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Mathematics

Necessary and Sufficient Conditions for Weighted Boundedness of Integral Transforms Defined on Product Spaces in Generalized Grand Lebesgue Spaces

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ABSTRACT. **The paper deals with one-weighted boundedness criteria for the integral transforms generated by the strong maximal functions, multiple conjugate functions and Hilbert transforms in grand Lebesgue spaces with respect to measurable functions. We characterize both weak and strong type weighted inequalities. Both cases of weighted spaces differing by position of the weight function in the norms are explored.** *© 2018 Bull. Georg. Natl. Acad. Sci.*

Key words: strong maximal function, multiple Hilbert transform, weights, generalized grand Lebesgue space

The grand Lebesgue spaces $L^{p)}$ first appeared in the paper by T. Iwaniec and C. Sbordone [1]. In this paper the authors explored the integrability problem of Jacobian under the minimal hypothesis. The generalized grand Lebesgue spaces $L^{p),\theta}$ were introduced by E. Greco, T. Iwaniec and C. Sbordone in [2], when they studied the nonhomogeneous n-harmonic equation $divA(x,\nabla u)=\mu$. In the theory of PDE's, it turns out that these spaces are the right spaces in which some nonlinear partial differential equations are appropriate in view of existence and uniqueness of solutions, of their regularity properties (see e. g. [2,3]). The study of mapping properties of integral operators of harmonic analysis in weighted grand Lebesgue spaces was started by the paper [4], where the necessary and sufficient condition ensuring the one-weight inequality for Hardy-Littlewood maximal functions was established. The similar problem for various type singular integral operators and potentials were explored in [5-11].

In the present paper we explore the boundedness problems for integral transforms defined on product spaces in the more general grand Lebesgue spaces introduced, in the unweighted case, by C. Capone, M. R. Formica and R. Giova [12]. Recently the in [13] the weighted boundedness criteria for Cauchy singular integral operator in weighted grand Lebesgue spaces with respect to measurable function were established.

Let Q be a bounded set in R^n. Let now w be a weight, i. e. almost everywhere positive integrable function on the set Q. Let $1<p<\infty$ and δ be a positive, measurable bounded function on $(0,p-1)$, $\delta(0+)=0$.

In the sequel we introduce two type weighted grand Lebesgue spaces with respect to measurable functions: in the first case the weight in the definition of the norm participates as a function generated a measure and the other case, when it plays a role of multiplier.

By $L_w^{p),\delta}(Q)$ we denote the set of all measurable functions on Q for which

$$\|f\|_{L_w^{p),\delta}(Q)}=\sup_{0<\varepsilon<p-1}\left(\delta(\varepsilon)\right)\left(\int_Q|f(x)|^{p-\varepsilon}w(x)dx\right)^{1/p-\varepsilon}<\infty.$$

In the following we will discuss also the generalized weighted grand Lebesgue space $\overline{L}_w^{p),\delta}(Q)$ defined by the norm

$$\|f\|_{\overline{L}_w^{p),\delta}(Q)}=\sup_{0<\varepsilon<p-1}\left(\left(\delta(\varepsilon)\right)\int_Q|f(x)w(x)|^{p-\varepsilon}dx\right)^{\frac{1}{p}}$$

Both these spaces are non-reflexive, non-separable and non-rearrangement invariant Banach function spaces. In the contrasts to the classical Lebesgue spaces with weights the spaces $L_w^{p),\theta}$ and $\overline{L}_w^{p),\theta}$ are different, non-reducible to each other.

For arbitrary Borel sets $e\in R^n$ we denote

$$we=\int_e w(x)dx.$$

Together with $L_w^{p),\delta}(Q)$ we are interested in the weak grand Lebesgue spaces with respect to the measurable functions $WL_w^{p),\delta}(Q)$, which we define by the quasi-norm

$$\|f\|_{WL_w^{p),\theta}}=\sup_{\lambda>0}\sup_{0<\lambda<p-1}\left(\delta(\varepsilon)w\{x\in Q:|f(x)|>\lambda\}\right)^{\frac{1}{p-\varepsilon}}.$$

It is clear that $L_w^{p),\theta}\mapsto WL_w^{p),\delta}$.

