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ON NONMEASURABILITY OF ADDITIVE FUNCTIONS

Let f be a real-valued function which is defined in $\mathbf{R}$ and additive, i.e. satisfied Cauchy's classical functional equation

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathbf{R}$, where, as usual, by $\mathbf{R}$ is denoted the set of all real numbers.

It is well known that every additive function

$$f : \mathbf{R} \to \mathbf{R}$$

which is not of the form

$$f(x) = k \cdot x,$$

for all $x \in \mathbf{R}$, satisfies the following conditions:

(a) f is nonmeasurable with respect to the standard Lebesgue measure on $\mathbf{R}$;

(b) the graph of f is dense in the plane $\mathbf{R}^2$.

There are many text-books, manuals and monographs devoted to this subject (see, [1], [2], [3]).

Let μ be a measure on E. As usual, we say that μ is diffused (or continuous) if it vanishes on all singletons in E (i.e., $\mu(\{x\}) = 0$ for each point $x \in E$).

For any set E, let M_E denote the class of all nonzero σ-finite diffused measures on E. Assuming some additional set-theoretical axioms, it is not difficult to demonstrate that there exists an absolutely nonmeasurable function $f : \mathbf{R} \to \mathbf{R}$ with respect to the class $M_{\mathbf{R}}$. Consequently, we can formulate the following statement.

Lemma 1. *There exists a nontrivial solution of the Cauchy functional equation absolutely nonmeasurable with respect to the class* $M_{\mathbf{R}}$.

The proof above-mentioned fact can be found in [4].

Theorem 1. *Among the nontrivial solutions of the Cauchy functional equations one can meet those which are absolutely nonmeasurable with respect to the class of all translation invariant measures on* $\mathbf{R}$*, extending the Lebesgue measure.*

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Let (E_1, S_1, μ_1) and (E_2, S_2, μ_2) be two measurable spaces equipped with sigma-finite measures. We recall that a graph $\Gamma \subset E_1 \times E_2$ is $(\mu_1 \times \mu_2)$-thick in $E_1 \times E_2$ if for each $(\mu_1 \times \mu_2)$-measurable set $Z \subset E_1 \times E_2$ with $(\mu_1 \times \mu_2)(Z) > 0$, the intersection $\Gamma \cap Z$ is not empty (see, [3]).

Theorem 2. *There exists an additive function*

$$f : \mathbf{R} \to \mathbf{R}$$

having the following property: for any sigma-finite diffused Borel measure μ *on* $\mathbf{R}$ *and for any sigma-finite measure* ν *on* $\mathbf{R}$*, the graph of* f *is a* $(\mu \times \nu)$*-thick subset of the Euclidean plane* $\mathbf{R}^2$*.*

Notice that Theorem 1 and Theorem 2 are generalizations of properties a) and b) from a certain point of view.

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