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ON SOME PROPERTIES OF CERTAIN DISCRETE POINT-SETS IN EUCLIDEAN SPACES

A system X of points in the Euclidean space $\mathbf{R}^n$ $(n \geq 1)$ is called discrete if every ball in $\mathbf{R}^n$ contains only finitely many points from X.

In general, a set X is discrete in a topological space E if every point $x \in X$ has a neighborhood U such that $X \bigcap U = \{x\}$.

If a space E has a countable base, then any discrete set in E is either finite or countably infinite. The standard example of an infinite discrete set on the real line $\mathbf{R}$ is the set $\{\frac{1}{m} : m \in N, m \geq 1\}$.

In any reasonable space, a finite set turns out to be discrete.

Discrete point-systems can be met in various fields of pure and applied mathematics. We may indicate several such directions in contemporary mathematics, for instance, discrete and computational geometry, classical number theory, combinatorics (finite or infinite), the theory of convex sets, etc.

The investigation of the combinatorial structure of various discrete and finite point-systems in Euclidean spaces is a rather attractive and important topic. Properties of various discrete point systems are considered in many works (see, for example, [1]–[5]).

Let D be a point-set (finite or infinite) in the n-dimensional Euclidean space $\mathbf{R}^n$.

We say that this D is a Diophantine set if the distance between any two points from D is a natural number.

The most simple example of an infinite Diophantine set is the set of all integer numbers in $\mathbf{R}$.

A point-set Y is a quasi-Diophantine set if the distance between any two points from Y is a rational number.

Lemma 1. *Let X be a finite quasi-Diophantine subset of $\mathbf{R}^n$. Then there exists a homothety h of $\mathbf{R}^n$ with integer coefficient such that the set $h(X)$ is Diophantine.*

Let X be a point-set in Euclidean space $\mathbf{R}^n$.

We shall say that a line segment l is an edge of X if there exist two points from X which are the end-points of l. This terminology is compatible with graph theory.

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Take any four points x, y, z and t from X such that the intersection of the line segments $[x, y]$ and $[z, t]$ is a singleton. The family of all singletons obtained in this manner will be denoted by $I(X)$.

Note that if X contains at least three points, then $X \subset I(X)$.

Remark 1. In general, we cannot assert that if X is a quasi-Diophantine set, then the set $I(X)$ is also quasi-Diophantine. We only can prove a somewhat weaker result (see Theorem 1 below).

We shall say that a line segment l is admissible for X if its end-points belong to $I(X)$ and there exists an edge of X containing l.

Theorem 1. *Let X be a finite quasi-Diophantine set in the space $\mathbf{R}^n$. Then the length of each admissible line segment for X is a rational number.*

The proof of Theorem 1 uses the method of induction on $\mathrm{card}(X)$, where $\mathrm{card}(X) \geq 4$.

The basis of induction, i.e., the case $\mathrm{card}(X) = 4$ essentially relies on the fact that if the points of quasi-Diophantine set $\{a, b, c, d\}$ are the vertices of quadrangular in $\mathbf{R}^n$ and $[a, b] \bigcap [c, d] = \{x\}$, then the set $\{a, b, c, d, x\}$ is a quasi-Diophantine set, too.

A well-known result of combinatorial geometry states that any infinite Diophantine subset D of the Euclidean plane $\mathbf{R}^2$ is necessarily collinear, i.e., all points of D belong to a certain line (see, for instance, [2], [4]). Notice also that the analogous proposition is not valid for finite Diophantine sets in $\mathbf{R}^2$ (see, for example, [2], [3]).

The following statements are true.

Lemma 2. *Let D be a Diophantine subset of an n-dimensional sphere of integer radius r, where $r \geq 2$. Then in the Euclidean space $\mathbf{R}^{n+2}$ there exists a Diophantine set D_1 containing D and such that $card(D_1 \setminus D) = 2$.*

Theorem 2. *For any natural number $n \geq 2$, there are Diophantine sets in $\mathbf{R}^n$ which have arbitrarily many points and do not lie in a hyperplane of $\mathbf{R}^n$.*

Theorem 3. *Let D be an infinite Diophantine set in the Euclidean space $\mathbf{R}^n$, where $n \geq 1$. Then all points of D are collinear.*

The proof of Theorem 3 is essentially concerned with some geometric properties of intersections of finite families each member of which is a hyperboloid in the Euclidean space $\mathbf{R}^n$. The main fact here is that the intersection of sufficiently many algebraic surfaces, which are is general position, always yields the empty set.

Remark 2. The analogue of the above-mentioned Theorem 3 is not valid for infinite-dimensional vector spaces. Indeed, the set of vectors $(\frac{e_i}{\sqrt{2}})_{i \in N}$ from the classical infinite-dimensional Hilbert space l_2, where $e_1, e_2, \dots, e_n, \dots$ are elements of orthonormal basis of l_2, is a Diophantine set in l_2, and it is clear that these points are not collinear (moreover, they do not lie in any finite-dimensional subspace of l_2).

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