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TRIGONOMETRIC APPROXIMATION BY ANGLE IN CLASSICAL WEIGHTED LORENTZ AND GRAND LORENTZ SPACES

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Abstract. In this paper, we present our results on an angular trigonometric approximation of functions of two variables in weighted Lorentz spaces and in that subspace of weighted grand Lorentz spaces in which C_0^∞ with a compact support is dense.

1. Introduction

The study of angular trigonometric approximation of 2π-periodic multivariate functions in classical Lebesgue spaces L^p $(1 < p < \infty)$ was initiated by K. Potapov (see, e.g., [7–9] and the review article [10]). Recently, these results were extended to the L^p $(1 < p < \infty)$ spaces with Muckenhoupt weights [1,2]. We aim to generalize the results of [1,2] to the classical weighted Lorentz spaces and to the certain subspace of weighted grand Lebesgue spaces.

In the sequel, by $\mathbb{T}^2$ we denote the torus $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$, where $\mathbb{T}$ is a circle $\{e^{i\varphi},\ \varphi \in [0, 2\pi)\}$. The function $w : \mathbb{T}^2 \longrightarrow \mathbb{R}^1$ is called a weight if w is a measurable on $\mathbb{T}^2$, positive almost everywhere and integrable. For a Borel measure $E \subset \mathbb{T}^2$, we define the absolute continuous measure

$$wE = \int\limits_E w(x,y)dxdy.$$

A weight function w is said to be of Muckenhoupt type class $\mathcal{A}_p(\mathbb{T}^2)$ if

$$\sup \bigg(\frac{1}{|J|}\int\limits_J w(x,y)dxdy\bigg)\bigg(\frac{1}{|J|}\int\limits_J w^{1-p'}(x,y)dxdy\bigg)^{p-1} < \infty, \quad p' = \frac{p}{p-1},$$

where the supremum is taken over all two-dimensional intervals with sides parallel to the coordinate axis.

In the sequel, we consider the set of measurable functions $f(x,y) : \mathbb{T}^2 \longrightarrow \mathbb{R}^1$ such that they are 2π-periodic with respect to each variable x and y.

2. Approximation in Weighted Lorentz Spaces

Definition 2.1. Let $1 < p, s < \infty$, w be the weight function defined on $\mathbb{T}^2$. We say that a measurable function f belongs to the weighted Lorentz space $L_w^{ps}(\mathbb{T}^2)$ (L_w^{ps} shortly) if the norm

$$\|f\|_{L_w^{ps}} = \bigg(s\int\limits_0^\infty \Big(w\Big((x,y) \in \mathbb{T}^2 :\ |f(x,y)| > \lambda\Big)\Big)^{s/p} \lambda^{s-1} d\lambda\bigg)^{1/s}$$

is finite.

The space L_w^{ps} is the Banach function space.

Let us introduce the notion a of modulus of smoothness

$$\Omega(f, \delta_1, \delta_2) = \sup \bigg\| \frac{1}{hk} \int\limits_{x-h}^{x+h} \int\limits_{y-k}^{y+k} f(t,s)dtds - f(x,y) \bigg\|_{L_w^{ps}}.$$

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By $P_{m,0}$ (respectively, by $P_{0,n}$) is denoted the set of all trigonometric polynomials of degree m (at most n) with respect to the variable x (variable y). Also, $P_{m,n}$ is defined as the set of all trigonometric polynomials of degree at most m with respect to the variable x and of degree at most n with respect to the variable y.

The best partial trigonometric approximation orders are defined as

$$E_{m,0}(f)_{L^{ps}_w} = \inf\{\|f - T\|_{L^{ps}_w} : T \in P_{m,0}\}.$$

Analogously,

$$E_{0,m}(f)_{L^{ps}_w} = \inf\{\|f - G\|_{L^{ps}_w} : G \in P_{0,n}\}.$$

Then the best angular approximation order is defined by the equality

$$E_{m,n}(f)_{L^{ps}_w} = \inf\{\|f - T - G\|_X : \mathbb{T} \in P_{m,0},\ G \in P_{0,n}\}.$$

The following assertions are true.

