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Mathematics

Fundamental Inequalities for Trigonometric Polynomials in New Function Spaces and Applications

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ABSTRACT. The paper deals with the fundamental inequalities for single variable trigonometric polynomials in new function spaces, variable exponent Morrey spaces and their applications to the trigonometric approximation in appropriate vanishing new function spaces. In present paper we explore trigonometric approximation problem of periodic functions in that subspace of variable exponent Morrey space, which is the closure of smooth functions by the norm of initial space. The generalized moduli of smoothness is defined via the Steklov means. In terms of this structural characteristic the direct and inverse inequalities of approximation are obtained. We state two-sided estimate of *K*-functional by generalized moduli of smoothness.

Key words: trigonometric polynomials, generalized moduli of smoothness, vanishing variable exponent Morrey spaces, best approximations, direct and inverse inequalities

The paper deals with approximation problems of 2®-periodic functions by trigonometric polynomials within the framework of new nonstandard function spaces, namely variable exponent Morrey spaces. The subject of our presentation are the following topics:

i) Fundamental inequalities for trigonometric polynomials in variable exponent Morrey spaces. We mean the Bernstein-Zygmund type inequality for fractional derivatives of trigonometric polynomials and Nikol'skii type inequality

ii) On the basis of the aforementioned inequalities we prove the direct and inverse type inequalities (in the Bernstein's terminology) for trigonometric approximations in approximable subspace of variable exponent Morrey spaces. The latter one is sited in the literature as vanishing variable exponent Morrey spaces. Moreover, the inverse type inequalities will be done in such form where on different sides of inequalities the space exponents are different.

Let $T=\left[-®,®\right]$ and $p:T\to\left(1,\infty\right)$ be 2®-periodic continuous on the real line function. By definition $p\in P\left(T\right)$ if $1<p_-\le p\left(x\right)\le p_+<\infty$ and $p_-=\min\limits_{x\in T}p\left(x\right)$, $p_+=\max\limits_{x\in T}p\left(x\right)$. A function $p\left(x\right)$ is called of class $P^{\log}(T)$ if it satisfies the log-continuity condition.

$$\left|p(x)-p(y)\right| \le \frac{A}{-\ln|x-y|}, \quad x \in T, \quad y \in T.$$

By $L^{p(\cdot)}(T)$ we denote the variable exponent Lebesgue space defined by the norm

$$\|f\|_{L^{p(\cdot)}(T)} = \inf_{\alpha>0}\left\{\alpha : \int_T \left|\frac{f(x)}{\alpha}\right|^{p(x)} dx \le 1\right\}.$$

When $p \in P(T)$ then $L^{p(\cdot)}$ is reflexive, separable Banach function space.

Let

$$I_{x,r} = (x-r, x+r) \cap T, \quad r>0.$$

The variable exponent Morrey spaces are defined as

$$M^{p(\cdot),\alpha(\cdot)} = \left\{ f : \ \|f\|_{M^{p(\cdot),\alpha(\cdot)}(T)} < +\infty \right\},$$

where

$$\|f\|_{M^{p(\cdot),\alpha(\cdot)}(T)} = \sup_{I_{x,r}} \left|I_{x,r}\right|^{-\alpha(x)} \|f\|_{L^{p(\cdot)}(I_{x,r})}.$$

Theorem 1 (Bernstein-Zygmund type inequality). *Let $\sim>0$ and let $p \in P \cap P^{\log}(T)$. Suppose that $\alpha(x) \in P^{\log}(T)$ and $0<\alpha_- \le \alpha(x) \le \alpha_+ <1$. Then for arbitrary trigonometric polynomial the following inequality*

$$\left\|T_n^{(\sim)}\right\|_{M^{p(\cdot),\alpha(\cdot)}} \le cn^{\sim} \left\|T_n\right\|_{M^{p(\cdot),\alpha(\cdot)}}$$

holds with a constant independent of T_n and n.

Here by $T_n^{(\sim)}$ the fractional derivative of order $\sim$ in Liouville sense is denoted.

Theorem 2 (Nikol'skii type inequality). *Let $p \in P \cap P^{\log}(T)$ and $\alpha_1(x) \in P^{\log}$. Suppose that $\frac{1}{p(x)} - \frac{1}{q(x)} = s > 0$ and $\alpha_2(x) = \frac{\alpha_1(x)q(x)}{p(x)}$, $\alpha_1(x) < \frac{1}{p(x)}$, $0 < s < \frac{1}{p_+}$. Then for arbitrary trigonometric polynomial the following inequality*

$$\left\|T_n\right\|_{M^{q(\cdot),\alpha_2(\cdot)}} \le cn^{\frac{1}{p(x)}-\frac{1}{q(x)}} \left\|T_n\right\|_{M^{p(\cdot),\alpha_1(\cdot)}}$$

holds with a constant independent of T_n and n.

