Transactions of A. Razmadze
Mathematical Institute
Vol. **175** (2021), issue 3, 431–432

ON THE BOUNDEDNESS OF PSEUDODIFFERENTIAL OPERATORS DEFINED BY AMPLITUDES IN GENERALIZED WEIGHTED GRAND LEBESGUE SPACES

ETERI GORDADZE AND VAKHTANG KOKILASHVILI

Abstract. In this article, we present weighted estimates for pseudo-differential operators with amplitudes which are only measurable in the spatial variables. The source of this investigation is the paper [2], in which weighted inequalities for the above-mentioned operators are established in classical $L^p(p>1)$ spaces with Muckenhoupt weights.

Our paper deals with the weighted inequalities for pseudo-differential operators with amplitudes in nonstandard Banach function space and generalized weighted grand Lebesgue spaces. Below, all the definitions concerning the amplitudes and symbols are taken from [2]. For a function $f\in C_0^\infty(R^n)$, a pseudo-differential operator is given formally by

$$T_af(x):=\frac{1}{(2n)^n}\int\limits_{R^n}\int\limits_{R^n} a(x,y,\xi)\exp^{i(x-y,\xi)} f(y)\,dy\,d\xi,$$

whose amplitude $(x,y,\xi)\mapsto a(x,y,\xi)$ is assumed to satisfy certain growth conditions. For the class of amplitudes we refer the reader to [1].

Let $1<p<\infty$, φ be a positive non-decreasing function on $(0,p-1)$ satisfying $\varphi(0+)=0$. The generalized weighted grand Lebesgue space $L_v^{p),\varphi}(R^n,w)$ is defined as the set of all measurable functions for which

$$||f||_{L_v^{p),\varphi}}(R^n,w)=\sup_{0<\epsilon<p-1}(\varphi(\epsilon)\int\limits_{R^n}|f(x)|^{p-\epsilon}w(x)v^\epsilon(x)\,dx)^{\frac{1}{p-\epsilon}}<+\infty,$$

where $wv^\epsilon\in L^1_{loc}(R^n)$ for all ϵ, $0<\epsilon<p-1$. The function $a:R^n\times R^n\times R^n$ is called an amplitude when it belongs to any one of the following sets. Let $m\in R,\rho\in[0,1]$ and $\delta\in[0,1]$.

Definition 1.

(i) We say that $a\in A^m_{\rho,\delta}$ when for each triple of multi-indices α, β and γ there exists a constant $C_{\alpha,\beta,\gamma}$ such that

$$|\partial_\xi^\alpha\partial_x^\beta\partial_y^\gamma a(x,y,\xi)|\le C_{\alpha,\beta,\gamma}\langle\xi\rangle^{m-|\rho|\alpha+\delta|\beta+\gamma|}.$$

(ii) We say that $a\in L^\infty A^m_\rho$ when for each multi-index α there exists a constant C_α such that

$$||\partial_\xi^\alpha a(.,.,\xi)||_{L^\infty(R^n\times R^n}\le C_\alpha\langle\xi\rangle^{m-\rho|\alpha|},$$

where $\langle\xi\rangle:=(a+|\xi|^2)^{\frac12}$. Here it is assumed only measurability in the (x,y)-variables.

Definition 2 ([2])**.** A function $a:R^n\times R^n\mapsto R^n$ is called a symbol when it belongs one of the following sets. Let $m\in R$, $\rho\in[0,1]$ and $\delta\in[0,1]$.

(i) We say that $a\in S^m_{\rho,\delta}$, when for each pair of multi-indices α and β there exists a constant $C_{\alpha,\beta}$ such that

$$|\partial_\xi^\alpha\partial_\xi^\beta a(x,\xi)|\le C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}.$$

2020 *Mathematics Subject Classification.* Primary 47G30, Secondary 35S05, 35015.

Key words and phrases. Weighted grand Lebesgue spaces; Pseudo-differential operators; Muckenhoupt A_p weights; Pseudo-differential amplitudes; Symbols.

(ii) We say that $a \in L^\infty S^m_\rho$ when for each multi-index α there exists a constant C_α such that

$$||\partial^\alpha_\xi a(.,\xi)||_{L^\infty} \le C_\alpha \langle\xi\rangle^{m-\rho|\alpha|}.$$

Therefore here it is assumed only measurability in the x-variable. The following statements are true.