Theorem 2.1. *Let $1 < p, s < \infty$, $w \in \mathcal{A}_p(\mathbb{T}^2)$. For $f \in L^{ps}_w$, the following inequality*

$$E_{m,n}(f)_{L^{ps}_w} \le c_1 \Omega\left(f, \frac{1}{m}, \frac{1}{n}\right)_{L^{ps}_w}$$

holds with a constant c_1, independent of f, m and n.

Theorem 2.2. *Let $1 < p, s < \infty$, $w \in \mathcal{A}_p(\mathbb{T}^2)$, $f \in L^{ps}_w$. Then*

$$\Omega\left(f, \frac{1}{m}, \frac{1}{n}\right)_{L^{ps}_w} \le \frac{c}{m^2 n^2} \sum_{i=0}^{m} \sum_{j=0}^{n} (i+1)(j+1) E_{ij}(f)_{L^{ps}_w}.$$

In what follows, we discuss some tools, contributing to the proving of aforementioned assertions.

Let $\sigma^{\alpha,\beta}_{mn}(f,x,y)$ ($\alpha > 0$, $\beta > 0$) be the Cesáro means of double Fourier trigonometric series of $f \in L^{ps}_w$.

Theorem 2.3. *Let $1 < p, s < \infty$, $w \in \mathcal{A}_p(\mathbb{T}^2)$. Then*

$$\|\sigma^{\alpha,\beta}_{mn}(f)\|_{L^{ps}_w} \le c\|f\|_{L^{ps}_w}$$

and

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \|\sigma^{\alpha,\beta}_{mn}(f) - f\|_{L^{ps}_w} = 0.$$

For the partial sums of double Fourier trigonometric series, we have

$$\|S_{mn}(f)\|_{L^{ps}_w} \le c\|f\|_{L^{ps}_w}$$

with a constant c, independent of $m, n \in \mathbb{N}$ and $f \in L^{ps}_w$.

Further, for $f \in L^{ps}_w$,

$$\lim_{n\to\infty} \|S_{n,n} - f\|_{L^{ps}_w} = 0.$$

In the sequel, under the derivatives we assume those in Weyl's sense.

Theorem 2.4 (Bernstein type inequalities)**.** *Let $1 < p, s < \infty$, $w \in \mathcal{A}_p(\mathbb{T}^2)$. Assume that $\alpha, \beta > 0$.*

Let $T_1 \in P_{m,0}$, $T_2 \in P_{0,n}$ and $T_3 \in P_{mn}$. Then for α, β order Weyl's derivatives, we have

$$\left\|\frac{\partial^\alpha}{\partial x^\alpha} T_1\right\|_{L^{ps}_w} \le c_1 m^\alpha \|T_1\|_{L^{ps}_w}$$

$$\left\|\frac{\partial^\beta}{\partial y^\beta} T_2\right\|_{L^{ps}_w} \le c_2 n^\beta \|T_2\|_{L^{ps}_w}$$

and

$$\left\|\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta} T_3\right\|_{L^{ps}_w} \le c_3 m^\alpha n^\beta \|T_3\|_{L^{ps}_w},$$

where the constants c_1, c_2 and c_3 are independent of m, n and of polynomial.

For one– and two-weighted Bernstein inequalities in Lebesgue spaces we refer to [5], Chapter 6.

Definition 2.2. Let $f \in L_w^{ps}$, $w \in \mathcal{A}_p(\mathbb{T}^2)$. The mixed K-functional is defined as

$$K(f,\delta,\varepsilon,p,s,w,2)_{L_w^{ps}}:$$

$$=\inf_{h_1 h_2,h}\left\{\|f-h_1-h_2-h\|_{L_w^{ps}}+\delta^2\left\|\frac{\partial^2 h_1}{\partial x^2}\right\|_{L_w^{ps}}+\varepsilon^2\left\|\frac{\partial^2 h_2}{\partial y^2}\right\|_{L_w^{ps}}+\delta^2\varepsilon^2\left\|\frac{\partial^4 h}{\partial x^2\partial y^2}\right\|_{L_w^{ps}}\right\},$$

where the infimum is taken from all h_1, h_2, h such that $h_1 \in W^{2,0}_{L_w^{ps}}$, $h_2 \in W^{0,2}_{L_w^{ps}}$, $h \in W^4_{L_w^{ps}}$.

