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Original article

On some methods of extending invariant and quasi-invariant measures

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Abstract

In the present paper an approach to some questions in the theory of invariant (quasi-invariant) measures is discussed. It is useful in certain situations, where given topological groups or topological vector spaces are equipped with various nonzero σ-finite left invariant (left quasi-invariant) measures.

Keywords: Invariant measure; Quasi-invariant measure; Extensions of measures; Surjective homomorphism

The measure extension problem is one of the most important questions in measure theory. It forms a basis for harmonic analysis, the theory of functions of a real variable, probability theory, the theory of dynamical systems, and many other domains of contemporary mathematics. An interesting and important direction in measure theory is concerned with the investigation of properties of various (countably-additive) extensions of initial measures.

In this connection, there are some well-known methods of extending invariant measures: Marczewski's method; the method of Kodaira and Kakutani; the method of Kakutani and Oxtoby; the method of surjective homomorphisms.

Various aspects of the theory of extensions of invariant (and, more generally, quasi-invariant) measures are widely presented in the works of many authors (see, [1–11]).

A measure μ defined on some G-invariant σ-algebra of subsets of $(G, \cdot)$ is called quasi-invariant with respect to G (briefly, G-quasi-invariant) if, for every μ-measurable set X and for each $g \in G$, the relation

$$\mu(X) = 0 \Leftrightarrow \mu(g \cdot X) = 0$$

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holds true. Moreover, if the equality

$$\mu(g \cdot X) = \mu(X)$$

is valid for any μ-measurable X and for any $g \in G$, then μ is called an invariant measure with respect to G (briefly, G-invariant measure).

Above-mentioned problem has the following three aspects:

(1) purely set-theoretical aspect;

(2) algebraic aspect;

(3) topological aspect.

A sufficiently general method of extending measures was suggested by Marczewski. This method is purely set-theoretical because no specific properties of given measurable space are used. According to a result of Marczewski, we can always extend Lebesgue measure to an isometrically-invariant countably additive measure (see, for example [1]).

A. Kharazishvili applied a purely algebraic method of surjective homomorphisms and solved the analogue of W.Sierpinski's problem for nonzero sigma-finite quasi-invariant (invariant) measures on arbitrary uncountable solvable groups (see, [4]).

An important special case of the method of surjective homomorphisms is the method of direct products which can be described as follows.

Suppose that two groups $(G, \cdot)$ and $(H, \cdot)$ are given and a set $X \subset G$ has a "nice" measure-theoretical property with respect to G. Then, in some situations, it turns out that the set $X \times H$ preserves this property with respect to the direct product $G \times H$. Notice that here $(G, \cdot)$ and $(H, \cdot)$ are arbitrary groups (not necessarily commutative).

Example 1. The method of direct products is essential for studying the property of metrical transitivity (ergodicity) of given measure. In particular, if an invariant measure μ_1 is metrically transitive with respect to a countable transformation group G_1 and an invariant measure μ_2 is metrically transitive with respect to a transformation group G_2, then the product measure $\mu_1 \times \mu_2$ is metrically transitive with respect to the product group $G_1 \times G_2$. Since the metrical transitivity of a measure is closely connected with the uniqueness property, one can conclude that the method of direct products turns out to be helpful for establishing the uniqueness property of a given invariant measure.

About Example 1 see, [12,13].

Example 2. The method of direct products is useful for obtaining some generalizations of W. Sierpinski's old result for an uncountable group $(G, \cdot)$, with the regular $card(G) = \alpha$. In particular, let $(G, \cdot)$ be an arbitrary group such that

$$G = G_1 \cdot G_2 \quad (G_1 \cap G_2 = \{e\})$$

where G_1 and G_2 are subgroups of G and $card(G_1) = \omega_1$ and e denotes the neutral element of G. If μ is a nonzero σ-finite G-quasi-invariant measure on G, then for each uncountable set $X \subset G_1$, there exist a G-quasi-invariant measure μ' on G extending μ and a set $Y \in I(\mu')$, for which we have

$$X \cdot Y = G \notin I(\mu'),$$

where $I(\mu')$ is the σ-ideal generated by all μ'-measure zero sets in G.

In particular, if $X \in I(\mu')$, then G is representable in the form of algebraic product of two μ'-measure zero sets.

About Example 2 see, [14].

Example 3. The method of direct products is also useful for constructing non-separable extensions of invariant measures given on infinite-dimensional topological groups or topological vector spaces. In the infinite-dimensional topological vector space $\mathbf{R}^{\mathbf{N}}$, a nonzero σ-finite invariant Borel measure χ was constructed, which is metrically transitive with respect to a dense vector subspace of $\mathbf{R}^{\mathbf{N}}$. On the other hand, in the Euclidean space $\mathbf{R}^n$ there exists a non-separable metrically transitive invariant measure μ extending the standard Lebesgue measure λ_n in $\mathbf{R}^n$. By applying the method of direct products it can be shown that the product measure $\chi \times \mu$ is non-separable, invariant with respect to a dense vector subspace of $\mathbf{R}^{\mathbf{N}}$, and metrically transitive with respect to the same subspace. Consequently, the completion of $\chi \times \mu$ has the uniqueness property.