The space $M^{p(\cdot),\alpha(\cdot)}$ is non-separable. The following statement is valid.

Proposition. The closure of the set of infinite differentiable functions in $M^{p(\cdot),\alpha(\cdot)}$ is characterized by the condition

$$\lim_{r\to 0} \sup_{I_{x,\circ}, 0<\circ\le r} \left|I_{x,\circ}\right|^{-\alpha(x)} \|f\|_{L^{p(\cdot)}(I_{x,\circ})} = 0.$$

We denote this subspace by $\tilde{M}^{p(\cdot),\alpha(\cdot)}(T)$. Thus $\tilde{M}^{p(\cdot),\alpha(\cdot)}(T)$ is an approximable by trigonometric polynomials subspace of $M^{p(\cdot),\theta(\cdot)}$.

For $f \in \tilde{M}^{p(\cdot),\theta(\cdot)}$ we introduce constructive and structural characteristics

$$E_n(f)_{M^{p(\cdot),\theta(\cdot)}} = \inf_{T_k} \| f - T_k \|_{M^{p(\cdot),\theta(\cdot)}} \quad k \le n$$

-the best approximation by trigonometric polynomials and

$$\Omega(f,\delta)_{M^{p(\cdot),\theta(\cdot)}} = \sup_{0<h\le\delta} \left\| f(x) - \frac{1}{2h}\int_{x-h}^{x+h} f(y)\,dy \right\|_{M^{p(\cdot),\theta(\cdot)}}$$

-generalized moduli of smoothness.

Theorem 3 (Jackson's first type direct inequality). *Let* $p(x) \in P \cap P^{\log}(T)$, $\theta(x) \in P^{\log}(T)$, $0 < \theta_- \le \theta(x) \le \theta_+ < 1$. *Then for* $f \in \tilde{M}^{p(\cdot),\theta(\cdot)}$ *the following inequality*

$$E_n(f)_{M^{p(\cdot),\theta(\cdot)}} \le c\,\Omega\left(f, \frac{1}{n}\right)_{M^{p(\cdot),\theta(\cdot)}}$$

holds with a constant c independent f and n.

Theorem 4 (Jackson's second type direct inequality). *Let* $\alpha > 0$ *and* $f^{(\alpha)} \in \tilde{M}^{p(\cdot),\theta(\cdot)}$. *Then under the hypothesis of previous theorem we have*

$$E_n(f)_{M^{p(\cdot),\theta(\cdot)}} \le \frac{c}{n^{\alpha}}\,\Omega\left(f^{(\alpha)}, \frac{1}{n}\right)_{M^{p(\cdot),\theta(\cdot)}}$$

with a constant c non-depending on f and n.

Theorem 5 (Inverse type inequality). *Let* $p(x) \in P \cap P^{\log}(T)$. *Suppose* $\theta(x) \in P^{\log}(T)$ *and* $0 < \theta_- \le \theta(x) \le \theta_+ < 1$. *Then for arbitrary* $f \in \tilde{M}^{p(\cdot),\theta(\cdot)}$ *the following inequality*

$$\Omega\left(f, \frac{1}{n}\right)_{M^{p(\cdot),\theta(\cdot)}} \le \frac{c}{n^2}\left(\sum_{\nu=1}^{n} \nu\, E_{\nu-1}(f)_{M^{p(\cdot),\theta(\cdot)}}\right)$$

holds with a constant independent of f *and* n.

Corollary. *Let* $f \in \tilde{M}^{p(\cdot),\theta(\cdot)}(T)$ *and*

$$E_n(f)_{M^{p(\cdot),\theta(\cdot)}} = O\left(\frac{1}{n}\right) \qquad > 0.$$

Then

$$\Omega(f,\delta)_{M^{p(\cdot),\theta(\cdot)}} = \begin{cases} O(\delta\ ), & 0 < \ \ < 2; \\ O\left(\delta^2 \log\frac{1}{\delta}\right), & \ \ = 2; \\ \delta^2, & \ \ > 2. \end{cases}$$

Let us introduce the Lipschitz type class of function.