In the sequel we will need also the definition of the Muckechoupt type weights. Let $\overline{J}$ be the set of all rectangles $I\subset R^n$ with the edges parallel to the coordinate axis.

For p, $1<p<\infty$, by $A_p(\overline{J})$ we denote the set of all weights for which

$$\sup\frac{1}{|J|}\int_J w(x)dx\left(\frac{1}{|J|}\int_J w^{1-p'}(x)dx\right)^{p-1}<+\infty,$$

where the least upper bound is taken with respect all $J\subset\overline{J}$. By $|J|$ is denoted the volume of J.

Our aim is to treat the following integral operators:

$$\overline{M}_{\overline{J}}f(x)=\sup_{\substack{J\subseteq I\\ x\in J}}\frac{1}{|J|}\int_J f(y)dy$$

_ strong maximal function,

$$H_nf(x)=\int_J f(y)\prod_{i=1}^{n}\frac{1}{x_i-y_i}dy_i,\qquad x=(x_1,\dots,x_n),\ \ y=(y_1,\dots,y_n),$$

_ multiple Hilbert transform and

$$\tilde{f}(x)=\int_{T^n}f(x_1+s_1,x_2+s_2,\dots,x_n+s_n)\prod_{i=1}^{n}ctg\frac{s_i}{2}ds,\qquad s=(s_1,\dots,s_n),$$

_ multiple conjugate function defined for the functions, 2π -periodic with respect to each variable.

Now we are ready to formulate the main results.

Theorem 1. *Let* $1 < p < \infty$. *Then the following statements are equivalent:*

i) $\overline{M}_{\overline{J}}$ *is bounded in* $L_w^{p),\delta}\left(\overline{J}\right)$,

ii) $\overline{M}_{\overline{J}}$ *is bounded from* $L_w^{p),\delta}\left(\overline{J}\right)$ *to* $WL_w^{p),\delta}\left(\overline{J}\right)$,

iii) H_n *is bounded in* $L^{p),\delta}\left(\overline{J}\right)$,

iv) H_n *is bounded from* $L_w^{p),\delta}\left(\overline{J}\right)$ *to* $WL_w^{p),\delta}\left(\overline{J}\right)$,

v) $w \in A_p\left(\overline{J}\right)$.

Theorem 2. *Let* $1 < p < \infty$. *The following statements are valid:*

i) $\tilde{f}$ *is bounded in* $L_w^{p),\delta}\left(T^n\right)$,

ii) $\tilde{f}$ *is bounded from* $L_w^{p),\delta}\left(T^n\right)$ *to* $WL_w^{p),\delta}\left(T^n\right)$,

iii) $w \in A_p\left(T^n\right)$.

Theorem 3. *Let* $1 < p < \infty$ *and* $w^p \in A_p\left(\overline{J}\right)$. *Then* $\overline{M}_{\overline{J}}$ *as well as* H_n *are bounded in* $L_w^{p),\delta}\left(\overline{J}\right)$.

Theorem 4. *Let* $1 < p < \infty$ *and* $w^p \in A_p\left(T^n\right)$. *Then the operator* $f \to \tilde{f}$ *is bounded in* $\overline{L}_w^{p),\delta}\left(T^n\right)$.

The sufficience parts of above cited theorems follow from the following general assertion.

Theorem 5. *Let* $1 < p < \infty$. *Suppose that a sublinear operator* T *is bounded in* L_w^p $(1 < p < \infty)$ *for arbitrary* $w \in A_p$. *Then* T *is bounded in* $L_w^{p),\delta}$ *for arbitrary functions* δ *by the condition from the definition of* $L_w^{p),\delta}$.

The proof of Theorem 5 is based on the Calderon-Zygmund interpolation theorem [14] and the openess property of A_p condition.

In the following paper we intend to give several applications of aforementioned statements to the approximation by trigonometric polynomials of periodic functions of several variables.

მათემატიკა

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