About Example 3 see, [5–7,15].

Example 4. By using the same method, it was constructed a non-locally compact non-commutative topological group for which there exists a nonzero Borel measure quasi-invariant with respect to some dense connected subgroup.

About Example 4 see, [16].

Let (G_1, μ_1) and (G_2, μ_2) be any two groups endowed with σ-finite invariant measures and let

$$\varphi : G_1 \to G_2$$

be a surjective homomorphism. Suppose that a general property $P(X)$ of a set $X \subset G_2$ is given. Sometimes, it turns out that

$$P(\varphi^{-1}(X)) \Leftrightarrow P(X).$$

In such a situation we say that $P(X)$ is stable under surjective homomorphisms.

In particular, if φ coincides with the canonical surjective homomorphism

$$pr_2 : H \times G_2 \to G_2,$$

then we may apply the method of direct products, where $H \subset G_1$ and the role of G_1 is played by $H \times G_2$.

Example 5. Let G be an arbitrary group and let $Y \subset G$. We say that Y is G-absolutely negligible in G if, for any σ-finite G-invariant (G-quasi-invariant) measure μ on G, there exists a G-invariant (G-quasi-invariant) measure $\hat{\mu}$ on G extending μ and such that $\hat{\mu}(Y) = 0$.

By using the method of surjective homomorphisms, it was shown that for any uncountable commutative group $(G, +)$, there exists two G-absolutely negligible subsets A and B such that their algebraic sum $A + B$ coincides the whole of G.

About Example 5 see, [17,18].

Example 6. The method of surjective homomorphisms is crucial for establishing the existence of a non-atomic non-separable σ-finite left invariant measure on an arbitrary uncountable solvable group. Also, it is possible to get a lower estimate of the topological weight of a nonseparable left-invariant measure given on an uncountable solvable group in terms of cardinalities of the factors of the composition series of this group.

About Example 6 see, [19,20].

The following simple statement is valid.

Lemma 1. *Let $(G_1, \cdot)$ and $(G_2, \cdot)$ be arbitrary uncountable groups. Let the group G_2 be equipped with a σ-finite G_2-left-invariant (left G_2-quasi-invariant) measure μ and let*

$$\varphi : G_1 \to G_2$$

be a surjective homomorphism. Consider the family of sets

$$S = \{\varphi^{-1}(Y) : Y \in dom(\mu)\},$$

and define a functional ν on this family by putting

$$\nu(\varphi^{-1}(Y)) = \mu(Y),$$

where $Y \in dom(\mu)$.

Then this functional is a measure satisfying the following relations:

(a) *S is a G_1-left-invariant σ-algebra of subsets of G_1;*

(b) *ν is a non-atomic σ-finite G_1-left-invariant measure on S.*

According to Lemma 1 we obtain the following statement.

Theorem 1. *Let $(G_1, \cdot)$ and $(G_2, \cdot)$ be arbitrary uncountable groups. Let the group G_2 be equipped with a σ-finite G_2-left-invariant (left G_2-quasi-invariant) measure μ and let*

$$\varphi : G_1 \to G_2$$

be a surjective homomorphism.

If a measure μ' is some σ-finite G_2-left-invariant (G_2-left-quasi-invariant) extension of measure μ on G_2, then ν' is σ-finite G_1-left-invariant (G_1-left-quasi-invariant) extension of the measure ν on G_1, where ν and ν' are measures respectively corresponding to μ and μ' under the surjective homomorphism φ.

Now, let $\{Y_i : i \in I\}$ be an uncountable family of μ-measurable subsets of G_2. Applying Lemma 1, we may write

$$\nu(\varphi^{-1}(Y_j \odot Y_k)) = \nu(\varphi^{-1}(Y_j \odot Y_k)) = \mu(Y_j \odot Y_k),$$

where $j \in I, k \in I$ and symbol "$\odot$" denotes any of the basic set-theoretical operations (union, intersection, difference, symmetrical difference and so on).

From the above general principle, we have the following statement:

Let $(G_1, \cdot)$ and $(G_2, \cdot)$ be arbitrary uncountable groups. Let the group G_2 be equipped with a σ-finite G_2-left-invariant (left G_2-quasi-invariant) measure μ and let

$$\varphi : G_1 \to G_2$$

be a surjective homomorphism of the group G_1 into the group G_2. Consider the family of sets

$$S = \{\varphi^{-1}(Y) : Y \in dom(\mu)\},$$

and define a functional ν on this family by putting

$$\nu(\varphi^{-1}(Y)) = \mu(Y),$$

where $Y \in dom(\mu)$.

If the measure μ has some set-theoretical property, then the measure ν has the same property.

A useful method of extending measures is by applying those mappings whose graphs are thick from the measure-theoretical point of view. Thus method was successfully applied by Kodaira and Kakutani in their famous construction of a nonseparable translation-invariant extension of the Lebesgue measure on **R** (see, [8]).

