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ON FUNCTIONS WITH THICK GRAPHS AND EXTENSIONS OF MEASURES

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It is well known that set-theoretical methods play a significant role in various questions of real analysis and measure theory. For instance, the method of transfinite recursion, Bernstein's construction of some pathological sets, Sierpinski's functions with thick graphs, etc. give rise to many important consequences in study of measurability properties of real-valued functions. In this connection, see, for instance, [1]–[6].

Let E_1 be a basic set, (E_2, S_2) be a measurable space and let M be a class of measures on E_1 (we assume, in general, that the domains of measures from M are various σ-algebras of subsets of E_1). For our further purposes, it is convenient to use the following notation:

$\mathbf{R}$=the set of all real numbers;

$\mathrm{dom}(\mu)$ = the σ-algebra on which a given measure μ is defined;

$\mathrm{ran}(f)$= the range of a given function f.

We shall say that a function $f : E_1 \to (E_2, S_2)$ is relatively measurable with respect to M if there exists at least one measure $\mu \in M$ such that f is measurable with respect to μ. Otherwise, we shall say that f is absolutely nonmeasurable with respect to M(see [4]).

Let (E_1, S_1, μ_1) and (E_2, S_2, μ_2) be measurable spaces equipped with σ-finite measures. We recall that a graph $\Gamma \subset E_1 \times E_2$ is $(\mu_1 \times \mu_2)$-thick in $E_1 \times E_2$ if, for each $(\mu_1 \times \mu_2)$-measurable set $Z \subset E_1 \times E_2$ with $(\mu_1 \times \mu_2)(Z) > 0$, we have $\Gamma \cap Z \neq \emptyset$ (cf. [2], [4], [6]).

Let E_1 be a set equipped with a σ-finite measure μ_1 and let $f : E_1 \to E_2$ be a function satisfying the following condition: there exists a probability measure μ_2 on $\mathrm{ran}(f)$ such that the graph of f is a $(\mu_1 \times \mu_2)$-thick subset of the product set $E_1 \times \mathrm{ran}(f)$.

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Take any set $Z \in \mathrm{dom}(\mu_1 \times \mu_2)$ and put

$$Z' = \{x \in E_1 : (x, f(x)) \in Z\},$$
$$S = \{Z' : Z \in \mathrm{dom}(\mu_1 \times \mu_2)\},$$
$$\mu'_1(Z') = (\mu_1 \times \mu_2)(Z) \quad (Z' \in S).$$

Then the functional μ'_1 satisfies the following relations:

1) μ'_1 is a measure on S extending μ_1;

2) f is measurable with respect to μ'_1.

In other words, f turns out to be relatively measurable with respect to the class $M(\mu_1)$ of all extensions of μ_1.

Example. There exists a function

$$g : \mathbf{R} \to \mathbf{R}$$

having the property that, for any σ-finite diffused Borel measure μ on $\mathbf{R}$ and for any σ-finite measure ν on $\mathbf{R}$, the graph of g is a $(\mu \times \nu)$-thick subset of the plane $\mathbf{R}^2$.

In this connection note that, from a certain point of view, the function g is universally measurable, i.e., for any σ-finite diffused Borel measure μ on $\mathbf{R}$ there exists a measure $\overline{\mu}$ such that:

(1) $\overline{\mu}$ extends μ;

(2) g is measurable with respect to $\overline{\mu}$.

A detailed proof of this fact can be found in [4]. Note that a Bernstein type construction is used in this proof.

Applying the method of transfinite recursion, the following statement can be proved.

Theorem. *Let (E_1, S_1, μ_1) and (E_2, S_2, μ_2) be two uncountable sets equipped with σ-finite measures and let* $\mathrm{card}(E_1) = \mathrm{card}(E_2) = \alpha$. *Suppose that there exists a family $\{Z_\xi : \xi < \alpha\}$ of subsets of $E_1 \times E_2$ satisfying the following conditions:*

1) *for any $(\mu_1 \times \mu_2)$-measurable set $Z \subset E_1 \times E_2$, with $(\mu_1 \times \mu_2)(Z) > 0$, there is an index $\xi < \alpha$ such that $Z_\xi \subset Z$;*

2) *for any set $X \subset E_1$ with $card(X) < \alpha$ and for any Z_ξ $(\xi < \alpha)$, we have $Z_\xi \setminus (X \times E_2) \neq \emptyset$.*

Then there exists a function $f : E_1 \to E_2$ whose graph is $(\mu_1 \times \mu_2)$-thick in $E_1 \times E_2$. Consequently, f is relatively measurable with respect to the class $M(\mu_1)$.

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