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On small sets from the measure-theoretical point of view

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Abstract

For nonzero invariant (quasi-invariant) σ-finite measures on an uncountable group $(G, \cdot)$, the behaviour of small sets with respect to the group operation in G is studied.
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Let G be an arbitrary group and μ be a nonzero σ-finite G-invariant (more generally, G-quasi-invariant) measure defined on some σ-algebra of subsets of G. We recall that the symbol $I(\mu)$ denotes the σ-ideal of subsets of G, generated by the family of all μ-measure zero sets. Members of $I(\mu)$ are usually called negligible sets with respect to the given measure μ. Quite often, they are also called small sets with respect to μ.

Let us introduce one important notion concerning the general theory of small (negligible) sets.

Let G be an arbitrary group and let $Y \subset G$. We say that Y is G-absolutely negligible in G if, for any σ-finite G-invariant (G-quasi-invariant) measure μ on G, there exists a G-quasi-invariant measure $\hat{\mu}$ on G extending μ and such that $\hat{\mu}(Y) = 0$.

For more detailed information about the above-mentioned notion see [1–5].

Notice that it is natural to introduce the notion of a small set not only with respect to a given invariant (quasi-invariant) measure but also with respect to a given class of invariant (quasi-invariant) measures (see, for example, [1,5,6]).

The following statement gives a purely geometrical characterization of absolutely negligible sets and plays an essential role the process of studying various properties of these sets.

Lemma 1. *Let $(G, \cdot)$ be an arbitrary uncountable group and let Y be a subset of G. Then the following two relations are equivalent:*

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(1) *Y is a G-absolutely negligible set in G;*
(2) *for each countable family $\{g_i : i \in I\}$ of elements from G, there exists a countable family $\{h_j : j \in J\}$ of elements from G, such that*

$$\cap_{j \in J}(h_j \cdot (\cup_{i \in I}(g_i \cdot Y))) = \emptyset.$$

For the proof of this lemma, see e.g. [1] or [4].

By applying a Hamel basis of the real line **R**, W. Sierpinski has established the following statement.

Proposition. *Let λ be the standard Lebesgue measure on **R**. Then there exist two sets $X \subset \mathbf{R}$ and $Y \subset \mathbf{R}$ satisfying the relations*

$$X \in I(\lambda), \ Y \in I(\lambda), \quad X + Y \notin dom(\lambda).$$

For more details, see [6]. Some generalization of this result for uncountable vector spaces over the field **Q** of all rational numbers and for quasi-invariant extensions of measures on such spaces can be found in [7]. Similar properties of algebraic sums of subsets of the real line **R** are also discussed in [4,8].

It is reasonable to ask whether similar statements hold in more general situations when no topology is considered on given group. Namely, it is natural to pose the following question:

Let $(G, \cdot)$ be an uncountable group equipped with a nonzero σ-finite G-invariant (G-quasi-invariant) measure μ. Do there exist two sets $X \in I(\mu)$ and $Y \in I(\mu)$ such that $X \cdot Y$ does not belong to $dom(\mu)$.

For an arbitrary uncountable commutative group $(G, +)$ and for a nonzero σ-finite complete G-invariant (G-quasi-invariant) measure μ we have a direct analogue of the second part of above-mentioned proposition by Sierpinski. In particular, the following statement is valid.

Theorem 1. *Let $(G, +)$ be an uncountable commutative group and let μ be a nonzero σ-finite G-invariant measure on G. There exists a G-invariant complete measure $\hat{\mu}$ on G extending μ and such that, for some two sets $X \in I(\hat{\mu})$ and $Y \in I(\hat{\mu})$, the relation*

$$X + Y \notin dom(\hat{\mu})$$

is satisfied.

The proof of Theorem 1 can be found, for instance in [3].

It seems to be interesting to generalize the above result (i.e. Theorem 1) to a wider class of uncountable groups $(G, \cdot)$ (not necessarily commutative). From this point of view the following statement can be formulated.

Theorem 2. *Let $(G, \cdot)$ be an uncountable solvable group and let μ be a nonzero σ-finite G-invariant measure on G. There exists a G-invariant complete measure $\hat{\mu}$ on G extending μ and such that, for some two sets $X \in I(\hat{\mu})$ and $Y \in I(\hat{\mu})$, the relation*

$$X + Y \notin dom(\hat{\mu})$$

is satisfied.

In connection with Theorem 2, let us remark that its proof is obtained by using the method of surjective homomorphisms (see [4,5] for a detailed description this method).

Finally notice that an analogous problem for arbitrary noncommutative groups is still open.

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