Definition. A function $f \in \tilde{M}^{p(\cdot),\theta(\cdot)}$ is said to be of class $Lip\ \left(M^{p(\cdot),\theta(\cdot)}\right)$ if

$$\Omega(f,\delta)_{M^{p(\cdot),\theta(\cdot)}} = O(\delta\ ), \qquad > 0..$$

Theorem 6. *Let* $p(x)\in P\cap P^{\log}(T)$, $\alpha(x)\in P^{\log}(T)$, $0<\alpha_{-}\le\alpha(x)\le\alpha_{+}<1$. *Then* $f\in Lip\ \left(M^{p(\cdot),\alpha(\cdot)}\right)$, $0<\quad<2$, *if and only if*

$$E_n(f)=O\left(\frac{1}{n}\right).$$

Now we present the inverse type inequality when on different sides of inequalities the space exponents are different.

Theorem 7. *Let the conditions of Theorem 2 be fulfilled. If* $f\in\tilde{M}^{p(\cdot),\alpha_1(\cdot)}$ *and the condition*

$$\sum\nu^{s-1}E_n(f)_{M^{p(\cdot),\alpha_1(\cdot)}}<\infty$$

holds. Then $f\in\tilde{M}^{q(\cdot),\alpha_2(\cdot)}$ *and the following inequalities are true:*

$$E_n(f)_{M^{q(\cdot),\alpha_2(\cdot)}}\le c\left((n+1)E_n(f)_{M^{p(\cdot),\alpha_1(\cdot)}}+\sum_{\nu=n+1}^{\infty}\nu^{s-1}E_\nu(f)_{M^{p(\cdot),\alpha_1(\cdot)}}\right)$$

and

$$\Omega\left(f,\frac{1}{n}\right)_{M^{q(\cdot),\alpha_2(\cdot)}}\le c\left(\frac{1}{n^2}\sum_{\nu=1}^{n}\nu^{s+1}E_{\nu-1}(f)_{M^{p(\cdot),\alpha_1(\cdot)}}+\sum_{\nu=n+1}^{\infty}\nu^{s-1}E_\nu(f)_{M^{p(\cdot),\alpha_1(\cdot)}}\right)$$

with a constant independent of f *and* n.

Similar results in classical Lebesgue spaces and for classical moduli of smoothness were obtained by S. B. Stechkin [1] and A. F. Timan [2].

The proofs of the aforementioned results are based on the Theorems 1 and 2 (Bernstein-Zygmund and Nikol'skii type inequalities), boundedness of harmonic analysis fundamental integral operators in variable exponent Morrey spaces [3] and on the following proposition.

Theorem 8. *Let* $p(x)\in P\cap P^{\log}(T)$, $\alpha(x)\in P^{\log}(T)$, $0<\alpha_{-}\le\alpha(x)\le\alpha_{+}<1$. *Then for arbitrary trigonometric polynomials the following two-sided estimates hold.*

$$c_1n^{-2}\left\|T_n''\right\|_{M^{p(\cdot),\alpha(\cdot)}}\le\Omega\left(T_n,\frac{1}{n}\right)_{M^{p(\cdot),\alpha(\cdot)}}\le c_2n^{-2}\left\|T_n''\right\|_{M^{p(\cdot),\alpha(\cdot)}}$$

with some constants c_1 *and* c_2 *non-depending on* T_n *and* n.

მათემატიკა

ფუნდამენტური უტოლობები ტრიგონომეტრიული პოლინომებისთვის ახალ ფუნქციურ სივრცეებში და მათი გამოყენება

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ნაშრომში დადგენილია ერთი ცვლადის ტრიგონომეტრიული პოლინომებისათვის ბერნშტეინ-ზიგმუნდისა და ნიკოლსკის ტიპის ფუნდამენტური უტოლობები ცვლადმაჩვენებლიან მორის სივრცეებში. აღნიშნული უტოლობები გამოყენებულია ცვლადმაჩვენებლიან მორის სივრცეების აპროქსიმებად ქვესივრცეებში პერიოდულ ფუნქციათა ტრიგონომეტრიული პოლინომებით მიახლოების ამოცანებში. სტატიაში შესწავლილია პერიოდული ფუნქციების ტრიგონომეტრიული პოლინომებით მიახლოების პრობლემა ცვლადმაჩვენებლიანი მორის სივრცის იმ ქვესივრცეში, რომელიც წარმოადგენს გლუვი ფუნქციების ჩაკეტვას გამოსავალი სივრცის ნორმით. განზოგადებული სიგლუვის მოდული განსაზღვრულია სტეკლოვის საშუალოების მეშვეობით. ამ სტრუქტურული მახასიათებლის ტერმინებში დადგენილია მიახლოების პირდაპირი და შებრუნებული უტოლობები. მიღებულია K-ფუნქციონალის ორმხრივი შეფასება განზოგადებული სიგლუვის მოდულით.

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