Theorem 1. *Let $1 < p < \infty$, $w \in A_p$ and let $v \in L^p(R^n, w)$, $v^\gamma \in A_p$ for some $\gamma > 0$. Assume that $\sigma \in L^\infty_1 S^m$, with $m < \frac{n}{2}(\rho - 1)$ and set $a(x,\xi) = e^{i|\xi|^{1-\rho}}\sigma(x,\xi)$, with $0 < \rho < 1$. Then T_a is bounded in $L^{p),\varphi_v}_v(R^n, w)$.*

Theorem 2. *Let $a(x,y,\xi) = e^{i|\xi|^{1-\rho}}\sigma(x,y,\xi)$ with $m < \frac{n}{2}(\rho - 1)$. Then under the condition on p, v and w of Theorem 1, the operator T_a is bounded in $L^{p),\varphi}_v(R^m, w)$.*

Definition 3 ([2])**.** The class $L^\infty S^m_{cl}$ consists of all the symbols which are bounded and measurable in the spatial variable and satisfy

(1) $||\partial^\alpha_\xi a(.,\xi)||_{L^\infty} \le c_\alpha \langle\xi\rangle^{m-|\alpha|}$, for each multi-index α;

(2) $a(x, t\xi) = t^m a(x,\xi), t \ge 1, |\xi| \ge 1$.

Theorem 3. *Let p, w and v satisfy the conditions of Theorem 1. Assume that $\sigma \in L^\infty S^{\frac{n(\rho-1)}{2}}_{cl}$ and set $a(x,\xi) = e^{i|\xi|^{1-rho}}\sigma(x,\xi)$ with $0 < \rho \le 1$. Then the operator T_a is bounded in $L^{p),\varphi}_v(R^n, w)$.*

Theorem 4. *Let $1 < p < \infty$, $w \in A_p$ and $v \in L^p(R^n, w), v^\gamma \in A_p$ for some $\gamma > 0$. Suppose $0 \le \rho < 1$, $m < n(\rho - 1)$ and $a \in L^\infty A^m_\rho$. Then the operator T_a is bounded in $L^{p),\varphi}_v(R^n, w)$.*

Theorem 5. *Let p, w and v be the same as in previous Theorem. Suppose that $a \in A^{n(p-1)}_{p,8}$ with $0 < \rho \le 1$, $0 \le S < 1$. Then T_a is bounded in $L^{p),\varphi}_v(R^n, w)$.*

Below, we announce weighted norm inequalities for the commutators of BMO functions for variation pseudodifferential operators.

Theorem 6. *Assume that p, w and v satisfy the conditions of Theorem 1. Suppose either:*

(a) *$a \in L^\infty A^m_\rho$ with $m < n(\rho - 1)$ and $0 \le \rho \le 1$; or*

(b) *$a(x,y,\xi) = e^{i|\xi|^{1-\rho}}\sigma(x,y,\xi)$ and $\sigma \in L^\infty A^m_\rho$ with $0 < \rho \le 1$ and $m < \frac{n}{2}(\rho - 1)$ or*

(c) *$a \in A^{n(\rho-1)}_{\rho,\delta}$ with $0 \le \delta < \xi$ and $0 < \rho \le 1$; or*

(d) *$a(x,\xi) = e^{i|\xi|^{1-\rho}}\delta \in L^\infty S^{\frac{n}{2}(\rho-1)}_{cl}$, $0 < \rho \le 1$.*

Then for $b \in BMO$ the operator $T_{\alpha,\beta} f = bTf - T(fb)$ is bounded in $L^{p),\varphi}_v(R^n, w)$.

References

1. L. Hörmander, On the L^2 continuity of pseudo-differential operators. *Comm. Pure Appl. Math.* **24** (1971), 529–535.
2. N. Michalowski, D. Rule, W. Staubach, Weighted norm inequalities for pseudo-pseudodifferential operators defined by amplitudes. *J. Funct. Anal.* **258** (2010), no. 12, 4183–4209.

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0186, Georgia

E-mail address: vakhtang.kokilashvili@tsu.ge