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ON RELATIVE MEASURABILITY OF REAL-VALUED FUNCTIONS WITH RESPECT TO SOME MEASURES IN THE SPACE $\mathbf{R}^{\mathbf{N}}$

Let E be a set and let M be a class of measures on E (we assume, in general, that the domains of measures from M are various σ-algebras of subsets of E). We shall say that a function

$$f : E \to \mathbf{R}$$

is relative measurable with respect to M if there exists at least one measure $\mu \in M$ such that f is measurable with respect to μ, where $\mathbf{R}$ is the set of all real numbers. Otherwise, we shall say that f is absolutely nonmeasurable with respect to M.

Example 1. Let μ be a measure on E and let $M(E)$ be the class of all nonzero σ-finite diffused measures on E. Let $f : E \to \mathbf{R}$ be a function and let, for some $t_0 \in \mathbf{R}$, the relation $\mathrm{card}(f^{-1}(t_0)) > \omega$ be satisfied, where ω denotes the first infinite cardinal number. In this case, we can assert that f is relative measurable with respect to the class $M(E)$.

In particular, if an original set E is such that $\mathrm{card}(E) > 2^{\omega}$, then every function $f : E \to \mathbf{R}$ is relative measurable with respect to $M(E)$.

The above-mentioned notation and example are discussed in [1] and [3].

Example 2. Let V be an equivalence relation on $\mathbf{R}$ whose all equivalence classes are at most countable. We shall say that $f : \mathbf{R} \to \mathbf{R}$ is a Vitali type function for V if $(r, f(r)) \in V$ for each $r \in \mathbf{R}$ and the set $\mathrm{ran}(f)$ is a selector of the partition of $\mathbf{R}$ determined by V. Let M_1 be the class of all translation invariant extensions of the Lebesgue measure λ on $\mathbf{R}$ and let M_2 be the class of all translation quasi-invariant extensions of λ on $\mathbf{R}$. Then there exists a Vitali type function for V which is relatively measurable with respect to the class M_2 and is absolutely nonmeasurable with respect to the class M_1.

Notice that if M denotes the class of all measures on $\mathbf{R}$ extending classical Lebesgue measure on $\mathbf{R}$, then every Vitali type function for V is relatively measurable with respect to M.

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In connection with Example 2, see [1], [3].

Example 3. Let M_1 be the class of all nonzero σ-finite separable measures on $\mathbf{R}$ and let M_2 be the class of all nonzero σ-finite non-separable measures on $\mathbf{R}$. If a function $f : E \to \mathbf{R}$ is relatively measurable with respect to the class M_2, then f is relatively measurable with respect to the class M_1.

The above-mentioned example is discussed in [5].

Let μ be a σ-finite measure given on a base set E. A subset X of E is called μ-thick if $\mu_*(E \setminus X) = 0$, where μ_* denotes the inner measure associated with μ (see, e. g., [1], [4]).

As usual, the symbol N denotes the set of all natural numbers and $\mathbf{R}^{\mathbf{N}}$ denotes the space of all real-valued sequences.

The following two statements are valid.

Theorem 1. *There exists a function*

$$f : \mathbf{R}^{\mathbf{N}} \to \mathbf{R}$$

having the following property: for any σ-finite diffused Borel measure μ on $\mathbf{R}^{\mathbf{N}}$ and for any σ-finite diffused Borel measure ν on $\mathbf{R}$, the graph of f is a $(\mu \times \nu)$-thick subset of $\mathbf{R}^{\mathbf{N}} \times \mathbf{R}$.

From the Theorem 1 we deduce the following statement.

Theorem 2. *The function*

$$f : \mathbf{R}^{\mathbf{N}} \to \mathbf{R}$$

is relatively measurable with respect to the class of all extension of any σ-finite diffused Borel measure μ on $\mathbf{R}^{\mathbf{N}}$.

Notice that there exists nonzero, σ-finite, diffused Borel measure χ on $\mathbf{R}^{\mathbf{N}}$, which is invariant with respect to an everywhere dense vector subspace of $\mathbf{R}^{\mathbf{N}}$ and, in addition, is metrical transitive (i. e., ergodic) with respect to the same subspace (see, for example [3]). Also, notice that on the space $\mathbf{R}^{\mathbf{N}}$ there exists maximally large class of σ-finite, non-separable measures, which are invariant with respect to an everywhere dense vector subspace of $\mathbf{R}^{\mathbf{N}}$ and extend the measure χ (see [7]).

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