Let (G_1, μ_1) and (G_2, μ_2) be any two groups endowed with σ-finite left-invariant measures.

We recall that a subset $\Gamma \subset G_1 \times G_2$ is $(\mu_1 \times \mu_2)$-thick in $G_1 \times G_2$ if, for each $(\mu_1 \times \mu_2)$-measurable set $Z \subset (G_1 \times G_2)$ with $(\mu_1 \times \mu_2)(Z) > 0$, we have $\Gamma \cap Z \neq \emptyset$.

Let

$$f : G_1 \to G_2$$

be a homomorphism.

We say that f is an almost surjective homomorphism if the graph of f is $(\mu_1 \times \mu_2)$-thick in $G_1 \times G_2$.

Our argument may be regarded as a certain combination of the method of Kodaira and Kakutani with the method of surjective homomorphisms.

The following theorems are true.

Theorem 2. *Let $(G_1, \cdot)$ and $(G_2, \cdot)$ be arbitrary uncountable groups and let the group G_2 be equipped with a G_2-left-invariant (left G_2-quasi-invariant) probability measure μ_2 and let*

$$f : G_1 \to G_2$$

be an almost surjective homomorphism of the group G_1 onto the group G_2.

Then there exist two measures μ_1 and μ_1' on G_1 such that:

(1) *μ_1 is a non-atomic σ-finite G_1-left-invariant measure on G_1;*

(2) *μ_1' extends μ_1;*

(3) *μ_1' is a G_1-left-invariant measure.*

Proof. Suppose that for sets $Y_1 \in dom(\mu_2)$ and $Y_2 \in dom(\mu_2)$ the following assertion is true:

$$f^{-1}(Y_1) = f^{-1}(Y_2).$$

Consequently, we have

$$f^{-1}(Y_1 \triangle Y_2) = \emptyset.$$

Therefore, we get

$$(Y_1 \triangle Y_2) \cap Gr(f) = \emptyset.$$

In view of the thickness of the graph $Gr(f)$ of f, we infer that

$$\mu_2(Y_1 \triangle Y_2) = 0.$$

Hence

$$\mu_2(Y_1) = \mu_2(Y_2).$$

This implies that the definition of μ_1 is correct.

In a manner similar to Lemma 1, we prove the left-invariance of measure μ_1 under the group G_1.

Now, for each $(\mu_1 \times \mu_2)$-measurable set $Z \subset G_1 \times G_2$, we denote

$$Z' = \{x \in G_1 : (x, f(x)) \in Z\}.$$

Further, we put

$$S = \{Z' : Z \in dom(\mu_1 \times \mu_2)\}.$$

It can easily be verified that S is a σ-algebra of subsets of G_1. We define a functional μ_1' on S by the formula

$$\mu_1'(Z') = (\mu_1 \times \mu_2)(Z) \qquad (Z \in dom(\mu_1 \times \mu_2)).$$

It is easy to show that the definition of μ_1' is correct in view of the $(\mu_1 \times \mu_2)$-thickness of the graph of f. Also, μ_1' turns out to be a measure on S, which extends the original measure μ_1.

This ends the proof of Theorem 2. □

The following auxiliary statement is true.

Lemma 2. *Let $(G, \cdot)$ be arbitrary uncountable group. Let the group G be equipped with a σ-finite G-left-quasi-invariant measure μ on a σ-algebra S and satisfying the equality $\mu(G) = +\infty$. Then on the same σ-algebra there exists a probability G-left-quasi-invariant measure ν such that the measures μ and ν are equivalent.*

Proof. Let $\{X_n : n \in \mathbf{N}\} \subseteq S$ be a countable family of pairwise disjoint sets such that

$$\cup_n X_n = G$$

and

$$0 < \mu(X_n) < +\infty$$

for each $n \in \mathbf{N}$, where $\mathbf{N}$ is the set of all natural numbers. Let us consider the measure ν on the σ-algebra S defined by the formula

$$\nu(X) = \Sigma_n \frac{1}{2^{n+1}} \cdot \frac{\mu(X \cap X_n)}{\mu(X_n)} \quad (X \in S).$$

It is clear that ν is a probability measure on S. If X is an arbitrary set from S, then $\nu(X) > 0$ if and only if $\mu(X) > 0$. In this case the measures μ and ν are equivalent.

Thus, the formulated Lemma is proved. □

From Threorem 2 and Lemma 2 the next statement can be obtained.

Theorem 3. *Let $(G_1, \cdot)$ and $(G_2, \cdot)$ be arbitrary uncountable groups and let the group G_2 be equipped with a nonzero σ-finite G_2-left-quasi-invariant measure μ_2 and let*

$$f : G_1 \to G_2$$

be an almost surjective homomorphism of the group G_1 onto the group G_2.

Then there exist two measures μ_1 and μ_1' such that:

(1) μ_1 is a non-atomic σ-finite G_1-left-quasi-invariant measure on G_1;

(2) μ_1' extends μ_1;

(3) μ_1' is a G_1-left-quasi-invariant measure.

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