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Mathematics

The Inverse Inequalities of Trigonometric Approximation in Weighted Variable Exponent Lebesgue Spaces with Different Space Norms

Vakhtang Kokilashvili

Academy Member, A. Razmadze Mathematical Institute, I. Javakhishvili Tbilisi State University, Tbilisi

ABSTRACT. The inverse type inequalities of trigonometric approximations are established in weighted variable exponent Lebesgue spaces with different space norms. *© 2015 Bull. Georg. Natl. Acad. Sci.*

Key words: *variable exponent Lebesgue spaces, best approximations, fractional moduli of smoothness, different space norms, fractional derivative, weights.*

Let $T=[-\pi,\pi]$ and let $p(x)$ be 2π-periodic function continuous on the real line. We suppose that $p(x)$ satisfies the local log-continuity condition, i. e. there exists a positive constant A such that for all $x,y\in R$, $|x-y|<\frac{1}{2}$ the inequality

$$|p(x)-p(y)|\le\frac{A}{-\log|x-y|}$$

holds.

In the sequel the class of 2π-periodic functions satisfying the log-continuity condition ia denoted by $P^{\log}$. Further, we say that $p\in P$ if $p_-=\inf_T|p(x)|>1$. Also, for $p\in P^{\log}\bigcap P$ the notation $p_+=\sup_T p(x)$ will be used.

The variable exponent Lebesgue spaces $L^{p(\cdot)}$ of 2π-periodic functions are defined by the norm

$$\|f\|_{p(\cdot)}=\inf_{\lambda>0}\left\{\lambda:\int_T\left|\frac{f(x)}{\lambda}\right|^{p(x)}dx\le 1\right\}.$$

When $p\in P\bigcap P^{\log}$ these spaces are reflexive, separable, non-rearrangement invariant Banach function spaces (for these spaces we refer e. g. [1, 2]).

For $f \in L^{p(\cdot)}(T)$ we consider the fractional moduli of smoothness:

$$\Omega_r(f,\delta)_{p(\cdot)} = \sup_{0<h_i,t\le\delta} \left\| \prod_{i=1}^{[r]} \left(I - A_{h_i}\right)\sigma_t^{\{r\}} f \right\|_{p(\cdot)}, \qquad r>0, \delta>0,$$

where

$$A_h f(x) := \frac{1}{h} \int_{x-h/2}^{x+h/2} f(t)dt, \qquad x \in T$$

and

$$\sigma_h^r f(x) := (I - A_h)^r f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(r+1)}{\Gamma(k+1)\Gamma(r-k+1)} \left(A_h\right)^k .$$

For similar structural characterization we refer the reader to [3; 4: Section 3.16], etc.

For further use, we need to make the following definition of fractional derivative in the Weyl sense. Let

$$f(x) \sim \sum_{k=1}^{\infty} \left(a_k \cos kx + b_k \sin kx\right) = \sum_{k\in Z^+} c_k e^{ikx},$$

where $Z^+ = \{\pm1, \pm2, \ldots\}$.

If $\alpha > 0$, then α -th order fractional integral of f is defined as

$$I_\alpha(x,f) = \sum_{k\in Z^+} c_k (ik)^{-\alpha} e^{ikx},$$

where

$$(\mathrm{ik})^{-\alpha} := |\mathrm{k}|^{-\alpha} \mathrm{e}^{\left(-\frac{1}{2}\right)\pi \mathrm{i}\alpha \,\mathrm{sign k}},$$

For $\alpha \in (0,1)$ let

$$f^{(\alpha)}(x) := \frac{d}{dx} I_{1-\alpha}(x,f)$$

and

$$f^{(\alpha+l)}(x) := \left(f^{(\alpha)}(x)\right)^{(l)},$$

if the right hand side exists, where $l \in Z^+$, see e. g. [5: Section 8].

For $\alpha > 0$ let $W_{p(\cdot)}^{\alpha}$ be the class of functions for which

$$\|f\|_{W_{p(\cdot)}^{\alpha}} = \|f\|_{p(\cdot)} + \left\|f^{(\alpha)}\right\|_{p(\cdot)} < \infty.$$

Further, by $E_n(f)_{p(\cdot)}$ we denote the best approximations of $f \in L^{p(\cdot)}$ by trigonometric polynomilas of degree not greater than *n*.

Now we are ready to give the main results of this paper.

Theorem 1. *Let* 2π *-periodic continuous on the real line functions* $p(x)$ *and* $q(x)$ *belong to* $P^{\log} \cap P$. *Suppose that*

$$\frac{1}{q(x)} = \frac{1}{p(x)} - s, \qquad x \in T,$$

where *s* is a positive constant on *T*.

Let $p^+ < \frac{1}{s}$ and

$$\sum_{\upsilon=1}^{\infty} \upsilon^{\gamma s-1} E_{\upsilon}^{\gamma}(f)_{p(\cdot)} < +\infty, \; \gamma = \min(2, q_-).$$

Then $f \in L^{q(\cdot)}(T)$ *and the following estimates hold:*

$$E_n(f)_{q(\cdot)} \le c\left(n^s E_n(f)_{p(\cdot)} + \left(\sum_{\upsilon=n+1}^{\infty} \upsilon^{\gamma s-1} E_{\upsilon}^{\gamma}(f)_{p(\cdot)} \right)^{1/\gamma} \right)$$

and

$$\Omega_r\left(f, \frac{1}{n}\right)_{q(\cdot)} \le c\left\{ \frac{1}{n^{2r}} \left(\sum_{\upsilon=1}^{n} \upsilon^{\gamma(2r+s)-1} E_{\upsilon-1}^{\gamma}(f)_{p(\cdot)} \right)^{1/\gamma} + \left(\sum_{\upsilon=n+1}^{\infty} \upsilon^{\gamma s-1} E_{\upsilon}^{\gamma}(f)_{p(\cdot)} \right)^{1/\gamma} \right\}$$

with a positive constant *c* independent of f and *n*.

