n(f)_{L_w^{p),\theta}}=\inf\left\|f-T_n\right\|_{L^{p),\theta}},$$

where the infimum is taken over all trigonometric polynomials of degree not greater than *n*.

For a further presentation of our results we need the definition of power-logarithmic order derivative. Let $\lambda_n=n^{\alpha}\ln^{\gamma}(n+1)$. Assume that α, γ and β are some positive numbers. We say that a function $f\in L_w^{p),\theta}$, $1<p<\infty$, $\theta>0$ and $w\in A_p(T)$, has a (λ,β) derivative $f^{(\lambda,\beta)}$ if the series

$$\sum_{k=1}^{\infty}\lambda_k\left(a_k(f)\cos k\left(x+\frac{\beta\pi}{2k}\right)+b_k(f)\sin k\left(x+\frac{\beta\pi}{2k}\right)\right)$$

is the Fourier series of function $f^{(\lambda,\beta)}$.

For the logarithmic-fractional order derivatives of periodic functions we refer to L.D.Kudryavtsev [8]

The following analog of the well-known Bernstein inequality is valid.

Theorem 4. *Let* $1 < p < \infty$, $\theta > 1$. *Let* $w \in A_p(T)$. *There exists a constant c* >0 *such that*

$$\left\|T_n^{(\lambda,\beta)}\right\|_{L_w^{p),\theta}} \le c\lambda_n \left\|T_n\right\|_{L_w^{p),\theta}}$$

for an arbitrary trigonometric polynomial T_n.

We emphasize the results of A.I.Stepanets [9,10], who proved the Bernstein type inequality in unweighted classical Lebesque spaces for the derivatives in general sense.

Theorem 5. *Let* $1 < p < \infty$ *and* $\theta > 1$. *Assume that* $f \in \overset{\bullet}{L}{}^{p),\theta}$ *has a derivative* $f^{(\lambda,\beta)} \in \overset{\bullet}{L}{}^{p),\theta}$. *Then for arbitrary* $r > 0$ *we have*

$$E_n(f)_{L^{p),\theta}} \le c\frac{1}{\lambda_n}\omega_r\left(f^{(\lambda,\beta)}, \frac{1}{n+1}\right)$$

with a positive constant c independent of f *and n.*

Theorem 6. *Let* $1 < p < \infty$ *and* $\theta > 1$. *Then for* $f \in \overset{\bullet}{L}{}^{p),\theta}$ *and r* >0 *the inequality*

$$\omega_r\left(f, \frac{1}{n}\right)_{L^{p),\theta}} \le \frac{c_r}{n^r}\sum_{k=1}^{n} k^{r-1}E_{k-1}(f)_{L^{p),\theta}}$$

holds with a positive constant c_r *independent of* f *and n.*

Theorem 7. *Let* $1 < p < \infty$ *and* $\theta > 1$. *Let the condition*

$$\sum_{k=1}^{\infty}\frac{\lambda_k}{k}E_{k-1}(f)_{L^{p),\theta}} < \infty$$

be satisfied for a function $f \in \overset{\bullet}{L}{}^{p),\theta}$. *Then there exists* $f^{(\lambda,\beta)} \in \overset{\bullet}{L}{}^{p),\theta}$ *and the inequality*

$$\omega_r\left(f^{(\lambda,\beta)}, \frac{1}{n}\right)_{L^{p),\theta}} \le c\left(\frac{1}{n^r}\sum_{k=1}^{n}\lambda_k \cdot k^{r-1}E_{k-1}(f)_{L^{p),\theta}} + \sum_{k=n+1}^{\infty}\frac{\lambda_k}{k}E_{k-1}(f)_{L^{p),\theta}}\right)$$

holds with a constant c independent of f *and n.*

In the Lebesgue spaces L^p $(1 < p < \infty)$, results analogous to Theorems 5, 6 and 7 are presented in [11] for the moduli of smoothness of *k*-th order $(k \in \mathbb{N})$.

Our next result concerns the rate of approximation by linear summability means in $\overset{\bullet}{L}{}^{p),\theta}$.

Theorem 8. *Let* $1 < p < \infty$ *and* $\theta > 1$. *For* $f \in \overset{\bullet}{L}{}^{p),\theta}$ *and* $\beta > 0$ *the inequality*

$$\left\|f(\cdot) - \sigma_n^{\beta}(f, \cdot)\right\|_{L^{p),\theta}} \le c\omega_r\left(f, \frac{1}{n+1}\right)_{L^{p),\theta}}$$

holds for a positive constant c independent of f *and n.*

A similar estimate holds for the deviation by the Abel-Poisson means.

The proof of Theorem 8 is based on the analog of the Marcinkiewicz multiplier theorem for $L_w^{p),\theta}(\mathbb{T})$.

Theorem 9. *Let* $1 < p < \infty$ *and* $\theta > 1$, *and let* $w \in A_p$. *Assume that* $f \in L_w^{p),\theta}$ *and its Fourier series is* (1).

Let the sequence $\left(\lambda_n\right)_{n=0}^{\infty}$ *satisfy the conditions*

$$\left|\lambda_n\right| \le M, \quad \sum_{k=2^{n-1}}^{2^n-1}\left|\lambda_k - \lambda_{k+1}\right| \le M$$

for some $M > 0$ *and arbitrary* $n \in \mathbb{N}$.

Then the trigonometric series

$$\frac{\lambda_0 a_0}{2} + \sum_{n=1}^{\infty} \lambda_n \left(a_n \cos nx + b_n \sin nx\right)$$

is the Fourier series of some $F \in L_w^{p),\theta}$ *with a constant c independent of* f *and the inequality*

$$\left\|F\right\|_{L^{p),\theta}} \le Mc\left\|f\right\|_{L^{p),\theta}}$$

holds with a constant c independent of f.

In their further paper the authors intend to throw light on the approximation problems both in weighted grand Lebesgue spaces and in multidimensional cases.

მათემატიკა

პერიოდული ფუნქციების მიახლოვების შესახებ გრანდ ლებეგის სივრცეების ჩარჩოებში

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ნაშრომში მოყვანილია წონიანი გრანდ ლებეგის $L^{p),\theta}$ ($1 < p < \infty$, $\theta > 0$) სივრცეების ფუნქციების β-რიგის ($\beta > 0$) ჩეზაროსა და აბელ-პუასონის საშუალოების მაჟორანტების შემოსაზღვრულობის კრიტერიუმები. ნაჩვენებია, რომ გრანდ ლებეგის სივრცეებში ფუნქციათა საშუალოდ უწყვეტობის თვისებას, განსხვავებით კლასიკური ლებეგის სივრცეებიდან, აღარ აქვს ადგილი. აღნიშნული სივრცის იმ ქვესივრცისათვის, სადაც გლუვი ფუნქციები ყველგან მკვრივია გრანდ ლებეგის სივრცის ნორმით, შემოღებულია წილადური რიგის სიგლუვის მოდული და მის ტერმინებში დამტკიცებულია ფუნქციათა კონსტრუქციული თეორიის პირდაპირი და შებრუნებული თეორემები. ზემოხსენებული ქვესივრცის ფუნქციებისათვის დადგენილია β-რიგის ($\beta > 0$) ჩეზაროსა და აბელ-პუასონის საშუალოებით მიახლოების რიგი. წონიან გრანდ ლებეგის სივრცეებში მიღებულია ტრიგონომეტრიული პოლინომების წარმოებულების შესახებ ს. ბერნშტეინის ცნობილი თეორემის ანალოგი.

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