Here we use the following notation:

$$W^{2,0}_{L_w^{ps}}=\left\{h_1 : \frac{\partial^2 h_1}{\partial x^2}\in L_w^{ps}\right\},$$

$$W^{0,2}_{L_w^{ps}}=\left\{h_2 : \frac{\partial^2 h_2}{\partial y^2}\in L_w^{ps}\right\}$$

and

$$W^{4}_{L_w^{ps}}=\left\{h : \frac{\partial^4 h}{\partial x^2\partial y^2}\in L_w^{ps}\right\}.$$

The following statement is true.

Theorem 2.5. *Let $f \in L_w^{ps}$, $1<p,s<\infty$, $w \in \mathcal{A}^p(\mathbb{T})^2$. Then the equivalence*

$$\Omega(f,\delta_1,\delta_2)_{L_w^{ps}} \approx K(f,\delta_1,\delta_2,p,s,w,2)_{L_w^{ps}}$$

holds with the equivalence constants, independent of f, δ_1 and δ_2.

It should be noted that the mixed K-functionals were explored in [6] and [11]. This notion turn out to be very useful in the approximation and interpolation theory.

3. Approximation in a Subspace of Weighted Grand Lorentz Spaces

Let $1<p<\infty$. By Φ_p we denote the set of positive measurable functions φ defined on $(0,p-1]$ which are nondecreasing, bounded with a condition $\lim\limits_{x\to 0+}\varphi(x)=0$.

Let $\varphi \in \Phi_p$. The fully measurable weighted grand Lebesgue space $L_w^{p),s,\varphi}(\mathbb{T}^2)$ is defined as a set of all measurable functions $f:\mathbb{T}^2 \longrightarrow \mathbb{R}^1$, for which the norm

$$\|f\|_{L_w^{p),s,\varphi}(\mathbb{T}^2)}=\sup_{0<\varepsilon<p-1}(\varphi(\varepsilon))^{\frac{1}{p-\varepsilon}}\|f\|_{L_w^{p-\varepsilon}(\mathbb{T}^2)},\quad 1<p<\infty,\quad \theta>0$$

is finite.

The space $L_w^{p),s,\varphi}(\mathbb{T}^2)$ is non-reflexive, non-separable Banach function space.

The subspace of $L_w^{p),s,\varphi}(\mathbb{T}^2)$, in which the smooth functions are dense, is characterized by the equality

$$\lim_{\varepsilon\to 0}(\varphi(\varepsilon))^{\frac{1}{p-\varepsilon}}\|f\|_{L_w^{p-\varepsilon}(\mathbb{T}^2)}=0.$$

Denote this subspace by $\widetilde{L}_w^{p),s,\varphi}(\mathbb{T}^2)$. We treat the angular trigonometric approximation of a function of two variables in this subspace.

Analogously to the previous section, for $f \in \widetilde{L}_w^{p),s,\varphi}(\mathbb{T}^2)$ we introduce the structural and constructive characteristics $\Omega(f,\delta_1,\delta_2)_{\widetilde{L}_w^{p),s,\varphi}(\mathbb{T}^2)}$ and $E_{m,0}(f)_{\widetilde{L}_w^{p),s,\varphi}(\mathbb{T}^2)}$, $E_{(0,n)}(f)_{\widetilde{L}_w^{p),s,\varphi}(\mathbb{T}^2)}$ and $E_{m,n}(f)_{\widetilde{L}_w^{p),s,\varphi}(\mathbb{T}^2)}$.

We claim that for $\widetilde{L}_w^{p),s,\varphi}(\mathbb{T}^2)$, the statements similar to Theorem 2.1 and Theorem 2.2 are valid.

Remark 3.1. The results, analogous to the above-mentioned, are true both for the function of several variables and for the modulus of smoothness of fractional order.

The proofs of the presented results based essentially on the boundedness of integral operators in the classical weighted Lorentz spaces and grand Lorentz spaces have been obtained recently in [6].

The detiled proofs will be published in Georgian Mathematikal journal.

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