Corollary 1. *Let*

$$E_{\upsilon}(f)_{p(\cdot)} = O\left(\frac{1}{\nu^{2r+s}} \right), \qquad s = \frac{1}{p(x)} - \frac{1}{q(x)},$$

then

$$\Omega_r\left(f, \frac{1}{n}\right)_{q(\cdot)} = O\left(\frac{(\ln n)^{1/\gamma}}{n^{2r}} \right), \qquad \gamma = \min(2, q_-).$$

Theorem 2. *Let under the conditions of Theorem 1 for some* $\alpha > 0$ *the condition*

$$\sum_{\upsilon=1}^{\infty} \upsilon^{\gamma(\alpha+s)-1} E_{\upsilon}^{\gamma}(f)_{p(\cdot)} < +\infty$$

is satisfied. Then $f \in W_{q(\cdot)}^{\alpha}$ *and*

$$\Omega_r\left(f^{(\alpha)}, \frac{1}{n}\right)_{q(\cdot)} \le c\left\{ \frac{1}{n^{2r}} \left(\sum_{\upsilon=1}^{n} \upsilon^{\gamma(2r+s+\alpha)-1} E_{\upsilon-1}^{\gamma}(f)_{p(\cdot)} \right)^{1/\gamma} + \left(\sum_{\upsilon=n+1}^{\infty} \upsilon^{\gamma(\alpha+s)-1} E_{\upsilon}^{\gamma}(f)_{p(\cdot)} \right)^{1/\gamma} \right\}$$

with a positive constant *c* independent of f and *n*.

The case $p(x) = q(x)$ was explored in [6, 7].

Now we consider the problem in more general setting, namely, in weighted variable exponent Lebesgue spaces.

For a weight function w by $L_w^{p(\cdot)}(T)$ we denote the Banach function space defined by the norm

$$\|f\|_{p(\cdot),w} = \|fw\|_{p(\cdot)}.$$

We will employ the weights of the class $A_{p(\cdot),q(\cdot)}$.

A weight function w is said to be of class $A_{p(\cdot),q(\cdot)}$ if there exists a positive constant *c* such that for every interval *I* of the real line, the inequality

$$\|w\chi_I\|_{q(\cdot)}\|w^{-1}\chi_I\|_{p'(\cdot)}\le c|I|^{1-s},\ s=\frac{1}{p(\cdot)}-\frac{1}{q(\cdot)}$$

holds.

In the sequel by $\overline{A}_{p(\cdot),q(\cdot)}$ we denote the set of weights, which are the restrictions of $A_{p(\cdot),q(\cdot)}$ class weights on $(-5\pi,5\pi)$ with the condition $w(x+2\pi)=w(x)$ for all $x\in(-3\pi,3\pi)$.

The general moduli of smoothness and the best approximations by trigonometric polynomials in $L_w^{p(\cdot)}$ are defined as

$$\Omega_r(f,\delta)_{p(\cdot),w}=\sup_{0<h_i,t\le\delta}\left\|\prod_{i=1}^{[r]}\left(I-A_{h_i}\right)\sigma_t^{\{r\}}f\right\|_{p(\cdot),w},\qquad r>0,\ \ \delta>0,$$

and

$$E_n(f)_{p(\cdot),w}=\inf_t\|f-t\|_{p(\cdot),w},$$

where the infimum is taken with respect to all trigonometric polynomials $t(x)$ of degree not greater than *n*.

Theorem 3. *Let the functions* $p(x)$, $q(x)$ *and numbers s, r satisfy the conditions of Theorem* 2. *Suppose* $w\in\overline{A}_{p(\cdot),q(\cdot)}$ and *for some* $\alpha>0$ *the series*

$$\sum_{\upsilon=1}^{\infty}\upsilon^{\gamma(\alpha+s)-1}E_\upsilon^\gamma(f)_{p(\cdot),w},\quad \gamma=\min(2,q_-),$$

converges. Then $f\in W_{q(\cdot),w}^{\alpha}$ *and*

$$\Omega_r\left(f^{(\alpha)},\frac{1}{n}\right)_{q(\cdot),w}\le c\left\{\frac{1}{n^{2r}}\left(\sum_{\upsilon=1}^{n}\upsilon^{\gamma(2r+s+\alpha)-1}E_{\upsilon-1}^\gamma(f)_{p(\cdot),w}\right)^{1/\gamma}+\left(\sum_{\upsilon=n+1}^{\infty}\upsilon^{\gamma(\alpha+s)-1}E_\upsilon^\gamma(f)_{p(\cdot),w}\right)^{1/\gamma}\right\}.$$

The proofs are based on the Littlewood-Paley decomposition theorem, Bernstein-Zygmund and Nikol'skii inequalities in weighted variable exponent Lebesgue spaces.

We claim that analogous to Theorem 2 result is valid for more general type of derivatives, discussed e. g. in [7, 8]. For the analogous results in the case of constant exponents $p=q$ and $w\equiv 1$ we refer to [9]. The detailed proofs and some applications we are going to give in the forthcoming paper in Georgian Math. J.

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ვ. კოკილაშვილი

აკადემიის წევრი, ი. ჯავახიშვილის სახელობის თბილისის სახელმწიფო უნივერსიტეტის ა. რაზმაძის მათემატიკის ინსტიტუტი, თბილისი

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